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Novel soliton hierarchies of Levi type and their bi-Hamiltonian structures

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ABSTRACT

We present two new matrix spectral problems, and construct the corresponding soliton hierarchies of Levi type with the aid of symbolic computation by Maple. Bi-Hamiltonian structures of the resulting soliton hierarchies are obtained by means of the trace identity. Thus it is shown that these soliton hierarchies are Liouville integrable.

1. Introduction

It is a very important topic to seek for new soliton hierarchies generated from certain matrix spectral problems in mathematical physics [1–37] because soliton hierarchies often possess bi-Hamiltonian structures which guarantee the existence of hereditary recursion operators [32] and Liouville integrability [33]. The so-called trace identity [12,16], or more generally the variational identity [27], provides a basic tool for generating Hamiltonian structures of soliton hierarchies. Typical examples of soliton hierarchies, which fit into the zero curvature formulation, include the Korteweg-de Vries hierarchy, the Ablowitz–Kaup–Newell–Segur hierarchy, the Kaup–Newell hierarchy, the Wadati–Konno–Ichikawa hierarchy, the Boiti–Pempinelli–Tu hierarchy, and the Levi hierarchy.

In Refs.[6,7,16,28], the Levi hierarchy associated with the following spatial spectral problem

$$\phi_x = U\phi, \qquad U = \begin{bmatrix} 0 & -p \\ -1 & \lambda - q \end{bmatrix}\phi, \qquad u = \begin{bmatrix} p \\ q \end{bmatrix}, \qquad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{1.1}$$

has been considered, with λ denoting the spectral parameter. Inspired by the form of the spectral matrix U in (1.1), we would like to construct two new spatial spectral problems of Levi type by introducing modifications to (1.1):

$$\phi_x = U\phi, \qquad U = \begin{bmatrix} 0 & -\lambda p \\ -\lambda & \lambda^2 - q \end{bmatrix}, \tag{1.2}$$

and

$$\phi_x = U\phi, \qquad U = \begin{bmatrix} 0 & -\lambda p \\ -\lambda & \lambda - q \end{bmatrix}. \tag{1.3}$$

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The remainder of this paper is organized as follows. In Sections 2 and 3, we construct the corresponding soliton hierarchies of Levi type with bi-Hamiltonian structures from the matrix spectral problems (1.2) and (1.3), respectively. In our analysis, we will use the software Maple to deal with some complicated computations. The last section is devoted to a few conclusions and discussions.

2. Soliton hierarchy of Levi type associated with (1.2)

In this section, we will construct a new soliton hierarchy of Levi type from the spatial spectral problem (1.2). Let W be of the following form

$$W = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \tag{2.1}$$

and we solve the stationary zero curvature equation $W_x = [U, W]$, which becomes

$$\begin{aligned} a_x &= \lambda(b - pc), \\ b_x &= -\lambda^2 b + 2\lambda pa + qb, \\ c_x &= \lambda^2 c - 2\lambda a - qc. \end{aligned} \tag{2.2}$$

According to the relations of the spectral parameter λ in these Eq. (2.2), we take the following Laurent series expansions

$$a = \sum_{k=0}^{\infty} a_k \lambda^{-2k}, \qquad b = \sum_{k=0}^{\infty} b_k \lambda^{-2k-1}, \qquad c = \sum_{k=0}^{\infty} c_k \lambda^{-2k-1}, \tag{2.3}$$

namely,

$$W = \sum_{k=0}^{\infty} W_k \lambda^{-2k}, \qquad W_k = \begin{bmatrix} a_k & b_k \lambda^{-1} \\ c_k \lambda^{-1} & -a_k \end{bmatrix}.$$

Substituting (2.3) into (2.2), we arrive at

$$\begin{aligned} & b_0 - 2pa_0 = 0, \\ & c_0 - 2a_0 = 0, \\ & a_{kx} = b_k - pc_k, \\ & b_{k+1} = -b_{kx} + qb_k + 2pa_{k+1}, \\ & c_{k+1} = c_{kx} + qc_k + 2a_{k+1}, \qquad k \geqslant 0. \end{aligned} \tag{2.4}$$

To determine the sequence of $\{a_k, b_k, c_k, k \geqslant 0\}$ recursively, we need rewrite $a_{k+1,x}$ as

$$a_{k+1,x} = b_{k+1} - pc_{k+1} = \left(-b_{kx} + qb_k + 2pa_{k+1}\right) - p(c_{kx} + qc_k + 2a_{k+1}) = -b_{kx} - pc_{kx} + qb_k - pqc_k. \tag{2.5}$$

