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Envelope bright- and dark-soliton solutions for the Gerdjikov–Ivanov model

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Abstract Within the context of the Madelung fluid description, investigation has been carried out on the connection between the envelope soliton-like solutions of a wide family of nonlinear Schrödinger equations and the soliton-like solutions of a wide family of Korteweg–de Vries or Korteweg–de Vries-type equations. Under suitable hypothesis for the current velocity, the Gerdjikov–Ivanov envelope solitons are derived and discussed in this paper. For a motion with the stationary profile current velocity, the fluid density satisfies a generalized stationary Gardner equation, which possesses *bright*- and *dark*-type (including *gray* and *black*) solitary waves due to associated parametric constraints, and finally envelope solitons are found correspondingly for the Gerdjikov–Ivanov model. Moreover, this approach may be useful for studying other nonlinear Schrödinger-type equations.

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1 Introduction

The proposal of the concept “quantum potential” is a valuable and seminal contribution to quantum mechanics [1,2]. The Madelung fluid description of quantum mechanics plays an important role in a number of applications, e.g., from stochastic mechanics to quantum cosmology and from the pilot waves theory to the hidden variables theory [3–5]. It has been employed to describe quantum effects in mesoscopic systems and in plasma physics and discuss quantum aspects of beam dynamics in high-intensity accelerators (see [6,7] and references therein). It has also been applied to such situations where the quantum formalism is a useful tool for describing the evolution of quantum-like systems or solving classical nonlinear evolution equations [8–24].

Within the context of Madelung fluid description, the complex wave function (say Ψ) is represented in terms of modulus and phase, and submitted into the Schrödinger equation to lead to the possible achievement of a pair of nonlinear fluid equations for the "density" $\rho = |\Psi|^2$ and the "current velocity" $v = \nabla \mathrm{Arg}(\Psi)$: one is the continuity equation (taking into account probability conservation) and the other one is a Navier-Stokes-like equation of motion [6,7,15–20]. Substitution of $\Psi(x,t) = \sqrt{\rho(x,t)}\, e^{\frac{i}{\hbar}\Theta(x,t)}$ into the one-dimensional Schrödinger equation

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x,t)}{\partial x^2} + m\, U(x)\Psi(x,t), \quad (1)$$

results in the following pair of coupled Madelung fluid equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho\, v) = 0, \quad (2a)$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - \frac{\hbar^2}{2m^2}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}\right) + \frac{\partial U}{\partial x} = 0, \quad (2b)$$

where $\rho = |\Psi|^2$ is the fluid density and $v = \frac{1}{m}\frac{\partial \Theta}{\partial x}$ is the current velocity. Equation (2a) is a continuity equation for the fluid density, while Eq. (2b) is an Euler equation or equation of motion for the fluid velocity and contains a force term proportional to the gradient of the "quantum potential", $\frac{\hbar^2}{2m^2}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}\right)$.

Madelung fluid description has been used to discuss families of generalized one-dimensional nonlinear Schrödinger equations [15–19], as follows:

- In Ref. [15], the following nonlinear Schrödinger-like equation

  $$i\mu\frac{\partial \Psi}{\partial t} = -\frac{\mu^2}{2}\frac{\partial^2 \Psi}{\partial x^2} + U\left[|\Psi|^2\right]\Psi, \quad (3)$$

  where $U\left[|\Psi|^2\right]$ and μ are as an arbitrary real functional of the complex wave function Ψ and an arbitrary real dispersion/diffraction coefficient, respectively, has been cast as

  $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho\, v) = 0, \quad (4a)$$

  $$-\rho\frac{\partial v}{\partial t} + v\frac{\partial \rho}{\partial t} + 2\left[c_0(t) - \int \frac{\partial v}{\partial t}\,\mathrm{d}x\right]\frac{\partial \rho}{\partial x} - \left(\rho\frac{\mathrm{d}U}{\mathrm{d}\rho} + 2U\right)\frac{\partial \rho}{\partial x} + \frac{\mu^2}{4}\frac{\partial^3 \rho}{\partial x^3} = 0, \quad (4b)$$

