

# Bäcklund transformation, multiple wave solutions and lump solutions to a $(3 + 1)$ -dimensional nonlinear evolution equation

Li-Na Gao · Yao-Yao Zi · Yu-Hang Yin ·  
Wen-Xiu Ma · Xing Lü

Received: 13 October 2016 / Accepted: 22 May 2017 / Published online: 19 June 2017  
© Springer Science+Business Media Dordrecht 2017

**Abstract** In this paper, a  $(3 + 1)$ -dimensional nonlinear evolution equation is cast into Hirota bilinear form with a dependent variable transformation. A bilinear Bäcklund transformation is then presented, which consists of six bilinear equations and involves nine arbitrary parameters. With multiple exponential function method and symbolic computation, nonresonant-typed one-, two-, and three-wave solutions are obtained. Furthermore, two classes of lump solutions to the dimensionally reduced cases with  $y = x$  and  $y = z$  are both derived. Finally, some figures are given to reveal the propagation of multiple wave solutions and lump solutions.

L.-N. Gao · Y.-Y. Zi · Y.-H. Yin · X. Lü (✉)  
Department of Mathematics, Beijing Jiao Tong University,  
Beijing 100044, People's Republic of China  
e-mail: XLV@bjtu.edu.cn  
e-mail: xinglv655@aliyun.com

W.-X. Ma  
Department of Mathematics and Statistics, University of  
South Florida, Tampa, FL 33620-5700, USA

W.-X. Ma  
College of Mathematics and Systems Science, Shandong  
University of Science and Technology, Qingdao 266590,  
Shandong, People's Republic of China

W.-X. Ma  
International Institute for Symmetry Analysis and Mathematical  
Modelling, Department of Mathematical Sciences, North-West  
University, Mafikeng Campus, Private Bag X 2046, Mmabatho  
2735, South Africa

**Keywords** Bäcklund transformation · Nonresonant multiple wave solutions · Lump solution · Symbolic computation

**Mathematics Subject Classification** 35Q51 · 35Q55 · 37K40

## 1 Introduction

Nonlinear evolution equations (NLEEs) including soliton equations play an important role in the areas of mathematical physics [1–7]. Generally speaking, it is very difficult to find exact solutions to NLEEs [8–24]. The transformed rational function method and multiple exponential function method provide two effective pathways to construct multiple wave solutions [13, 14]. If we can get the Hirota bilinear form for a NLEE, then we can derive the exact solutions with multiple exponential function algorithm. Furthermore, the Bäcklund transformation (BT) can also be used in solution aspects [20]. Based on a known solution, we can obtain another solution by using BT.

In this paper, we will study the following  $(3 + 1)$ -dimensional NLEE [7, 12] as

$$u_{y\ell} - u_{xxxy} - 3(u_x u_y)_x - 3u_{xx} + 3u_{zz} = 0, \quad (1)$$

which was proposed firstly in Ref. [7, 12], and the resonant behavior of multiple wave solutions has been investigated [12]. With symbolic computation, two classes of lump solutions have been derived to the dimensionally reduced equations in  $(2 + 1)$ -dimensions

with  $z = y$  and  $z = t$ , respectively, by searching for positive quadratic function solutions to associated bilinear equation [7]. It is important to study other properties for Eq. (1) such as BT, nonresonant multiple wave solutions, and lump dynamics with novel dimensional reductions.

The structure of this paper is as follows: In Sect. 2, we will construct a BT for Eq. (1) based on its bilinear form, and as an application, we will derive some exact solutions via this BT. Note that the BT consists of six bilinear equations and involves nine arbitrary parameters. Nonresonant-typed multiple wave solutions will be solved in Sect. 3 by use of multiple exponential function method. In Sect. 4, we will give two classes of lump solutions to the dimensionally reduced equations in  $(2+1)$ -dimensions with  $y = x$  and  $y = z$ , respectively. Finally, Sect. 5 presents discussions and conclusions, and we will plot some figures to describe the characteristics of multiple wave solutions and lump solutions.

## 2 Bilinear BT

### 2.1 Construction of BT

Substitution of the dependent variable transformation  $u = 2(\ln f)_x$  with  $f = f(x, y, z, t)$  into Eq. (1) yields the bilinear representation for Eq. (1) as

$$(D_t D_y - D_x^3 D_y - 3D_x^2 + 3D_z^2) f \cdot f = 0, \quad (2)$$

where  $D_t D_y$ ,  $D_x^3 D_y$ ,  $D_x^2$ , and  $D_z^2$  are all the bilinear derivative operators [20] defined by

$$D_x^\alpha D_y^\beta D_t^\gamma (\rho \cdot \varrho) = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^\alpha \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^\beta \times \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^\gamma \rho(x, y, t) \varrho(x', y', t') \Big|_{x'=x, y'=y, t'=t}. \quad (3)$$

To construct a bilinear BT by means of Eq. (2), we consider

$$2P \equiv 2f^2 \left( D_t D_y - D_x^3 D_y - 3D_x^2 + 3D_z^2 \right) g \cdot g - 2g^2 \left( D_t D_y - D_x^3 D_y - 3D_x^2 + 3D_z^2 \right) f \cdot f, \quad (4)$$

in which  $g = g(x, y, z, t)$  is another solution to Eq. (2).

