



Solitary waves with the Madelung fluid description: A generalized derivative nonlinear Schrödinger equation

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ABSTRACT

Within the framework of the Madelung fluid description, we will derive *bright* and *dark* (including *gray*- and *black-soliton*) envelope solutions for a generalized derivative nonlinear Schrödinger model $i \frac{\partial \Psi}{\partial t} = \frac{\partial^2 \Psi}{\partial x^2} + i a \frac{\partial}{\partial x} (|\Psi|^2 \Psi) + b |\Psi|^2 \Psi$, by virtue of the corresponding solitary wave solutions for the stationary Gardner equations. Note that we only consider the motion with stationary-profile current velocity case and exclude the motion with constant current velocity case for $a \neq 0$; on the other hand, our results are derived under suitable assumptions for the current velocity associated with corresponding boundary conditions of the fluid density, and under corresponding parametric constraints.

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1. Introduction

The Madelung fluid description theory [1–3] has been successfully used to discuss families of generalized one-dimensional nonlinear Schrödinger equations [4–10]. The outline of this approach can be summarized as: (i) derive the basic equations in the Madelung fluid description, among which one is a continuity equation for the fluid density, and the other is an Euler equation (or equation of motion) for the fluid velocity; (ii) under suitable hypothesis for the current velocity, transform the motion equation into solvable stationary nonlinear ordinary differential equation, and (iii) discuss the solitary wave solutions for the stationary nonlinear ordinary differential equation and construct the corresponding envelope solitons for the original nonlinear Schrödinger equations. Moreover, the types of the obtained solitary waves and the associated envelope solitons are usually fruitful, e.g., *bright*-, *black*- and *gray-soliton*-types [5–10].

Within the framework of the Madelung fluid description, the complex wave function (say Ψ) is represented in terms of modulus and phase. Substitution of $\Psi(x, t) = \sqrt{\rho(x, t)} e^{\frac{i}{\hbar} \Theta(x, t)}$ into the one-dimensional Schrödinger equation

$$i \hbar \frac{\partial \Psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + m U(x) \Psi(x, t), \quad (1)$$

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leads to the following pair of coupled Madelung fluid equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \quad (2a)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} - \frac{\hbar^2}{2m^2} \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} \right) + \frac{\partial U}{\partial x} = 0, \quad (2b)$$

where $\rho = |\Psi|^2$ is the fluid density and $v = \frac{1}{m} \frac{\partial \Theta}{\partial x}$ is the current velocity. Eq. (2a) is a continuity equation for the fluid density, while Eq. (2b) is the equation of motion for the fluid velocity and contains a force term proportional to the gradient of the “quantum potential”, $\frac{\hbar^2}{2m^2} \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} \right)$. Under suitable hypothesis for the current velocity, Eq. (2b) may be transformed into the stationary Korteweg–de Vries, modified Korteweg–de Vries or Gardner equations, which are directly solvable and possess abundant types of solitary wave solutions. Correspondingly, a number of envelope solitons can be constructed for the original equations.

Applications of the Madelung fluid description in studying families of generalized one-dimensional nonlinear Schrödinger equations include the following:

- An investigation to deepen the connection between the families of nonlinear Schrödinger equations and the Korteweg–de Vries equations is carried out in Ref. [5], and the cubic nonlinear Schrödinger equation,

$$i\mu \frac{\partial \Psi}{\partial t} = -\frac{\mu^2}{2} \frac{\partial^2 \Psi}{\partial x^2} + |\Psi|^2 \Psi, \quad (3)$$

has been put in correspondence with the standard Korteweg–de Vries equation.

- A modified nonlinear Schrödinger equation with a quartic nonlinear potential in the modulus of the wave function, as follows:

$$i\mu \frac{\partial \Psi}{\partial t} = -\frac{\mu^2}{2} \frac{\partial^2 \Psi}{\partial x^2} + q_1 |\Psi|^2 \Psi + q_2 |\Psi|^4 \Psi, \quad (4)$$

with q_1 and q_2 as real constants, has been studied within the framework of the Madelung fluid description in Ref. [6] and put in correspondence with a modified Korteweg–de Vries equation (see Eq. (48) in Ref. [6]). By considering different boundary conditions, up-shifted-bright-, upper-shifted-bright-, gray- and dark-type solitary waves are finally found for the modified Korteweg–de Vries equation, which can be cast into envelope solitons for Eq. (4) conversely.