Thus the last two equations in (2.4) are transformed to

$$\begin{aligned} & b_{k+1} = -b_{kx} + qb_k + 2p\partial^{-1}(-b_{kx} - pc_{kx} + qb_k - pqc_k), \\ & c_{k+1} = c_{kx} + qc_k + 2\partial^{-1}(-b_{kx} - pc_{kx} + qb_k - pqc_k), \qquad k \geqslant 0. \end{aligned}$$

Then we take the initial values

$$a_0 = 1, \qquad b_0 = 2p, \qquad c_0 = 2, \tag{2.6}$$

and impose the integration conditions

$$a_k|_{u=0} = b_k|_{u=0} = c_k|_{u=0} = 0, \qquad k \geqslant 1, \tag{2.7}$$

to guarantee the uniqueness of the sequence. So, by using the symbolic computation software Maple, the first few sets can be computed as follows:

$$\begin{aligned} & a_1 = -2p, \qquad b_1 = -2p_x - 4p^2 + 2pq, \qquad c_1 = -4p + 2q; \\ & a_2 = 2p_x + 6p^2 - 4pq, \\ & b_2 = 2p_{xx} - 2pq_x - 4p_x q + 12pp_x + 12p^3 - 12p^2 q + 2pq^2, \\ & c_2 = 2q_x + 12p^2 - 12pq + 2q^2; \\ & a_3 = -2p_{xx} - 12pp_x + 6p_x q - 20p^3 + 24p^2 q - 6pq^2, \\ & b_3 = -2p_{xxx} - 60p_x p^2 + 48pp_x q - 6p_x q^2 + 12p^2 q_x - 6pqq_x - 16pp_{xx} + 6p_{xx} q \\ & \qquad - 12p_x^2 + 6p_x q_x + 2pq_{xx} - 40p^4 + 60p^3 q - 24p^2 q^2 + 2pq^3, \\ & c_3 = -4p_{xx} + 2q_{xx} + 6qq_x - 12pq_x - 40p^3 + 60p^2 q - 24pq^2 + 2q^3. \end{aligned}$$

We point out that the localness of the above functions is not an accident, and in fact, all the functions $\{a_k, b_k, c_k, k \geqslant 0\}$ are local. First from $W_x = [U, W]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{tr}\left(W^2\right) = 2\mathrm{tr}(WW_x) = 2\mathrm{tr}(W[U,W]) = 0,$$

and so, due to $\mathrm{tr}(W^2) = 2(a^2 + bc)$, we can obtain

$$a^2 + bc = (a^2 + bc)|_{u=0} = 1,$$

the last step of which follows from the initial data (2.6). Then, by using the Laurent expansions (2.3), a balance of coefficients of λ^k for each $k \geqslant 3$ tells that

$$a_{k+1} = -\frac{1}{2}\left[\sum_{\substack{i+j=k+1\\ i,j\geqslant 1}} a_i a_j + \sum_{i+j=k}(b_i c_j)\right], \qquad k \geqslant 3. \tag{2.8}$$

Based on the recursion relations of the last two equations in (2.4) and the above Eq. (2.8), an application of the mathematical induction finally shows that all functions $\{a_k, b_k, c_k, k \geqslant 4\}$ are differential rational functions in u, and so, they are all local.

Now, taking

$$V^{[n]} = \lambda(\lambda^{2n+1}W)_+ + \Delta_n = \lambda(\lambda^{2n+1}W)_+ + \begin{bmatrix} 0 & 0 \\ 0 & -f_n \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{n} a_k \lambda^{2(n-k)+2} & \sum_{k=0}^{n} b_k \lambda^{2(n-k)+1} \\ \sum_{k=0}^{n} c_k \lambda^{2(n-k)+1} & -\sum_{k=0}^{n} a_k \lambda^{2(n-k)+2} - f_n \end{bmatrix},$$

where P_+ denotes the polynomial part of P in λ, the zero curvature equations for $n \geqslant 0$

$$U_{t_n} - V_x^{[n]} + \left[U, V^{[n]}\right] = 0 \tag{2.9}$$

give

$$\begin{aligned} f_n &= -c_{nx} - qc_n, \\ p_{t_n} &= -b_{nx} + pf_n + qb_n = -b_{nx} - p(c_{nx} + qc_n) + qb_n = a_{n+1,x}, \\ q_{t_n} &= f_{nx} = -c_{nxx} - (qc_n)_x = -c_{n+1,x} + 2a_{n+1,x}, \qquad n \geqslant 0. \end{aligned}$$