By using exchange formulas, symbolic computation on Eq. (4) leads to

$$\begin{aligned} -2P &= -2[(D_t D_y - D_x^3 D_y - 3D_x^2 + 3D_z^2) g \cdot g] f^2 \\ &\quad + 2g^2 [(D_t D_y - D_x^3 D_y - 3D_x^2 + 3D_z^2) f \cdot f] \\ &= -2[(D_t D_y g \cdot g) f^2 - g^2 (D_t D_y f \cdot f)] \\ &\quad + 2[(D_x^3 D_y g \cdot g) f^2 - g^2 (D_x^3 D_y f \cdot f)] \\ &\quad + 6[(D_x^2 g \cdot g) f^2 - g^2 (D_x^2 f \cdot f)] \\ &\quad - 6[(D_z^2 g \cdot g) f^2 - g^2 (D_z^2 f \cdot f)] \\ &= D_x (3D_x^2 D_y g \cdot f) \cdot f g + D_x (3D_x^2 g \cdot f) \cdot (D_y f \cdot g) \\ &\quad + D_x (6D_x D_y g \cdot f) \cdot (D_x f \cdot g) + D_x (12D_x g \cdot f) \cdot f g \\ &\quad + D_y (D_x^3 g \cdot f) \cdot f g + D_y (3D_x^2 g \cdot f) \cdot (D_x f \cdot g) \\ &\quad + D_t (-4D_y g \cdot f) \cdot f g - 12D_z (D_z g \cdot f) \cdot f g \\ &= D_x [(3D_x^2 D_y + \lambda_1 D_y \\ &\quad + \lambda_2 + 12D_x + 12\lambda_8 D_z) g \cdot f] \cdot f g \\ &\quad + D_y [(D_x^3 + \lambda_3 - \lambda_1 D_x - 4D_t - 12\lambda_9 D_z) g \cdot f] \cdot f g \\ &\quad + D_x [(3D_x^2 + \lambda_4 D_y + \lambda_6) g \cdot f] \cdot (D_y f \cdot g) \\ &\quad + D_y [(3D_x^2 + \lambda_5 D_x - \lambda_6) g \cdot f] \cdot (D_x f \cdot g) \\ &\quad + D_x [(6D_x D_y + 6\lambda_7 D_x) g \cdot f] \cdot (D_x f \cdot g) \\ &\quad - 12D_z [(D_z + \lambda_8 D_x + \lambda_9 D_y) g \cdot f] \cdot f g, \end{aligned} \quad (5)$$

where we have introduced nine arbitrary coefficients of  $\lambda_i$  ( $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$ ). To this stage, equation decoupling of Eq. (5) gives rise to an alternative BT for Eq. (2) as

$$\begin{cases} B_1 g \cdot f = (3D_x^2 D_y + \lambda_1 D_y + \lambda_2 + 12D_x + 12\lambda_8 D_z) g \cdot f = 0, \\ B_2 g \cdot f = (D_x^3 + \lambda_3 - \lambda_1 D_x - 4D_t - 12\lambda_9 D_z) g \cdot f = 0, \\ B_3 g \cdot f = (3D_x^2 + \lambda_4 D_y + \lambda_6) g \cdot f = 0, \\ B_4 g \cdot f = (3D_x^2 + \lambda_5 D_x - \lambda_6) g \cdot f = 0, \\ B_5 g \cdot f = (6D_x D_y + 6\lambda_7 D_x) g \cdot f = 0, \\ B_6 g \cdot f = (D_z + \lambda_8 D_x + \lambda_9 D_y) g \cdot f = 0, \end{cases} \quad (6)$$

which consists of six bilinear equations and involves nine arbitrary parameters.

### 2.2 Application of BT

We take  $f = 1$  as a solution to Eq. (2), which corresponds to the solution  $u = 2(\ln f)_x = 0$  to Eq. (1). Solving BT of Eq. (6), we obtain six linear partial differential equations as

$$\begin{cases} 3g_{xxy} + 12g_x + \lambda_1 g_y + 12\lambda_8 g_z + \lambda_2 g = 0, \\ g_{xxx} - 4g_t - \lambda_1 g_x - 12\lambda_9 g_z + \lambda_3 g = 0, \\ 3g_{xx} + \lambda_5 g_x - \lambda_6 g = 0, \\ g_{xy} + \lambda_7 g_x = 0, \\ g_z + \lambda_8 g_x + \lambda_9 g_z = 0, \\ 3g_{xx} + \lambda_4 g_y + \lambda_6 g = 0. \end{cases} \quad (7)$$

In the following, we will derive two classes of exact solutions to Eq. (1) by solving Eq. (7) with symbolic computation.

### 2.2.1 Exponential function solution to Eq. (7)

Firstly, we consider exponential function solutions to Eq. (7) by taking  $g = 1 + \varepsilon e^\theta$  with  $\theta = kx + ly + mz - wt$ , where  $\varepsilon, k, l, m$ , and  $w$  are all constants.