- Within the context of the Madelung fluid description, a nonlinear Schrödinger equation containing a sum of cubic, anti-cubic and quintic nonlinearities as

$$i\mu \frac{\partial \Psi}{\partial t} = -\frac{\mu^2}{2} \frac{\partial^2 \Psi}{\partial x^2} + q_0 |\Psi|^{-4} \Psi + q_1 |\Psi|^2 \Psi + q_2 |\Psi|^4 \Psi, \quad (5)$$

with q_0 , q_1 and q_2 as real constants has been studied with the achievement of upper-shifted bright envelope soliton-like solution [7].

- Similar discussions have been given for a class of derivative nonlinear Schrödinger-type equations (see Eqs. (1.8) and (1.9) in Ref. [8]). For a motion with stationary-profile current velocity, the fluid density satisfies a generalized stationary Gardner equation, and solitary wave solutions are found. For the completely integrable cases, they are compared with existing solutions in literatures.
- Summarily, Ref. [9] gives a review on the results of investigations dealing with a connection between the envelope soliton-like solutions of a wide family of nonlinear Schrödinger equations and the soliton-like solutions of a wide family of Korteweg–de Vries equations. In two different fluid motion regimes (uniform current velocity and stationary-profile current velocity variation, respectively), bright- and gray-/dark- soliton-like solutions of those equations are found.
- Under suitable hypothesis for the current velocity, the Gerdjikov–Ivanov envelope solitons are similarly discussed [10]. For a motion with stationary-profile current velocity, the fluid density satisfies a generalized stationary Gardner equation, which possesses bright-, gray- and dark-type solitary waves due to corresponding parametric constraints, and finally associated envelope solitons are found for the Gerdjikov–Ivanov model.

In the present paper, we will investigate a generalized derivative nonlinear Schrödinger equation (or named mixed nonlinear Schrödinger equation) enjoying the following form [11–14]:

$$i \frac{\partial \Psi}{\partial t} = \frac{\partial^2 \Psi}{\partial x^2} + ia \frac{\partial}{\partial x} (|\Psi|^2 \Psi) + b |\Psi|^2 \Psi, \quad (6)$$

with $a (\neq 0)$ and $b (\neq 0)$ as arbitrary real constants. Eq. (6) describes nonlinear propagation of Alfvén wave with a small but non-vanishing wave number [11], and its inverse scattering scheme is proposed [12]. It is a generalization of the cubic nonlinear Schrödinger equation, the Kaup–Newell equation [15], the Chen–Lee–Liu equation [16] or Eq. (1.8) in Ref. [8], and its Lax pair and Darboux transformation with soliton-like solutions have been given in Ref. [13], while infinitely many conservation laws have been derived in Ref. [14]. Note that Eq. (6) is an integrable reduction of the coupled AKNS–Kaup–Newell hierarchy, therefore, it is integrable for whatever constants a and b [17].

Setting $\Psi(x, t) = \sqrt{\rho(x, t)} e^{\frac{i}{k} \Theta(x, t)}$ with $k \neq 0$ in Eq. (6), we obtain the continuity equation for the fluid density ρ as

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x} \left(\frac{2}{k} \rho v + \frac{3}{2} a \rho^2 \right) = 0, \quad (7a)$$

and the equation of motion for the fluid velocity $v = \frac{\partial \Theta}{\partial x}$ as

$$\frac{\partial v}{\partial t} - \frac{2}{k} v \frac{\partial v}{\partial x} + k \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} \right) + \frac{\partial}{\partial x} (a \rho v + b k \rho) = 0. \quad (7b)$$