Consequently, we have obtained a new soliton hierarchy of Levi type

$$\begin{bmatrix} p \\ q \end{bmatrix}_{t_n} = K_n = \begin{bmatrix} a_{n+1,x} \\ -c_{n+1,x} + 2a_{n+1,x} \end{bmatrix}, \qquad n \geqslant 0. \tag{2.10}$$

The first and second nonlinear examples in this hierarchy (2.10) are as follows:

$$\begin{aligned} p_{t_1} &= 2p_{xx} - 4pq_x - 4p_x q + 12pp_x, \\ q_{t_1} &= 4p_{xx} - 2q_{xx} + 4p_x q + 4pq_x - 4qq_x, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} p_{t_2} &= -2p_{xxx} - 60p_x p^2 + 48pqp_x - 6p_x q^2 + 24p^2 q_x - 12pqq_x - 12pp_{xx} + 6p_{xx}q - 12p_x^2 + 6p_x q_x, \\ q_{t_2} &= -2q_{xxx} - 24pqp_x + 12p_x q^2 - 12p^2 q_x + 24pqq_x - 6q^2 q_x - 24pp_{xx} + 12p_{xx}q - 24p_x^2 \\ &\quad + 24p_x q_x + 12pq_{xx} - 6qq_{xx} - 6q_x^2. \end{aligned} \tag{2.12}$$

By setting $p = 0$, the first nonlinear example (2.11) reduces to the well-known Burgers equation $q_{t_1} = -2(q_{xx} + 2qq_x)$. For $q = 2p$, the second one reduces to mKdV equation $p_{t_2} = -2(p_{xxx} - 6p^2 p_x)$.

Next we consider Hamiltonian structures via the following trace identity [12,16],

$$\frac{\delta}{\delta u}\int \mathrm{tr}\left(\frac{\partial U}{\partial \lambda}W\right)\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}\left(\frac{\partial U}{\partial u}W\right), \qquad \gamma = -\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln\left|\mathrm{tr}\left(W^2\right)\right|. \tag{2.13}$$

It is easy to see

$$\begin{aligned} \frac{\partial U}{\partial \lambda} &= \begin{bmatrix} 0 & -p \\ -1 & 2\lambda \end{bmatrix}, \qquad \mathrm{tr}\left(W\frac{\partial U}{\partial \lambda}\right) = -2\lambda a - pc - b, \\ \frac{\partial U}{\partial p} &= \begin{bmatrix} 0 & -\lambda \\ 0 & 0 \end{bmatrix}, \qquad \mathrm{tr}\left(W\frac{\partial U}{\partial p}\right) = -\lambda c, \\ \frac{\partial U}{\partial q} &= \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \mathrm{tr}\left(W\frac{\partial U}{\partial q}\right) = a. \end{aligned}$$

Now the corresponding trace identity becomes

$$\frac{\delta}{\delta u}\int(-2\lambda a-pc-b)\mathrm{d}x=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{bmatrix}-\lambda c\\ a\end{bmatrix}.$$

Balancing coefficients of each power of λ in the above equality, we have

$$\frac{\delta}{\delta u}\int(-2a_{n+1}-pc_n-b_n)\mathrm{d}x=(\gamma-2n)\begin{bmatrix}-c_n\\ a_n\end{bmatrix},\qquad n\geqslant 0.$$

Checking a particular case with $n=1$ yields $\gamma=0$, and thus we obtain

$$\frac{\delta}{\delta u}\int\left(\frac{2a_{n+2}+pc_{n+1}+b_{n+1}}{2n+2}\right)\mathrm{d}x=\begin{bmatrix}-c_{n+1}\\ a_{n+1}\end{bmatrix},\qquad n\geqslant 0.$$

Consequently, we obtain the following Hamiltonian structure for the soliton hierarchy (2.10)

$$\begin{bmatrix}p\\ q\end{bmatrix}_{t_n}=K_n=\begin{bmatrix}a_{n+1,x}\\ -c_{n+1,x}+2a_{n+1,x}\end{bmatrix}=J\frac{\delta\mathcal{H}_n}{\delta u},\qquad n\geqslant 1,\tag{2.14}$$

with the Hamiltonian operator

$$J=\begin{bmatrix}0&\partial\\ \partial&2\partial\end{bmatrix}$$

and the Hamiltonian functionals

$$\mathcal{H}_0=\int(-2p+q)\mathrm{d}x,$$
$$\mathcal{H}_n=\int\left(\frac{2a_{n+2}+pc_{n+1}+b_{n+1}}{2n+2}\right)\mathrm{d}x,\qquad n\geqslant 1.$$