Selecting  $\lambda_2 = \lambda_3 = \lambda_6 = 0$ , and solving Eq. (7), we have

$$\begin{cases} 3k^2l + 12k + \lambda_1l + 12\lambda_8m = 0, \\ k^3 + 4w - \lambda_1k - 12\lambda_9m = 0, \\ 3k^2 + \lambda_5k = 0, \\ 3k^2 + \lambda_4l = 0, \\ kl + \lambda_7k = 0, \\ m + \lambda_8k + \lambda_9l = 0. \end{cases} \quad (8)$$

One choice of solutions to Eq. (8) is as follows

$$\begin{cases} \lambda_1 = \frac{12\lambda_8^2k + 12\lambda_8\lambda_9l - 3k^2l - 12k}{l}, \quad \lambda_4 = -\frac{3k^2}{l}, \\ \lambda_5 = -3k, \quad \lambda_7 = -l, \quad m = -(\lambda_8k + \lambda_9l), \\ \omega = \frac{3\lambda_8^2k^2 - 3\lambda_9^2l^2 - k^3l - 3k^2}{l} \end{cases} \quad (9)$$

then,

$$\begin{aligned} u &= 2(\ln g)_x \\ &= \frac{2k\varepsilon e^{kx+ly-(\lambda_8k+\lambda_9l)z-\frac{3\lambda_8^2k^2-3\lambda_9^2l^2-k^3l-3k^2}{l}t}}{1 + \varepsilon e^{kx+ly-(\lambda_8k+\lambda_9l)z-\frac{3\lambda_8^2k^2-3\lambda_9^2l^2-k^3l-3k^2}{l}t}}, \end{aligned} \quad (10)$$

is a solution to Eq. (1).

### 2.2.2 First-order polynomial solution to Eq. (7)

Secondly, we take a first-order polynomial solution into consideration as

$$g = kx + ly + mz - wt, \quad (11)$$

where  $\varepsilon, k, l, m$ , and  $w$  are all constants.

Selecting  $\lambda_i = 0$  ( $1 \leq i \leq 7$ ) and putting Eq. (11) into (7), we obtain the following algebraic equations

$$\begin{cases} 12k + 12\lambda_8m = 0, \\ 4w - 12\lambda_9m = 0, \\ m + \lambda_8k + \lambda_9l = 0, \end{cases} \quad (12)$$

which result in  $3k^2 - lw - 3m^2 = 0$ .

To this stage,

$$\begin{aligned} u &= 2(\ln g)_x \\ &= \frac{2k}{kx + ly + mz - wt} \quad \text{with } 3k^2 - lw - 3m^2 = 0, \end{aligned} \quad (13)$$

is a solution to Eq. (1).

## 3 Multiple wave solutions

### 3.1 One-wave solution

Substituting

$$f = 1 + \varepsilon e^\theta, \quad \theta = kx + ly + mz - wt,$$

with  $\varepsilon, k, l, m$ , and  $w$  as constants into Eq. (2), we can obtain

$$w = -k^3 + \frac{3m^2 - 3k^2}{l}, \quad (14)$$

which is called the dispersion relation. As a result, the one-wave solution to Eq. (1) can be written as

$$u = \frac{2\varepsilon k e^{kx+ly+mz-wt}}{1 + \varepsilon e^{kx+ly+mz-wt}}. \quad (15)$$

### 3.2 Two-wave solution

Substituting

$$\begin{aligned} f &= 1 + \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2} \\ &\quad + \varepsilon_1 \varepsilon_2 a_{12} e^{\theta_1 + \theta_2}, \quad \theta_i = k_i x + l_i y + m_i z - w_i t, \end{aligned}$$

with  $\varepsilon_1, \varepsilon_2, k_i, l_i, m_i$ , and  $w_i$  ( $i = 1, 2$ ) as constants into Eq. (2), we get

$$w_i = -k_i^3 + \frac{3m_i^2 - 3k_i^2}{l_i}, a_{12} = \frac{b_{12}}{c_{12}}, \quad (16)$$

where

$$\begin{aligned} b_{12} = & k_1^3 l_1 + k_2^3 l_2 - k_1^3 l_2 - k_2^3 l_1 - 3k_1^2 k_2 l_1 - 3k_1 k_2^2 l_2 \\ & + 3k_1^2 k_2 l_2 + 3k_1 k_2^2 l_1 + l_1 w_1 + l_2 w_2 - w_1 l_2 \\ & - w_2 l_1 + 3k_1^2 + 3k_2^2 - 6k_1 k_2 - 3m_1^2 \\ & - 3m_2^2 + 6m_1 m_2, \end{aligned} \quad (17)$$

$$\begin{aligned} c_{12} = & -(k_1 + k_2)^3 (l_1 + l_2) - (l_1 + l_2)(w_1 + w_2) \\ & + 3(m_1 + m_2)^2 - 3(k_1 + k_2)^2. \end{aligned} \quad (18)$$

where

$$\begin{aligned} b_{ij} = & k_i^3 l_i + k_j^3 l_j - k_i^3 l_j - k_j^3 l_i - 3k_i^2 k_j l_i - 3k_i k_j^2 l_j \\ & + 3k_i^2 k_j l_j + 3k_i k_j^2 l_i + l_i w_i + l_j w_j - w_i l_j - w_j l_i \\ & + 3k_i^2 + 3k_j^2 - 6k_i k_j - 3m_i^2 - 3m_j^2 + 6m_i m_j, \\ c_{ij} = & -(k_i + k_j)^3 (l_i + l_j) - (l_i + l_j)(w_i + w_j) \\ & + 3(m_i + m_j)^2 - 3(k_i + k_j)^2. \end{aligned}$$