In the sense of Refs. [5–10], Eq. (7b) is transformed into

$$\begin{aligned} -v \frac{\partial \rho}{\partial t} + \rho \frac{\partial v}{\partial t} + (5a + k) \rho v \frac{\partial \rho}{\partial x} + 3bk \rho \frac{\partial \rho}{\partial x} + a \rho^2 \frac{\partial v}{\partial x} + \frac{k}{2} \frac{\partial^3 \rho}{\partial x^3} + 2 \frac{\partial \rho}{\partial x} \int \frac{\partial v}{\partial t} dx \\ + 2c_0(t) \frac{\partial \rho}{\partial x} = 0, \end{aligned} \quad (7c)$$

where $c_0(t)$ is an arbitrary function of t .

Eqs. (7a) and (7c) constitute the basic equations for the subsequent discussion, based on which we will discuss envelope solitons for Eq. (6) within the framework of the Madelung fluid description.

2. Motion with stationary-profile current velocity

Under the assumption that both the quantities ρ and v involved in Eqs. (7a) and (7b) are functions of the combined variable $\xi = x - u_0 t$ with u_0 being a real constant, we cast Eq. (7a) into

$$u_0 \frac{d\rho}{d\xi} + \frac{d}{d\xi} \left(\frac{2}{k} \rho v + \frac{3}{2} a \rho^2 \right) = 0, \quad (8)$$

and integrate once of Eq. (8) with respect to ξ , by taking the integration constant as A_0 to obtain

$$v = \frac{k}{2} \left(-u_0 - \frac{3}{2} a \rho + \frac{A_0}{\rho} \right). \quad (9)$$

Substitution of Eq. (9) into Eq. (7c) leads to

$$\begin{aligned} \left(2akA_0 - \frac{k}{2} u_0^2 + \frac{k^2}{2} A_0 + 2c_0 \right) \frac{d\rho}{d\xi} + \left(3bk - \frac{k^2}{2} u_0 - ak u_0 \right) \rho \frac{d\rho}{d\xi} \\ - \left(\frac{3}{4} a k^2 + \frac{9}{2} k a^2 \right) \rho^2 \frac{d\rho}{d\xi} + \frac{k}{2} \frac{d^3 \rho}{d\xi^3} = 0. \end{aligned} \quad (10)$$

Remarks:

- Attention should be paid to the expression of the fluid velocity v , namely, Eq. (9). In the case of $a = 0$ with $A_0 = 0$, Eq. (6) reduces to the cubic nonlinear Schrödinger equation, and we can consider the motion with constant current velocity case for the nonlinear stationary states. When $a = 0$ and $A_0 \neq 0$, we can consider the motion with stationary-profile current velocity, which has been given in Ref. [5].
- When $a \neq 0$, Eq. (6) denotes a mixed nonlinear Schrödinger equation [12], and we study the motion with stationary-profile current velocity for the nonlinear stationary states, no matter $A_0 = 0$ or not. This is our goal in the subsequent section.
- Note that the fluid density ρ satisfies a generalized stationary Gardner equation, i.e., Eq. (10), which is not the Korteweg–de Vries or modified Korteweg–de Vries equation.

3. Solitary waves versus envelope solitons

In this section, with the constraint $\rho > 0$ and under suitable assumptions for the current velocity associated with corresponding boundary conditions of ρ , we will investigate different types of solitary waves for Eq. (10). Then, fruitful envelope solitons can be correspondingly given for the generalized derivative nonlinear Schrödinger equation, our Eq. (6).

Case I: Assume ρ satisfies the boundary conditions in the ξ -space as $\lim_{\xi \rightarrow \pm\infty} \rho(\xi) = 0$, and it follows from Eq. (9) that $A_0 = 0$ and $v = \frac{k}{2} (-u_0 - \frac{3}{2} a \rho)$.