  with $c_0(t)$ as an arbitrary function of t. Under the hypothesis of stationary fluid, it is revealed that the cubic nonlinear Schrödinger equation can be put in correspondence with the standard Korteweg–de Vries equation in such a way that the soliton solutions of the latter are the squared modulus of the envelope soliton solution of the former. The conditions for different types of envelope solitons (bright, dark or gray ones) are also discussed.
- In Ref. [16], a modified nonlinear Schrödinger equation with a quartic nonlinear potential in the modulus of the wave function has been studied within the framework of Madelung fluid description, i.e., setting $U\left[|\Psi|^2\right] = q_1|\Psi|^2 + q_2|\Psi|^4$ with q_1 and q_2 as real constants in Eq. (3). By considering different boundary conditions, up-shifted bright soliton, upper-shifted bright soliton, gray soliton and dark soliton are finally found and can be cast into envelope solitons conversely.
- In Ref. [17], the existence of envelope soliton-like solutions of a nonlinear Schrödinger equation containing an anti-cubic nonlinearity ($|\Psi|^{-4}\Psi$) plus a 'regular' nonlinear part is investigated. In particular, in the case that the regular nonlinear part consists of a sum of cubic and quintic nonlinearities, i.e., setting $U\left[|\Psi|^2\right] = Q_0|\Psi|^{-4} + q_1|\Psi|^2 + q_2|\Psi|^4$ with Q_0, q_1 and q_2 as real constants in Eq. (3), an upper-shifted bright envelope soliton-like solution is explicitly found.
- In Ref. [18], a review is given on the results of investigations dealing with the connection between the envelope soliton-like solutions of a wide family of nonlinear Schrödinger equations and the soliton-like solutions of a wide family of Korteweg–de Vries equations. In two different fluid motion regimes (uniform current velocity and stationary profile current velocity variation, respectively), bright- and gray-/dark-soliton-like solutions of those equations are found.
- In Ref. [19], a similar discussion is given for a class of derivative nonlinear Schrödinger-type equations. For a motion with the stationary profile current velocity, the fluid density satisfies a generalized stationary Gardner equation, and solitary wave solutions are found. For the completely integrable cases, these solutions are compared with existing solutions in the literature.

In the present paper, we will investigate the Gerdjikov–Ivanov envelope solitons with the framework of Madelung fluid description. Various optical envelope soliton problems have been proposed and studied in the series literature [25–34]. In Sect. 2, we will pay attention to the Gerdjikov–Ivanov model and derive the basic equations of Madelung fluid: the continuity equation for the fluid density and motion equation for the fluid velocity.

It will be found in Sect. 3 that for a motion with the stationary profile current velocity, the fluid density satisfies a generalized stationary Gardner equation. Under suitable hypothesis for the current velocity and due to associated parametric constraints, in Sect. 4, we will present *bright*- and *dark*-type (including *gray* and *black*) solitary waves for the stationary Gardner equation, and finally, associated envelope solitons will be constructed correspondingly for the Gerdjikov–Ivanov model. Sect. 5 will be our conclusions.

2 Gerdjikov–Ivanov model and basic equations

The Gerdjikov–Ivanov model studied hereby is given as [35–38]

$$i\,\frac{\partial \Psi}{\partial t}+\frac{\partial^2 \Psi}{\partial x^2}-i\,\Psi^2\,\frac{\partial \Psi^*}{\partial x}+\frac{1}{2}\,|\Psi|^4\Psi=0, \tag{5}$$

where $*$ denotes the complex conjugation and $\Psi = \Psi(x,t)$ is the complex wave function. Eq. (5) is also called the derivative nonlinear Schrödinger III equation and can be transformed into the Kaup–Newell or the Chen–Lee–Liu equation by corresponding gauge transformations [37,38]. With the help of gauge transformation of the spectral problem, a Darboux transformation for Eq. (5) has been constructed with the achievement of the explicit soliton-like solutions in Ref. [37]. In Ref. [38], the spectral problem with the associated Gerdjikov–Ivanov hierarchy of nonlinear evolution equations is presented, and its bi-Hamiltonian structure, finite-dimensional integrable systems and N-fold Darboux transformation have also been studied.