Finally, the three-wave solution to Eq. (1) is

$$u = \frac{2 \left[ \sum_{i=1}^3 k_i \varepsilon_i e^{\theta_i} + \sum_{1 \leq i < j \leq 3} (k_i + k_j) \varepsilon_i \varepsilon_j a_{ij} e^{\theta_i + \theta_j} + (k_1 + k_2 + k_3) \varepsilon_1 \varepsilon_2 \varepsilon_3 a_{123} e^{\theta_1 + \theta_2 + \theta_3} \right]}{1 + \sum_{i=1}^3 \varepsilon_i e^{\theta_i} + \sum_{1 \leq i < j \leq 3} \varepsilon_i \varepsilon_j a_{ij} e^{\theta_i + \theta_j} + \varepsilon_1 \varepsilon_2 \varepsilon_3 a_{123} e^{\theta_1 + \theta_2 + \theta_3}}, \quad (22)$$

As a result, the two-wave solution to Eq. (1) can be written as

$$u = \frac{2 \left[ k_1 \varepsilon_1 e^{\theta_1} + k_2 \varepsilon_2 e^{\theta_2} + a_{12} (k_1 + k_2) \varepsilon_1 \varepsilon_2 e^{\theta_1 + \theta_2} \right]}{1 + \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2} + a_{12} \varepsilon_1 \varepsilon_2 e^{\theta_1 + \theta_2}}, \quad (19)$$

where  $w_i$  and  $a_{12}$  are determined by Eq. (16).

### 3.3 Three-wave solution

Following the derivation of one- and two-wave solutions, we assume

$$\begin{aligned} f = & 1 + \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2} + \varepsilon_3 e^{\theta_3} + \varepsilon_1 \varepsilon_2 a_{12} e^{\theta_1 + \theta_2} \\ & + \varepsilon_1 \varepsilon_3 a_{13} e^{\theta_1 + \theta_3} \\ & + \varepsilon_2 \varepsilon_3 a_{23} e^{\theta_2 + \theta_3} + \varepsilon_1 \varepsilon_2 \varepsilon_3 a_{123} e^{\theta_1 + \theta_2 + \theta_3}, \end{aligned} \quad (20)$$

where  $\theta_i = k_i x + l_i y + m_i z - w_i t$ , ( $i = 1, 2, 3$ ),  $a_{123} = a_{12} a_{13} a_{23}$ , and  $\varepsilon_1, \varepsilon_2, \varepsilon_3, k_i, l_i, m_i, w_i$  are constants. With symbolic computation, we get

$$w_i = -k_i^3 + \frac{3m_i^2 - 3k_i^2}{l_i}, \quad a_{ij} = \frac{b_{ij}}{c_{ij}}, \quad (i = 1, 2, 3), \quad (21)$$

where  $\varepsilon_1, \varepsilon_2, k_i, l_i, m_i$  are arbitrary constants, and  $w_i$  and  $a_{ij}$  are determined by Eq. (21).

## 4 Lump solutions

In this section, we will search for positive quadratic function solutions to dimensionally reduced bilinear Eq. (2) via taking  $y = x$  or  $y = z$ , correspondingly to construct lump solutions to dimensionally reduced forms of Eq. (1). We begin with the assumption

$$f = g^2 + h^2 + a_9, \quad (23)$$

and

$$\begin{aligned} g = & a_1 x + a_2 z + a_3 t + a_4, \\ h = & a_5 x + a_6 z + a_7 t + a_8, \end{aligned}$$

where  $a_i$  ( $1 \leq i \leq 9$ ) are all real parameters to be determined. To construct the lump solutions, we note that the conditions guaranteeing the well definedness of  $f$ , positiveness of  $f$  and localization of  $u$  in all directions in the space need to be satisfied.

### 4.1 Lump solutions to reduction with $y = x$

With  $y = x$ , the dimensionally reduced form of the bilinear Eq. (2) turns out to be

$$\left(D_t D_x - D_x^4 - 3D_x^2 + 3D_z^2\right) f \cdot f = 0, \quad (24)$$

that is,

$$\begin{aligned} & (f_{xt} f - f_t f_x) - \left(f_{xxx} f - 4f_{xx} f_x + 3f_{xx}^2\right) \\ & - 3\left(f_{xx} f - f_x^2\right) + 3\left(f_{zz} f - f_z^2\right) = 0, \end{aligned} \quad (25)$$

which is transformed into

$$u_{xt} - u_{xxx} - 6u_x u_{xx} - 3u_{xx} + 3u_{zz} = 0. \quad (26)$$

through the link between  $f$  and  $u$ :

$$u = 2 \left[ \ln f(x, z, t) \right]_x = 2 \frac{f_x(x, z, t)}{f(x, z, t)}. \quad (27)$$