Consequently, Eq. (10) becomes

$$\left(2c_0 - \frac{k}{2} u_0^2 \right) \frac{d\rho}{d\xi} + \left(3bk - \frac{k^2}{2} u_0 - ak u_0 \right) \rho \frac{d\rho}{d\xi} - \left(\frac{3}{4} a k^2 + \frac{9}{2} k a^2 \right) \rho^2 \frac{d\rho}{d\xi} + \frac{k}{2} \frac{d^3 \rho}{d\xi^3} = 0, \quad (11)$$

which can be integrated twice with respect to ξ and take the integration constant as zero to give

$$\left(\frac{d\rho}{d\xi}\right)^2 = \rho^2 \left[\left(\frac{3}{2}a^2 + \frac{k}{4}a\right)\rho^2 - \left(2b - \frac{2}{3}au_0 - \frac{k}{3}u_0\right)\rho - \left(\frac{4}{k}c_0 - u_0^2\right) \right]. \quad (12)$$

With the constraint $\frac{4}{k}c_0 - u_0^2 > 0$ and $\frac{3}{2}a^2 + \frac{k}{4}a > 0$, the positive solitary wave solution of Eq. (10) can be written as

$$\rho = \frac{\alpha_0}{\alpha_1 + \sqrt{\alpha_1^2 + 4\alpha_0\alpha_2} \cosh[\sqrt{\alpha_0}(\xi + \xi_0)]}, \quad (13)$$

with $\alpha_0 = \frac{4}{k}c_0 - u_0^2$, $\alpha_1 = \frac{2}{3}au_0 + \frac{k}{3}u_0 - 2b$, $\alpha_2 = \frac{3}{2}a^2 + \frac{k}{4}a$ and ξ_0 as an integration constant.

Therefore,

$$v = \frac{d\Theta}{d\xi} = \frac{k}{2} \left(-u_0 - \frac{3a\alpha_0}{2\alpha_1 + 2\sqrt{\alpha_1^2 + 4\alpha_0\alpha_2} \cosh[\sqrt{\alpha_0}(\xi + \xi_0)]} \right), \quad (14)$$

and

$$\begin{aligned} \Theta(x, t) &= -\frac{k}{2}u_0\xi - \frac{3}{4}ak \int \rho d\xi - 2c_0t - \Theta_0 \\ &= -\frac{k}{2}u_0x + \left(\frac{k}{2}u_0^2 - 2c_0\right)t - \frac{3ak}{4\sqrt{\alpha_2}} \arctan \left[\frac{\sqrt{\alpha_1^2 + 4\alpha_0\alpha_2} - \alpha_1}{2\sqrt{\alpha_0\alpha_2}} \right. \\ &\quad \left. \times \tanh\left(\frac{\sqrt{\alpha_0}}{2}x - \frac{\sqrt{\alpha_0}}{2}u_0t + \frac{\sqrt{\alpha_0}}{2}\xi_0\right) \right] - \Theta_0, \end{aligned} \quad (15)$$

$$\Psi(x, t) = \sqrt{\rho(x, t)} e^{\frac{i}{k}\Theta(x, t)}, \quad (16)$$

where Θ_0 is an initial phase (integration constant).

In this case, the solitary wave solution of the stationary Gardner equation [i.e., Eq. (10)] possesses a *bright-soliton*-type profile [see expression (13) above]. When $\alpha_1 = 0$, expression (13) reduces to the standard sech-type bright soliton. Finally, the *bright-soliton*-type envelope solution of the generalized derivative nonlinear Schrödinger equation [see Eq. (6) above] can be derived via expression (16) with $\rho(x, t)$ and $\Theta(x, t)$ listed in Eqs. (13) and (15), respectively.

Case II: For the case of $\lim_{\xi \rightarrow \pm\infty} \rho(\xi) \neq 0$, we express $\rho(\xi) = \rho_0 + \rho_1(\xi)$ with $\lim_{\xi \rightarrow \pm\infty} \rho_1(\xi) = 0$ and $\rho_0 > 0$ as a constant, which determines that $v = \frac{k}{2}[-u_0 - \frac{3}{2}a(\rho_0 + \rho_1(\xi)) + \frac{A_0}{\rho_0 + \rho_1(\xi)}]$.