It is now a direct computation that all members in (2.10) are bi-Hamiltonian. Firstly, by means of

$$\begin{aligned}-c_{n+1}&=-c_{nx}-qc_n-2a_{n+1}\\&=-c_{nx}-qc_n-2\partial^{-1}(-b_{nx}-pc_{nx}+qb_n-pqc_n)\\&=-c_{nx}-qc_n+2(a_{nx}+pc_n)+2\partial^{-1}p\partial c_n-2\partial^{-1}q(a_{nx}+pc_n)+2\partial^{-1}pqc_n\\&=(\partial+q-2p-2\partial^{-1}p\partial)(-c_n)+(2\partial-2\partial^{-1}q\partial)a_n,\end{aligned}$$

$$\begin{aligned}a_{n+1}&=\partial^{-1}(-b_{nx}-pc_{nx}+qb_n-pqc_n)\\&=(p+\partial^{-1}p\partial)(-c_n)+(-\partial+\partial^{-1}q\partial)a_n,\end{aligned}$$

we have

$$\Psi=\begin{bmatrix}\partial+q-2p-2\partial^{-1}p\partial&2\partial-2\partial^{-1}q\partial\\ p+\partial^{-1}p\partial&-\partial+\partial^{-1}q\partial\end{bmatrix}$$

satisfying $\begin{bmatrix}-c_{n+1}\\ a_{n+1}\end{bmatrix}=\Psi\begin{bmatrix}-c_n\\ a_n\end{bmatrix}$. Then we arrive at

$$u_{t_n}=K_n=J\frac{\delta\mathcal{H}_n}{\delta u}=M\frac{\delta\mathcal{H}_{n-1}}{\delta u},\qquad n\geqslant 1,\tag{2.15}$$

where the second Hamiltonian operator M is given by

$$M=J\Psi=\begin{bmatrix}\partial p+p\partial&-\partial^2+q\partial\\ \partial^2+\partial q&0\end{bmatrix}.$$

Thus, it is direct to see that the new soliton hierarchy of Levi type (2.10) is Liouville integrable [33], i.e., it possesses infinitely many independent commuting symmetries and conservation laws. In particular, we have the Abelian symmetry algebra of symmetries,

$$[K_l,K_m]=K_l'(u)[K_m]-K_m'(u)[K_l]=0,\qquad l,m\geqslant 0\tag{2.16}$$

and the Abelian algebras of conserved functionals,

$$\{\mathcal{H}_l,\mathcal{H}_m\}_J=\int\left(\frac{\delta\mathcal{H}_l}{\delta u}\right)^T J\frac{\delta\mathcal{H}_m}{\delta u}\mathrm{d}x=0,\qquad l,m\geqslant 0\tag{2.17}$$

and

$$\{\mathcal{H}_l,\mathcal{H}_m\}_M=\int\left(\frac{\delta\mathcal{H}_l}{\delta u}\right)^T M\frac{\delta\mathcal{H}_m}{\delta u}\mathrm{d}x=0,\qquad l,m\geqslant 0.\tag{2.18}$$

We point out that the novel hierarchy (2.10) is among soliton hierarchies resulting from Hamiltonian pairs directly [38].

3. Soliton hierarchy of Levi type associated with (1.3)

Here we will construct the new soliton hierarchy of Levi type from the spatial spectral problem (1.3). Let W be of the form (2.1), and then the stationary zero curvature equation $W_x = [U, W]$ becomes

$$\begin{aligned} a_x &= \lambda(b - pc), \\ b_x &= -\lambda b + 2\lambda pa + qb, \\ c_x &= \lambda c - 2\lambda a - qc. \end{aligned} \tag{3.1}$$

Further, taking the Laurent series expansions,

$$a = \sum_{k=0}^{\infty} a_k \lambda^{-k}, \qquad b = \sum_{k=0}^{\infty} b_k \lambda^{-k}, \qquad c = \sum_{k=0}^{\infty} c_k \lambda^{-k}, \tag{3.2}$$

namely,

$$W = \sum_{k=0}^{\infty} W_k \lambda^{-k}, \qquad W_k = \begin{bmatrix} a_k & b_k \\ c_k & -a_k \end{bmatrix},$$

we arrive at

$$\begin{aligned} & b_0 - pc_0 = 0, \\ & c_0 - 2a_0 = 0, \\ & b_0 - 2pa_0 = 0, \\ & a_{kx} = b_{k+1} - pc_{k+1}, \\ & b_{k+1} = -b_{kx} + qb_k + 2pa_{k+1}, \\ & c_{k+1} = c_{kx} + qc_k + 2a_{k+1}, \qquad k \geqslant 0. \end{aligned} \tag{3.3}$$