Submitting Eq. (23) into (25), we obtain the following set of constraining equations for the parameters:

$$\begin{cases} a_1 = a_1, & a_2 = a_2, \\ a_3 = \frac{3a_1(a_1^2 - a_2^2 + a_5^2 + a_6^2) - 6a_2 a_5 a_6}{a_1^2 + a_5^2}, \\ a_4 = a_4, & a_5 = a_5, & a_6 = a_6, \\ a_7 = \frac{3a_5(a_1^2 + a_2^2 + a_5^2 - a_6^2) - 6a_1 a_2 a_6}{a_1^2 + a_5^2}, \\ a_8 = a_8, & a_9 = \frac{(a_1^2 + a_5^2)^3}{(a_1 a_6 - a_2 a_5)^2} \end{cases} \quad (28)$$

which needs to satisfy  $|a_6| > |a_5|$  and  $a_1 a_6 - a_2 a_5 \neq 0$ . The positive quadratic function solution to Eq. (25) is

$$\begin{aligned} f = & \left( a_1 x + a_2 z + \frac{3a_1(a_1^2 - a_2^2 + a_5^2 + a_6^2) - 6a_2 a_5 a_6}{a_1^2 + a_5^2} t + a_4 \right)^2 \\ & + \left( a_5 x + a_6 z + \frac{3a_5(a_1^2 + a_2^2 + a_5^2 - a_6^2) - 6a_1 a_2 a_6}{a_1^2 + a_5^2} t + a_8 \right)^2 \\ & + \frac{(a_1^2 + a_5^2)^3}{(a_1 a_6 - a_2 a_5)^2}, \end{aligned} \quad (29)$$

which, in turn, generates a class of lump solutions to dimensionally reduced Eq. (2) through the transformation  $u = 2(\ln f)_x$  as

$$u^{(I)} = \frac{4(a_1 g + a_5 h)}{f}, \quad (30)$$

where the function  $f$  is defined by Eq. (29), and the functions  $g$  and  $h$  are given as follows:

$$\begin{aligned} g = & a_1 x + a_2 z \\ & + \frac{3a_1(a_1^2 - a_2^2 + a_5^2 + a_6^2) - 6a_2 a_5 a_6}{a_1^2 + a_5^2} t + a_4, \\ h = & a_5 x + a_6 z \\ & - \frac{3a_5(a_1^2 + a_2^2 + a_5^2 - a_6^2) - 6a_1 a_2 a_6}{a_1^2 + a_5^2} t + a_8. \end{aligned}$$

## 4.2 Lump solutions to reduction with $y = z$

With  $y = z$ , the dimensionally reduced form of the bilinear Eq. (2) turns out to be

$$\left(D_t D_z - D_x^3 D_z - 3D_x^2 + 3D_z^2\right) f \cdot f = 0, \quad (31)$$

that is,

$$\begin{aligned} & (f_{tz} f - f_t f_z) - (f_{xxx} f - f_{xx} f_z \\ & - 3f_{xx} f_x + 3f_{xx} f_x) \\ & - 3\left(f_{xx} f - f_x^2\right) + 3\left(f_{zz} f - f_z^2\right) = 0, \end{aligned} \quad (32)$$

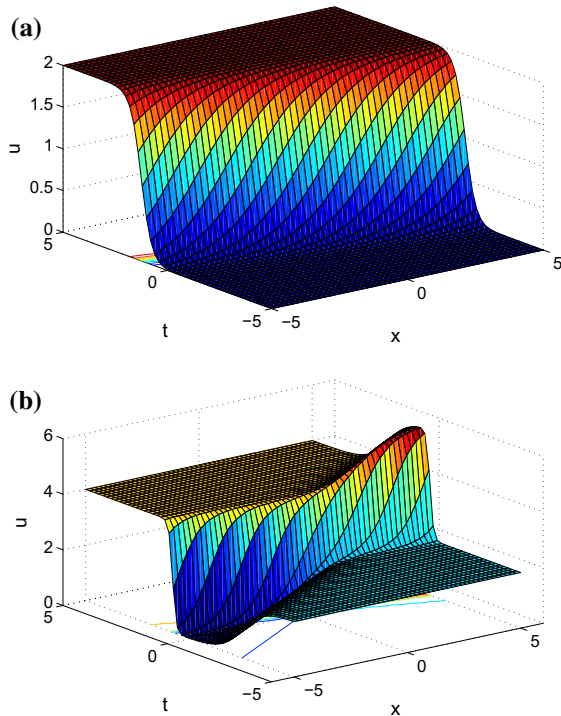
which is transformed into

$$u_{zt} - u_{xxx} - 3(u_x u_z)_x - 3u_{xx} + 3u_{zz} = 0, \quad (33)$$

through the link of Eq. (27) between  $f$  and  $u$ .