According to Eq. (10) with $\rho(\xi) = \rho_0 + \rho_1(\xi)$, we have

$$\begin{aligned} &\left[2akA_0 - \frac{k}{2}u_0^2 + \frac{k^2}{2}A_0 + 2c_0 - \left(ak u_0 + \frac{k^2}{2}u_0 - 3kb \right) \rho_0 \right. \\ &\quad \left. - \left(\frac{3}{4}ak^2 + \frac{9}{2}ka^2 \right) \rho_0^2 \right] \frac{d\rho_1}{d\xi} + \left[3kb - ak u_0 - \frac{k^2}{2}u_0 - \left(\frac{3}{2}ak^2 + 9ka^2 \right) \rho_0 \right] \rho_1 \frac{d\rho_1}{d\xi} \\ &\quad - \left(\frac{3}{4}ak^2 + \frac{9}{2}ka^2 \right) \rho_1^2 \frac{d\rho_1}{d\xi} + \frac{k}{2} \frac{d^3\rho_1}{d\xi^3} = 0, \end{aligned} \quad (17)$$

which is also a generalized stationary Gardner equation and can be similarly solved as Eq. (11) in Case I. The key point lies in that the sign of ρ_1 , which can be negative or positive, is different from the sign of ρ in Case I. Just this point inspires us to study other types of solutions.

Integrating twice of Eq. (17) with respect to ξ and taking the integration constants as zero, we obtain

$$\begin{aligned} \left(\frac{d\rho_1}{d\xi}\right)^2 &= \rho_1^2 \left[\rho_1^2 \left(\frac{3}{2}a^2 + \frac{1}{4}ak \right) + \rho_1 \left(\frac{2}{3}au_0 + \frac{k}{3}u_0 - 2b + 6a^2\rho_0 + ak\rho_0 \right) \right. \\ &\quad \left. + \left(u_0^2 - 4aA_0 - kA_0 - \frac{4}{k}c_0 + 2au_0\rho_0 + ku_0\rho_0 - 6b\rho_0 + 9a^2\rho_0^2 + \frac{3}{2}ak\rho_0^2 \right) \right]. \end{aligned} \quad (18)$$

Case II-I: Under the constraint $u_0^2 - 4aA_0 - kA_0 - \frac{4}{k}c_0 + 2au_0\rho_0 + ku_0\rho_0 - 6b\rho_0 + 9a^2\rho_0^2 + \frac{3}{2}ak\rho_0^2 < 0$ and $\frac{3}{2}a^2 + \frac{1}{4}ak > 0$, the positive solution of Eq. (18) is taken as

$$\rho_1 = \frac{2\beta_0}{\beta_1 + \sqrt{\beta_1^2 + 4\beta_0\beta_2} \cosh[\sqrt{\beta_0}(\xi + \xi_{01})]} > 0, \quad (19)$$

with $\beta_0 = 4aA_0 - u_0^2 + kA_0 + \frac{4}{k}c_0 - 2au_0\rho_0 - ku_0\rho_0 + 6b\rho_0 - 9a^2\rho_0^2 - \frac{3}{2}ak\rho_0^2$, $\beta_1 = \frac{2}{3}au_0 + \frac{k}{3}u_0 - 2b + 6a^2\rho_0 + ak\rho_0$, $\beta_2 = \frac{3}{2}a^2 + \frac{1}{4}ak$ and ξ_{01} as an integration constant.