To determine the sequence of $\{a_k, b_k, c_k, k \geqslant 0\}$ recursively, we need to represent a_{k+1} by $\{b_k, c_k\}$, using the last two equations in (3.3). In order to achieve this purpose, we rewrite a_{kx} as

$$a_{kx} = b_{k+1} - pc_{k+1} = \left(-b_{kx} + qb_k + 2pa_{k+1}\right) - p(c_{kx} + qc_k + 2a_{k+1}) = -b_{kx} - pc_{kx} + qb_k - pqc_k, \qquad k \geqslant 0. \tag{3.4}$$

Thus for each $k \geqslant 0$, we can change $a_{k+1,x}$ to

$$\begin{aligned} a_{k+1,x} &= -b_{k+1,x} - pc_{k+1,x} + qb_{k+1} - pqc_{k+1} \\ &= -\left[-b_{kxx} + (qb_k)_x + 2(pa_{k+1})_x\right] - p\left[c_{kxx} + (qc_k)_x + 2a_{k+1,x}\right] + q(-b_{kx} + qb_k + 2pa_{k+1}) - pq(c_{kx} + qc_k + 2a_{k+1}). \end{aligned}$$

From this equation, we can have

$$(4p+1)a_{k+1,x} + 2p_x a_{k+1} = b_{kxx} - pc_{kxx} - (qb_k)_x - p(qc_k)_x - qb_{kx} + q^2 b_k - pqc_{kx} - pq^2 c_k,$$

namely,

$$\left(\sqrt{4p+1}\,a_{k+1}\right)_x = \frac{1}{\sqrt{4p+1}}\left[b_{kxx} - pc_{kxx} - (qb_k)_x - p(qc_k)_x - qb_{kx} + q^2 b_k - pqc_{kx} - pq^2 c_k\right].$$

It means that we arrive at

$$\begin{aligned} a_{k+1} = \frac{1}{\sqrt{4p+1}} \partial^{-1} \frac{1}{\sqrt{4p+1}} & \left[b_{kxx} - pc_{kxx} - (qb_k)_x - p(qc_k)_x \right. \\ & \left. - qb_{kx} + q^2 b_k - pqc_{kx} - pq^2 c_k\right], \qquad k \geqslant 0. \end{aligned} \tag{3.5}$$

So by means of this relation (3.5) and the last two equations of (3.3), we can compute $\{a_k, b_k, c_k, k \geqslant 1\}$ recursively from a given initial values $\{b_0, c_0, a_0\}$.

Let us take the initial values

$$a_0 = \frac{1}{\sqrt{4p+1}}, \qquad b_0 = \frac{2p}{\sqrt{4p+1}}, \qquad c_0 = \frac{2}{\sqrt{4p+1}}, \tag{3.6}$$

which are determined by the initial conditions (3.3) and (3.4) with $k = 0$. To guarantee the uniqueness of $\{a_k, b_k, c_k\}$, we also need to impose the integration conditions

$$a_k|_{u=0} = b_k|_{u=0} = c_k|_{u=0} = 0, \qquad k \geqslant 1.$$

Then, by means of the symbolic computation software Maple, the first few sets can be computed as follows:

$$
\begin{aligned}
a_1 &= \frac{2p_x - 4pq}{(4p+1)^{3/2}}, \qquad b_1 = \frac{2pq - 2p_x}{(4p+1)^{3/2}}, \qquad c_1 = \frac{2q}{(4p+1)^{3/2}};\\
a_2 &= \frac{-2}{(4p+1)^{7/2}}\left(4pp_{xx} + p_{xx} - 12pqp_x - 5p_x^2 - 3p_xq + 12p^2q^2 + 3pq^2\right),\\
b_2 &= \frac{2}{(4p+1)^{7/2}}\left(8p^2p_{xx} + 6pp_{xx} + p_{xx} - 16p^3q_x + 16p^2qp_x - 8p^2q_x - 4pqp_x\right.\\
&\quad \left. -14pp_x^2 - pq_x - 2qp_x - 6p_x^2 - 8p^3q^2 + 2p^2q^2 + pq^2\right),\\
c_2 &= \frac{-2}{(4p+1)^{7/2}}\left(8pp_{xx} + 2p_{xx} - 16p^2q_x - 8pq_x - 10p_x^2 - q_x + 8p^2q^2 - 2pq^2 - q^2\right);\\
a_3 &= \frac{-2}{(4p+1)^{9/2}}\left(-16p^3q^3 + 12p^2q^3 + 24p^2q^2p_x + 32p^3q_{xx} + 4pq^3 - 18pq^2p_x - 16p^2qp_{xx}\right.\\
&\quad + 20pqp_x^2 - 48p^2p_xq_x + 16p^2q_{xx} - 6q^2p_x + 12pqp_{xx} - 30p_x^2q - 4pp_xq_x - 16p^2p_{xxx}\\
&\quad \left. +80pp_xp_{xx} - 70p_x^3 + 2pq_{xx} + 4p_{xx}q + 2p_xq_x - 8pp_{xxx} + 20p_xp_{xx} - p_{xxx}\right),\\
b_3 &= \frac{2}{(4p+1)^{9/2}}\left(-48p^3qq_x - 24p^2qq_x - 3pqq_x + 8p^2qp_{xx} + 14pqp_{xx} + 80pp_xp_{xx} + 72p^2q^2p_x\right.\\
&\quad + 6pq^2p_x + 14pp_xq_x + 8p^2p_xq_x - 50pqp_x^2 - p_{xxx} + 16p^3q_{xx} + 8p^2q_{xx} - 16p^2p_{xxx} + pq_{xx}\\
&\quad \left. +3p_{xx}q - 8pp_{xxx} + 20p_xp_{xx} - 24p^3q^3 - 2p^2q^3 + pq^3 - 30p_x^2q + 3p_xq_x - 3p_xq^2 - 70p_x^3\right),\\
c_3 &= \frac{2}{(4p+1)^{9/2}}\left(-24p^2q^3 - 2pq^3 + 48p^2qq_x + 16p^2q_{xx} + q^3 + 24pqq_x - 40pqp_{xx} + 70p_x^2q\right.\\
&\quad \left. -40pp_xq_x + 8pq_{xx} + 3qq_x - 10p_{xx}q - 10p_xq_x + q_{xx}\right).
\end{aligned}
$$

To prove the localness of $\{a_k, b_k, c_k, k \geqslant 4\}$, we have

$$
a_{k+1} = \frac{b_k - pc_{kx} - pqc_k - qb_k}{4p+1} - \frac{1}{2\sqrt{4p+1}}\sum_{\substack{i+j=k+1\\ i,j\geqslant 1}}\left(a_ia_j + b_ic_j\right), \qquad k \geqslant 3. \tag{3.7}
$$

Based on this recursion relation (3.7) and the last two recursion relations in (3.3), an application of the mathematical induction finally shows that all functions $\{a_k, b_k, c_k, k \geqslant 4\}$ are rational differential polynomials in u, namely, they are all local.

Now, taking

$$
V^{[n]} = \lambda^2(\lambda^nW)_+ + \Delta_n = \lambda^2(\lambda^nW)_+ + \begin{bmatrix} 0 & \lambda h_n \\ \lambda g_n & -f_n \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^n a_k\lambda^{n-k+2} & \sum_{k=0}^n b_k\lambda^{n-k+2} + \lambda h_n \\ \sum_{k=0}^n c_k\lambda^{n-k+2} + \lambda g_n & -\sum_{k=0}^n a_k\lambda^{n-k+2} - f_n \end{bmatrix},
$$

the zero curvature equations $U_{t_n} - V_x^{[n]} + \left[U, V^{[n]}\right] = 0, n \geqslant 0$, give

$$
\begin{aligned}
g_n &= c_{nx} + qc_n,\\
h_n &= -b_{nx} + qb_n,\\
f_n &= -g_{nx} - qg_n\\
&= -c_{nxx} - (qc_n)_x - q(c_{nx} + qc_n)\\
&= -c_{n+1,x} + 2a_{n+1,x} - qc_{nx} - q^2c_n\\
&= -c_{n+2} + 2a_{n+2} + 2a_{n+1,x} + 2qa_{n+1}\\
&= -c_{n+2} - 2pc_{n+2} - 2\partial^{-1}p\partial c_{n+2} + 2\partial^{-1}q\partial a_{n+1} + 2qa_{n+1},\\
p_{t_n} &= -h_{nx} + qh_n + pf_n,\\
q_{t_n} &= f_{nx}, \qquad n \geqslant 0.
\end{aligned}
$$