Substituting  $f = g^2 + h^2 + a_9$  into Eq. (32), we obtain the following set of constraining equations for the parameters:

$$\begin{cases} a_1 = a_1, & a_2 = a_2, \\ a_3 = \frac{3a_2(a_1^2 - a_2^2 - a_5^2 - a_6^2) + 6a_1 a_5 a_6}{a_2^2 + a_6^2}, \\ a_4 = a_4, & a_5 = a_5, & a_6 = a_6, \\ a_7 = \frac{-3a_6(a_1^2 + a_2^2 - a_5^2 + a_6^2) + 6a_1 a_2 a_5}{a_2^2 + a_6^2}, & a_8 = a_8, \\ a_9 = -\frac{(a_2^2 + a_6^2)(a_1^2 + a_5^2)(a_1 a_2 + a_5 a_6)}{(a_1 a_6 - a_2 a_5)^2} \end{cases} \quad (34)$$



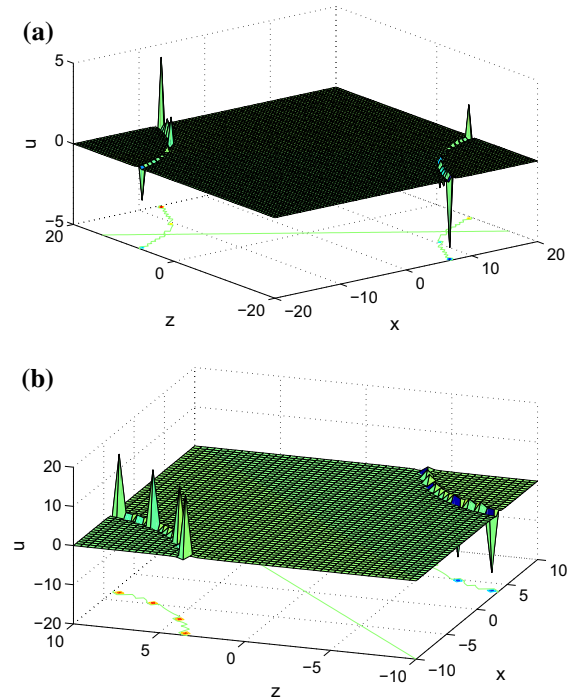
**Fig. 1** **a** Characteristics of the one-wave solution via Eq. (15) with  $\varepsilon = 1$ ,  $k = 1$ ,  $l = 1$ ,  $m = 0$ ,  $w = -4$ , and  $y = z = 0$ ; **b** characteristics of the two-wave solution via Eq. (19) with  $\varepsilon_1 = 1$ ,  $\varepsilon_2 = 1$ ,  $k_1 = 1$ ,  $k_2 = 2$ ,  $l_1 = 3$ ,  $l_2 = 3$ ,  $m_1 = 2$ ,  $m_2 = 0$ ,  $w_1 = 2$ ,  $w_2 = -12$ , and  $y = z = 0$

which needs to satisfy  $a_1 a_6 - a_2 a_5 \neq 0$  and  $a_1 a_2 + a_5 a_6 < 0$ . The positive quadratic function solution to Eq. (32) is

$$f = \left( a_1 x + a_2 z + \frac{3a_2 (a_1^2 - a_2^2 - a_5^2 - a_6^2) + 6a_1 a_5 a_6}{a_2^2 + a_6^2} t + a_4 \right)^2 + \left( a_5 x + a_6 z + \frac{-3a_6 (a_1^2 + a_2^2 - a_5^2 + a_6^2) + 6a_1 a_2 a_5}{a_2^2 + a_6^2} t + a_8 \right)^2 - \frac{(a_2^2 + a_6^2)(a_1^2 + a_5^2)(a_1 a_2 + a_5 a_6)}{(a_1 a_6 - a_2 a_5)^2}, \quad (35)$$

which, in turn, generates a class of lump solutions to dimensionally reduced Eq. (2) through the transformation  $u = 2 (\ln f)_x$  as

$$u^{(II)} = \frac{4(a_1 g + a_5 h)}{f}, \quad (36)$$



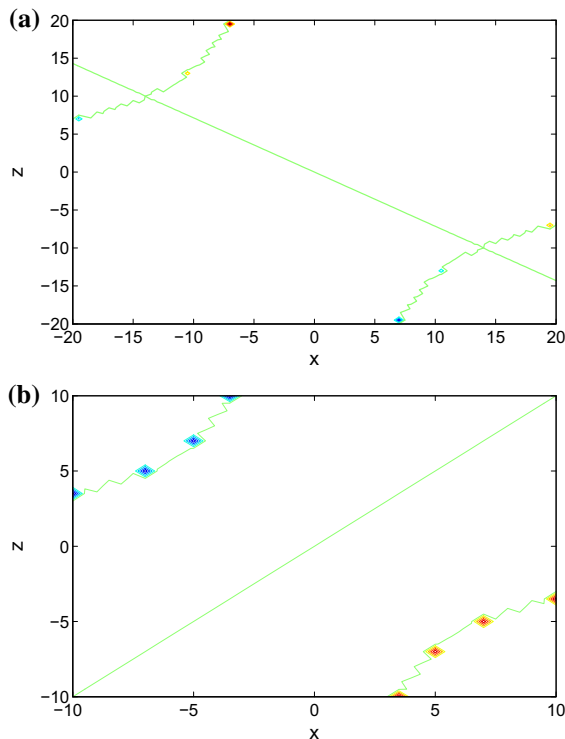
**Fig. 2** **a** Characteristics of lump solution  $u^{(I)}$  via Eq. (30) with  $a_1 = 1$ ,  $a_2 = 2$ ,  $a_4 = 0$ ,  $a_5 = 3$ ,  $a_6 = 4$ ,  $a_8 = 0$ , and  $t = 0$ ; **b** characteristics of lump solution  $u^{(II)}$  via Eq. (36) with  $a_1 = 1$ ,  $a_2 = 2$ ,  $a_4 = 0$ ,  $a_5 = -3$ ,  $a_6 = 4$ ,  $a_8 = 0$ , and  $t = 0$