Therefore, the solitary wave solution of Eq. (10) is in the form of

$$\rho = \rho_0 + \frac{2\beta_0}{\beta_1 + \sqrt{\beta_1^2 + 4\beta_0\beta_2} \cosh\left[\sqrt{\beta_0}(\xi + \xi_{01})\right]}, \quad (20)$$

and subsequently,

$$\begin{aligned} \Theta(x, t) = & \left(\frac{kA_0}{2\rho_0} - \frac{k}{2}u_0 - \frac{3ak}{4}\rho_0 \right)x + \left(\frac{k}{2}u_0^2 + \frac{3ak}{4}\rho_0u_0 - \frac{kA_0u_0}{2\rho_0} - 2c_0 \right)t + \frac{kA_0\xi_{01}}{2\rho_0} - \Theta_{01} \\ & - \frac{kA_0}{\rho_0\sqrt{\beta_0}\sqrt{1+A\rho_0}\sqrt{1+B\rho_0}} \operatorname{arctanh} \left[\frac{\sqrt{1+B\rho_0}}{\sqrt{1+A\rho_0}} \tanh \left[\frac{\sqrt{\beta_0}}{2}(x - u_0t + \xi_{01}) \right] \right] \\ & - \frac{3ak}{2\sqrt{\beta_2}} \operatorname{arctanh} \left[\sqrt{\frac{\beta_1 - \sqrt{\beta_1^2 + 4\beta_0\beta_2}}{\beta_1 + \sqrt{\beta_1^2 + 4\beta_0\beta_2}}} \tanh \left[\frac{\sqrt{\beta_0}}{2}(x - u_0t + \xi_{01}) \right] \right], \end{aligned} \quad (21)$$

$$\Psi(x, t) = \sqrt{\rho(x, t)} e^{\frac{i}{k}\Theta(x, t)}, \quad (22)$$

where $A = \frac{\beta_1 + \sqrt{\beta_1^2 + 4\beta_0\beta_2}}{2\beta_0}$ and $B = \frac{\beta_1 - \sqrt{\beta_1^2 + 4\beta_0\beta_2}}{2\beta_0}$ with Θ_{01} and ξ_{01} as two integration constants.

It is shown in expression (20) that the solution ρ enjoys a *bright-soliton* profile. By means of expressions (21) and (22), we can correspondingly discuss different solution cases for Eq. (6).

Under the condition $\frac{2}{3}au_0 + \frac{k}{3}u_0 - 2b + 6a^2\rho_0 + ak\rho_0 = 0$, expression (20) reduces to

$$\rho = \rho_0 \left\{ 1 + \varepsilon \operatorname{sech} \left[\sqrt{\beta_0}(x - u_0t - \xi_{01}) \right] \right\}, \quad (23)$$

with

$$\varepsilon = \sqrt{\frac{16akA_0 - 4ku_0^2 + 4k^2A_0 + 16c_0 - 8ka u_0\rho_0 - 4k^2u_0\rho_0 + 24kb\rho_0 - 36ka^2\rho_0^2 - 6ak^2\rho_0^2}{6ka^2\rho_0^2 + ak^2\rho_0^2}}.$$

- With $0 < \varepsilon < 1$, expression (23) represents a *up-shifted bright-soliton*, whose maximum amplitude is $\rho_0(1 + \varepsilon)$ and up-shifted by the quantity ρ_0 .
- With $\varepsilon = 1$, expression (23) represents a *upper-shifted bright-soliton*, whose maximum amplitude is $2\rho_0$ and up-shifted by the quantity ρ_0 .

Having obtained different forms and types of the solitary waves for stationary Gardner equation via expression (23), we can correspondingly investigate the associated different types of envelope solitons for the original generalized derivative nonlinear Schrödinger model, i.e., Eq. (6) by means of expression (22) (details ignored here).

Case II-II: With the constraint $u_0^2 - 4aA_0 - kA_0 - \frac{4}{k}c_0 + 2au_0\rho_0 + ku_0\rho_0 - 6b\rho_0 + 9a^2\rho_0^2 + \frac{3}{2}ak\rho_0^2 < 0$ and $\frac{3}{2}a^2 + \frac{1}{4}ak > 0$, the negative solution of Eq. (18) can be taken as

$$\rho_1 = \frac{-2\beta_0}{\sqrt{\beta_1^2 + 4\beta_0\beta_2} \cosh\left[\sqrt{\beta_0}(\xi + \xi_{02})\right] - \beta_1} < 0, \quad (24)$$