Then we compute

$$
\begin{aligned}
p_{t_n} &= -h_{nx} + qh_n + pf_n\\
&= -h_{nx} + qh_n + p(-g_{nx} - qg_n)\\
&= b_{nxx} - (qb_n)_x + q(-b_{nx} + qb_n) - p[c_{nxx} + (qc_n)_x] - pq(c_{nx} + qc_n)\\
&= a_{n+1,x} + 2(pa_{n+1})_x + 2pa_{n+1,x}\\
&= \sqrt{4p+1}\left(\sqrt{4p+1}a_{n+1}\right)_x, \qquad n \geqslant 0,
\end{aligned}
\tag{3.8}
$$

$$\begin{aligned} q_{t_n} &= f_{nx} \\ &= \partial\left(-c_{n+2} - 2pc_{n+2} - 2\partial^{-1}p\partial c_{n+2} + 2\partial^{-1}q\partial a_{n+1} + 2qa_{n+1}\right) \\ &= (-\partial - 2\partial p - 2p\partial)c_{n+2} + (2q\partial + 2\partial q)a_{n+1} \\ &= -\sqrt{4p+1}\left(\sqrt{4p+1}c_{n+2}\right)_x + (2q\partial + 2\partial q)a_{n+1}, \qquad n \geqslant 0. \end{aligned} \tag{3.9}$$

Consequently, we have obtained a new soliton hierarchy of WKI type

$$\begin{bmatrix} p \\ q \end{bmatrix}_{t_n} = K_n = \begin{bmatrix} \sqrt{4p+1}\left(\sqrt{4p+1}a_{n+1}\right)_x \\ -\sqrt{4p+1}\left(\sqrt{4p+1}c_{n+2}\right)_x + (2q\partial + 2\partial q)a_{n+1} \end{bmatrix}, \qquad n \geqslant 0. \tag{3.10}$$

The first example in this soliton hierarchy (3.10) is as follows:

$$\begin{aligned} p_{t_0} &= \frac{2}{(4p+1)^{3/2}}\left(4pp_{xx} + p_{xx} - 8p^2q_x - 2pq_x - 2p_xq - 4p_x^2\right), \\ q_{t_0} &= \frac{2}{(4p+1)^{7/2}}\left(32p^2p_{xx} + 16pp_{xxx} + 2p_{xxx} - 2qq_x - 128p^3qq_x - 96p^2qq_x - 24pqq_x\right. \\ &\quad + 64p^2qp_{xx} + 32pqp_{xx} - 144pp_xp_{xx} + 32p^2q^2p_x + 16pq^2p_x + 48pp_xq_x + 96p^2p_xq_x \\ &\quad \left. - 96pqp_x^2 - q_{xx} - 64p^3q_{xx} - 48p^2q_{xx} - 12pq_{xx} + 4p_{xx}q - 36p_xp_{xx} - 24p_x^2q + 6p_xq_x + 2p_xq^2 + 120p_x^3\right). \end{aligned}$$

Through a similar computation, we have

$$\frac{\delta}{\delta u}\int\left(\frac{b_{n+2} + pc_{n+2} + a_{n+2}}{n+1}\right)\mathrm{d}x = \begin{bmatrix} -c_{n+2} \\ a_{n+1} \end{bmatrix}, \qquad n \geqslant 0,$$

via the trace identity (2.13). Thus we obtain the following Hamiltonian structures for the soliton hierarchy (3.10):

$$\begin{bmatrix} p \\ q \end{bmatrix}_{t_n} = K_n = \begin{bmatrix} \sqrt{4p+1}\left(\sqrt{4p+1}a_{n+1}\right)_x \\ -\sqrt{4p+1}\left(\sqrt{4p+1}c_{n+2}\right)_x + (2q\partial + 2\partial q)a_{n+1} \end{bmatrix} = J\begin{bmatrix} -c_{n+2} \\ a_{n+1} \end{bmatrix} = J\frac{\delta\mathcal{H}_{n+1}}{\delta u}, \qquad n \geqslant 0, \tag{3.11}$$

with the Hamiltonian operator

$$J = \begin{bmatrix} 0 & \sqrt{4p+1}\partial\sqrt{4p+1} \\ \sqrt{4p+1}\partial\sqrt{4p+1} & 2q\partial + 2\partial q \end{bmatrix} \tag{3.12}$$

and the Hamiltonian functionals

$$\begin{aligned} \mathcal{H}_0 &= \int\left(\frac{2q}{1+\sqrt{4p+1}} - \frac{4pq}{\sqrt{4p+1}\left(1+\sqrt{4p+1}\right)^2}\right)\mathrm{d}x, \\ \mathcal{H}_{n+1} &= \int\left(\frac{b_{n+2} + pc_{n+2} + a_{n+2}}{n+1}\right)\mathrm{d}x, \qquad n \geqslant 0. \end{aligned} \tag{3.13}$$