where the function  $f$  is defined by Eq. (35), and the functions  $g$  and  $h$  are shown as follows:

$$g = a_1 x + a_2 z + \frac{3a_2 (a_1^2 - a_2^2 - a_5^2 - a_6^2) + 6a_1 a_5 a_6}{a_2^2 + a_6^2} t + a_4, \\ h = a_5 x + a_6 z + \frac{-3a_6 (a_1^2 + a_2^2 - a_5^2 + a_6^2) + 6a_1 a_2 a_5}{a_2^2 + a_6^2} t + a_8.$$

## 5 Discussions and conclusion

High-dimensional problems in soliton theory attract much more attention in recent research. For example, by using multiple exp-function method and symbolic computation, one-wave, two-wave, and three-wave solutions have been presented to  $(3 + 1)$ -dimensional generalized KP and BKP equations [4, 13, 14]. Resonant behavior of multiple wave solutions and lump

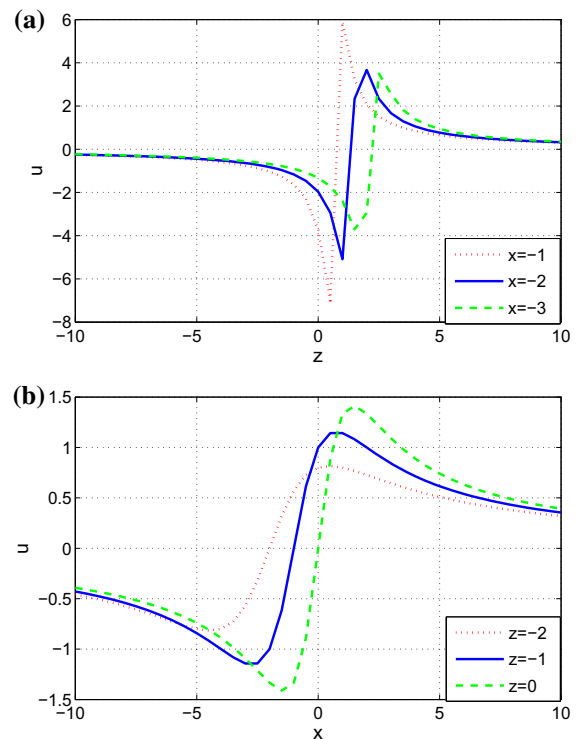


**Fig. 3** **a** Contour of lump solution  $u^{(I)}$ ; **b** contour of lump solution  $u^{(II)}$

dynamics has been studied for a  $(3 + 1)$ -dimensional NLEE [7, 19].

In this paper, we have firstly transformed the  $(3 + 1)$ -dimensional nonlinear partial differential equation, that is, Eq. (1) into Hirota bilinear form by a dependent transformation. Then, a bilinear BT has been constructed [see Eq. (6)], which consists of six bilinear equations and includes nine arbitrary parameters. As an application, we have derived two classes of exact solutions [see Eqs. (10) and (13)] to Eq. (1) by using this BT. Moreover, with multiple exp-function method and symbolic computation, nonresonant-typed multiple wave solutions have been given to Eq. (1) including one-wave, two-wave, and three-wave solutions [see Eqs. (15), (19), and (22)]. Characteristics of the one-wave and two-wave solutions are shown in Fig. 1.

Finally, two classes of lump solutions have been investigated to the dimensionally reduced forms of Eq. (1) with  $y = x$  and  $y = z$ , respectively, i.e., Eqs. (26) and (33). We found no lump solution in the form of  $f = g^2 + h^2 + a_9$  to reduced Eq. (1) via taking  $y = t$ . To reveal the lump dynamics, 3-dimensional



**Fig. 4** **a** Plot of  $x$  curves of lump solution  $u^{(I)}$  with  $x = -1$ ,  $x = -2$ , and  $x = -3$ ; **b** plot of  $z$  curves of lump solution  $u^{(II)}$  with  $z = 0$ ,  $z = -1$ , and  $z = -2$

plots, density plots and 2-dimensional curves with particular choices of the involved parameters in the potential function  $u$  are given in Figs. 2, 3, and 4, respectively.

**Acknowledgements** This work is supported by the Open Fund of IPOC (BUPT) under Grant No. IPOC2016B008. Y. H. Yin is supported by the Project of National Innovation and Entrepreneurship Training Program for College Students under Grant No. 170170007. W. X. Ma is supported in part by the National Natural Science Foundation of China under Grant Nos. 11371326 and 11271008, Natural Science Foundation of Shanghai under Grant No. 11ZR1414100, Zhejiang Innovation Project of China under Grant No. T200905, the First-class Discipline of Universities in Shanghai and the Shanghai University Leading Academic Discipline Project (No. A13-0101-12-004), and the Distinguished Professorship at Shanghai University of Electric Power.