Thus,

$$\rho = \rho_0 + \frac{2\beta_0}{\beta_1 - \sqrt{\beta_1^2 + 4\beta_0\beta_2} \cosh\left[\sqrt{\beta_0}(\xi + \xi_{02})\right]}, \quad (25)$$

$$\begin{aligned} \Theta(x, t) = & \left(\frac{kA_0}{2\rho_0} - \frac{k}{2}u_0 - \frac{3ak}{4}\rho_0 \right)x + \left(\frac{k}{2}u_0^2 + \frac{3ak}{4}\rho_0u_0 - \frac{kA_0u_0}{2\rho_0} - 2c_0 \right)t + \frac{kA_0\xi_{02}}{2\rho_0} - \Theta_{02} \\ & - \frac{kA_0}{\rho_0\sqrt{\beta_0}\sqrt{1+A\rho_0}\sqrt{1+B\rho_0}} \operatorname{arctanh} \left[\frac{\sqrt{1+B\rho_0}}{\sqrt{1+A\rho_0}} \tanh \left[\frac{\sqrt{\beta_0}}{2}(x - u_0t + \xi_{02}) \right] \right] \\ & - \frac{3ak}{2\sqrt{\beta_2}} \operatorname{arctanh} \left[\sqrt{\frac{\beta_1 + \sqrt{\beta_1^2 + 4\beta_0\beta_2}}{\beta_1 - \sqrt{\beta_1^2 + 4\beta_0\beta_2}}} \tanh \left[\frac{\sqrt{\beta_0}}{2}(x - u_0t + \xi_{02}) \right] \right], \end{aligned} \quad (26)$$

Table 1
Solitary wave types.

Solitary wave types for stationary Gardner Eq. (10)	Parametric constraints
Bright-soliton-type	$A_0 = 0;$ $\frac{4}{k} c_0 - u_0^2 > 0;$ $\frac{3}{2} a^2 + \frac{k}{4} a > 0;$
Bright-soliton-type	$\rho_0 > 0;$ $\frac{3}{2} a^2 + \frac{1}{4} a k > 0;$ $u_0^2 - 4 a A_0 - k A_0 - \frac{4}{k} c_0 + 2 a u_0 \rho_0 + k u_0 \rho_0$ $-6 b \rho_0 + 9 a^2 \rho_0^2 + \frac{2}{2} a k \rho_0^2 < 0;$
Up-shifted-bright-type	$\rho_0 > 0;$ $0 < \varepsilon < 1;$ $\frac{3}{2} a^2 + \frac{1}{4} a k > 0;$ $\frac{2}{3} a u_0 + \frac{k}{3} u_0 - 2 b + 6 a^2 \rho_0 + a k \rho_0 = 0;$ $u_0^2 - 4 a A_0 - k A_0 - \frac{4}{k} c_0 + 2 a u_0 \rho_0 + k u_0 \rho_0$ $-6 b \rho_0 + 9 a^2 \rho_0^2 + \frac{2}{2} a k \rho_0^2 < 0;$
Upper-shifted-bright-type	$\rho_0 > 0;$ $\varepsilon = 1;$ $\frac{3}{2} a^2 + \frac{1}{4} a k > 0;$ $\frac{2}{3} a u_0 + \frac{k}{3} u_0 - 2 b + 6 a^2 \rho_0 + a k \rho_0 = 0;$ $u_0^2 - 4 a A_0 - k A_0 - \frac{4}{k} c_0 + 2 a u_0 \rho_0 + k u_0 \rho_0$ $-6 b \rho_0 + 9 a^2 \rho_0^2 + \frac{2}{2} a k \rho_0^2 < 0;$
Gray-soliton-type	$0 < \delta < 1;$ Expression (28); $\frac{3}{2} a^2 + \frac{1}{4} a k > 0;$ $u_0^2 - 4 a A_0 - k A_0 - \frac{4}{k} c_0 + 2 a u_0 \rho_0 + k u_0 \rho_0$ $-6 b \rho_0 + 9 a^2 \rho_0^2 + \frac{2}{2} a k \rho_0^2 < 0;$
Black-soliton-type	$\delta = 1;$ Expression (28); $\frac{3}{2} a^2 + \frac{1}{4} a k > 0;$ $u_0^2 - 4 a A_0 - k A_0 - \frac{4}{k} c_0 + 2 a u_0 \rho_0 + k u_0 \rho_0$ $-6 b \rho_0 + 9 a^2 \rho_0^2 + \frac{2}{2} a k \rho_0^2 < 0;$