It is now a direct computation that all members in (3.10) are bi-Hamiltonian. By means of

$$\begin{aligned} -c_{n+2} &= -c_{n+1,x} - qc_{n+1} - 2a_{n+2} \\ &= -c_{n+1,x} - qc_{n+1} - 2\frac{1}{\sqrt{4p+1}}\partial^{-1}\frac{1}{\sqrt{4p+1}}\left[b_{n+1,xx} - pc_{n+1,xx} - (qb_{n+1})_x\right. \\ &\quad \left. - p(qc_{n+1})_x - qb_{n+1,x} + q^2b_{n+1} - pqc_{n+1,x} - pq^2c_{n+1}\right] \\ &= \left[\partial + q - \frac{2}{\sqrt{4p+1}}\partial^{-1}\frac{1}{\sqrt{4p+1}}\left(p\partial^2 - \partial^2p + p\partial q + q\partial p + pq\partial + \partial pq\right)\right](-c_{n+1}) \\ &\quad + \left[\frac{2}{\sqrt{4p+1}}\partial^{-1}\frac{1}{\sqrt{4p+1}}\left(-\partial^3 + \partial q\partial + q\partial^2 - q^2\partial\right)\right]a_n \\ &\equiv \Psi_{11}(-c_{n+1}) + \Psi_{12}a_n, \qquad n \geqslant 0, \end{aligned}$$

$$\begin{aligned} a_{n+1} &= \partial^{-1}\left(-b_{n+1,x} - pc_{n+1,x} + qb_{n+1} - pqc_{n+1}\right) \\ &= \left(-1 + \partial^{-1}q\right)\left(a_{nx} + pc_{n+1}\right) - \partial^{-1}p\partial c_{n+1} - \partial^{-1}pqc_{n+1} \\ &= \left(p + \partial^{-1}p\partial\right)(-c_{n+1}) + \left(-\partial + \partial^{-1}q\partial\right)a_n \\ &\equiv \Psi_{21}(-c_{n+1}) + \Psi_{22}a_n, \qquad n \geqslant 0, \end{aligned}$$

we have

$$\begin{aligned}
&M=\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}=J\begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix},\\
&M_{11}=\partial p+p\partial+2\partial p^2+2\partial p\partial^{-1}p\partial+2p\partial p+2p^2\partial^2,\\
&M_{12}=-\partial^2+q\partial-2\partial p\partial+2\partial p\partial^{-1}q\partial-2p\partial^2+2pq\partial,\\
&M_{21}=\partial^2+\partial q+2\partial p\partial+2\partial q\partial^{-1}p\partial+2\partial^2 p+2\partial pq,\\
&M_{22}=-2\partial^3+2\partial q\partial^{-1}q\partial.
\end{aligned} \tag{3.14}$$

Thus, we arrive at the bi-Hamiltonian structures:

$$u_{t_n}=K_n=J\frac{\delta\mathcal{H}_{n+1}}{\delta u}=M\frac{\delta\mathcal{H}_n}{\delta u},\qquad n\geqslant 0, \tag{3.15}$$

where the second Hamiltonian operator M is given by (3.14). So, it is now direct to see that the new soliton hierarchy of Levi type (3.10) is Liouville integrable [33], i.e., it possesses infinitely many independent commuting symmetries and conservation laws. In particular, we have the Abelian symmetry algebra of symmetries (2.16) and the Abelian algebras of conserved functionals (2.17) and (2.18).

4. Conclusions and discussions

There are many soliton hierarchies obtained by means of matrix spectral problems based on the Lie algebras such as $\mathrm{sl}(2,\mathbb{R})$ and $\mathrm{so}(3,\mathbb{R})$ [1–35]. In this paper, we have introduced the two new matrix spectral problems (1.2), (1.3) and the two novel soliton hierarchies of Levi type (2.15), (3.15) have been derived, together with their bi-Hamiltonian structures. We uses the software Maple to deal with some complicated computations.

In a subsequent study, we will consider some further problems: nonlinearization with Bargmann symmetry constraints, constructing these soliton hierarchies (2.15), (3.15) with self-consistent sources, constructing multi-component integrable couplings based on certain non-semisimple Lie algebras, seeking for Darboux transformations and exact solutions of these hierarchies, and classifying multi-component integrable systems with certain Lie algebras.

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