## References

1. Zhang, Y., Dong, H., Zhang, X., Yang, H: Rational solutions and lump solutions to the generalized  $(3 + 1)$ -dimensional shallow water-like equation. *Comput. Math. App.* **73**, 246 (2017)

2. Mani Rajan, M.S., Mahalingam, A.: Nonautonomous solitons in modified inhomogeneous Hirota equation: soliton control and soliton interaction. *Nonlinear Dyn.* **79**, 2469 (2015)
3. Biswas, A., Khalique, C.M.: Stationary solutions for nonlinear dispersive Schrödinger equation. *Nonlinear Dyn.* **63**, 623 (2011)
4. Ma, W.X., Abdeljabbar, A.: A bilinear Bäcklund transformation of a  $(3 + 1)$ -dimensional generalized KP equation. *Appl. Math. Lett.* **25**, 1500 (2012)
5. Zhang, Y., Ma, W.X.: Rational solutions to a KdV-like equation. *Appl. Math. Comput.* **256**, 252 (2015)
6. Zhang, Y.F., Ma, W.X.: A study on rational solutions to a KP-like equation. *Z. Naturforsch.* **70a**, 263 (2015)
7. Lü, X., Ma, W.X.: Study of lump dynamics based on a dimensionally reduced Hirota bilinear equation. *Nonlinear Dyn.* **85**, 1217 (2016)
8. Wang, D.S., Wei, X.: Integrability and exact solutions of a two-component Korteweg-de Vries system. *Appl. Math. Lett.* **51**, 60 (2016)
9. Dai, C.Q., Wang, Y.Y., Zhang, X.F.: Spatiotemporal localizations in  $(3 + 1)$ -dimensional PT-symmetric and strongly nonlocal nonlinear media. *Nonlinear Dyn.* **83**, 2453 (2016)
10. Dai, C.Q., Wang, Y.Y.: Controllable combined Peregrine soliton and Kuznetsov–Ma soliton in PT-symmetric nonlinear couplers with gain and loss. *Nonlinear Dyn.* **80**, 715 (2015)
11. Dai, C.Q., Wang, Y.Y., Liu, J.: Spatiotemporal Hermite–Gaussian solitons of a  $(3 + 1)$ -dimensional partially nonlocal nonlinear Schrödinger equation. *Nonlinear Dyn.* **84**, 1157 (2016)
12. Gao, L.N., Zhao, X.Y., Zi, Y.Y., Yu, J., Lü, X.: Resonant behavior of multiple wave solutions to a Hirota bilinear equation. *Comput. Math. Appl.* **72**, 1225 (2016)
13. Ma, W.X., Fan, E.G.: Linear superposition principle applying to Hirota bilinear equations. *Comput. Math. Appl.* **61**, 950 (2011)
14. Ma, W.X., Zhang, Y., Tang, Y.N., Tu, J.Y.: Hirota bilinear equations with linear subspaces of solutions. *Appl. Math. Comput.* **218**, 7174 (2012)
15. Dai, C.Q., Xu, Y.J.: Exact solutions for a Wick-type stochastic reaction Duffing equation. *Appl. Math. Model.* **39**, 7420 (2015)
16. Dai, C.Q., Fan, Y., Zhou, G.Q., Zheng, J., Cheng, L.: Vector spatiotemporal localized structures in  $(3 + 1)$ -dimensional strongly nonlocal nonlinear media. *Nonlinear Dyn.* **86**, 999 (2016)
17. Ma, W.X., Qin, Z.Y., Lü, X.: Lump solutions to dimensionally reduced p-gKP and p-gBKP equations. *Nonlinear Dyn.* **84**, 923 (2016)
18. Ma, W.X.: Lump solutions to the Kadomtsev–Petviashvili equation. *Phys. Lett. A* **379**, 1975 (2015)
19. Lü, X., Ma, W.X., Yu, J., Lin, F.H., Khalique, C.M.: Envelope bright- and dark-soliton solutions for the Gerdjikov–Ivanov model. *Nonlinear Dyn.* **82**, 1211 (2015)
20. Hirota, R.: *The Direct Method in Soliton Theory*. Cambridge Univ. Press, Cambridge (2004)
21. Lü, X., Ma, W.X., Yu, J., Khalique, C.M.: Solitary waves with the Madelung fluid description: a generalized derivative nonlinear Schrödinger equation. *Commun. Nonlinear Sci. Numer. Simul.* **31**, 40 (2016)
22. Lü, X., Ma, W.X., Chen, S.T., Khalique, C.M.: A note on rational solutions to a Hirota–Satsuma-like equation. *Appl. Math. Lett.* **58**, 13 (2016)
23. Lü, X., Ma, W.X., Zhou, Y., Khalique, C.M.: Rational solutions to an extended Kadomtsev–Petviashvili-like equation with symbolic computation. *Comput. Math. Appl.* **71**, 1560 (2016)
24. Lü, X., Lin, F.: Soliton excitations and shape-changing collisions in alpha helical proteins with interspine coupling at higher order. *Commun. Nonlinear Sci. Numer. Simul.* **32**, 241 (2016)