$$\Psi(x, t) = \sqrt{\rho(x, t)} e^{\frac{1}{k} \Theta(x, t)}, \quad (27)$$

where Θ_{02} and ξ_{02} are two integration constants.

Notice that the solution ρ_1 in expression (24) enjoys a *non-bright-soliton* profile. By virtue of expressions (25) and (26), we can discuss different solution cases for Eq. (10), correspondingly for the original Eq. (6) via expression (27).

Under the following condition

$$\begin{cases} \frac{2}{3} a u_0 + \frac{k}{3} u_0 - 2 b + 6 a^2 \rho_0 + a k \rho_0 = 0, \\ \rho_0 > \sqrt{\frac{\beta_0}{\beta_2}}, \end{cases} \quad (28)$$

expression (25) reduces to

$$\rho = \rho_0 \left\{ 1 - \delta \operatorname{sech} \left[\sqrt{u_0^2 - A_0 + \frac{4}{k} c_0 - 3 u_0 \rho_0 - \frac{11}{2} \rho_0^2} (x - u_0 t - \xi_{01}) \right] \right\}, \quad (29)$$

with

$$\delta = \sqrt{\frac{16 a k A_0 - 4 k u_0^2 + 4 k^2 A_0 + 16 c_0 - 8 k a u_0 \rho_0 - 4 k^2 u_0 \rho_0 + 24 k b \rho_0 - 36 k a^2 \rho_0^2 - 6 a k^2 \rho_0^2}{6 k a^2 \rho_0^2 + a k^2 \rho_0^2}}.$$

- With $0 < \delta < 1$, expression (29) represents a *gray-soliton*, whose minimum amplitude is $\rho_0(1 - \delta)$ and reaches asymptotically the upper limit ρ_0 .
- With $\delta = 1$, expression (29) represents a *black-soliton*, whose minimum amplitude is zero and reaches asymptotically the upper limit ρ_0 .

Different forms and types of the solitary waves for stationary Gardner equation can be obtained via expression (29), and as a result, we can investigate the associated different types of envelope solitons for the original generalized derivative nonlinear Schrödinger model, i.e., Eq. (6) by means of expression (27) (details ignored here).

4. Concluding remarks

Within the context of the Madelung fluid description, investigations on the connection between the family of nonlinear Schrödinger equations and the one of Korteweg–de Vries equations, modified Korteweg–de Vries equations or Gardner equations has been carried out (see Refs. [4–10] for details). Under suitable hypothesis for the current velocity, the nonlinear Schrödinger-type equations can be put in correspondence with the Korteweg–de Vries equation, modified Korteweg–de Vries equation or Gardner equation in such a manner that the soliton solutions of the latter are the squared modulus of the envelope soliton solutions of the former. This method provides new insights and represents an alternative key of reading of the *bright*-, *dark*- and *gray-soliton*-type envelope solutions based on the fluid language.

In the present paper, we have derived *bright* and *dark* (including *gray*- and *black-soliton*) envelope solutions for the generalized derivative nonlinear Schrödinger model [see Eq. (6) above], by solving corresponding stationary Gardner equations. Note that we have only considered the motion with stationary-profile current velocity case and excluded the motion with constant current velocity case for $a \neq 0$ in Eq. (9); on the other hand, our results are derived under suitable assumptions for the current velocity associated with corresponding boundary conditions of ρ , and under corresponding parametric constraints. The types of the solitary waves with associated parametric constraints have been given in Table 1. As an application of the Madelung fluid description approach, we hope that our work in this paper is helpful for the study of other nonlinear Schrödinger-type models [18,19].

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