Introducing $\Psi(x,t)=\sqrt{\rho(x,t)}\,e^{\frac{i}{k}\Theta(x,t)}$ with $k\neq 0$ into Eq. (5), we obtain the continuity equation for the fluid density ρ as

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}\left(\frac{2}{k}\,\rho\,v-\frac{1}{2}\,\rho^2\right)=0, \tag{6a}$$

and the equation of motion for the fluid velocity $v=\frac{\partial \Theta}{\partial x}$ as

$$\frac{\partial v}{\partial t}+\frac{2}{k}\,v\,\frac{\partial v}{\partial x}=k\,\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho}}\,\frac{\partial^2\sqrt{\rho}}{\partial x^2}\right) +\frac{\partial}{\partial x}\left(\rho\,v+\frac{2}{k}\,\rho^2\right). \tag{6b}$$

Symbolic-computation manipulation on Eq. (6b) gives rise to

$$-\,v\,\frac{\partial \rho}{\partial t}+\rho\,\frac{\partial v}{\partial t}-2\,\rho\,v\,\frac{\partial \rho}{\partial x}-\rho^2\,\frac{\partial v}{\partial x}-2k\,\rho^2\,\frac{\partial \rho}{\partial x} -\frac{k}{2}\,\frac{\partial^3 \rho}{\partial x^3}+2\,c_0(t)\,\frac{\partial \rho}{\partial x}+2\,\frac{\partial \rho}{\partial x}\int \frac{\partial v}{\partial t}\,\mathrm{d}x=0, \tag{6c}$$

where $c_0(t)$ is an arbitrary function of t.

Hereby, Eqs. (6a) and (6c) constitute the basic equations for the subsequent discussion.

3 Motion with the stationary profile current velocity

Assuming that both the quantities ρ and v involved in Eqs. (6a) and (6b) are functions of the combined variable $\xi = x - u_0\,t$ with u_0 being a real constant, we can cast Eq. (6a) into

$$-u_0\,\frac{\mathrm{d}\rho}{\mathrm{d}\xi}+\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{2}{k}\,\rho\,v-\frac{1}{2}\,\rho^2\right)=0, \tag{7}$$

and integrate once of Eq. (7) with respect to ξ to obtain

$$v=\frac{k}{2}\left(u_0+\frac{1}{2}\,\rho+\frac{A_0}{\rho}\right), \tag{8}$$

by taking the integration constant as A_0. Substitution of Eq. (8) into Eq. (6c) leads to

$$\left(\frac{k}{2}\,u_0^2-\frac{k}{2}\,A_0+2\,c_0\right)\frac{\mathrm{d}\rho}{\mathrm{d}\xi}-\frac{3k}{2}\,u_0\,\rho\,\frac{\mathrm{d}\rho}{\mathrm{d}\xi} -\frac{11k}{4}\,\rho^2\,\frac{\mathrm{d}\rho}{\mathrm{d}\xi}-\frac{k}{2}\,\frac{\mathrm{d}\rho^3}{\mathrm{d}\xi^3}=0. \tag{9}$$

Remark **(a)** According to Eq. (8), it is clear that $v\neq$ constant no matter $A_0=0$ or not. That is to say, we can only consider the motion with the stationary profile current velocity case and the motion with the constant current velocity case is excluded, which is different from that in Refs. [15–18].

(b) Note that the fluid density ρ satisfies a *generalized stationary Gardner equation*, i.e., Eq. (9), which is not the Korteweg–de Vries or modified Korteweg–de Vries equation, but a mixed one.

4 Solitary waves versus envelope solitons

To be followed, with the constraint $\rho > 0$ and under suitable assumptions for the current velocity associated with corresponding boundary conditions of ρ, we will investigate different types of solitary waves for Eq. (9).

Case I: In case ρ satisfies the boundary conditions in the ξ-space as $\lim_{\xi\to\pm\infty}\rho(\xi)=0$, it follows from Eq. (8) that $A_0=0$ and $v=\frac{k}{2}(u_0+\frac{1}{2}\rho)$.

Then, Eq. (9) becomes

$$\left(\frac{k}{2}u_0^2+2c_0\right)\frac{\mathrm{d}\rho}{\mathrm{d}\xi}-\frac{3k}{2}u_0\rho\frac{\mathrm{d}\rho}{\mathrm{d}\xi}-\frac{11k}{4}\rho^2\frac{\mathrm{d}\rho}{\mathrm{d}\xi}-\frac{k}{2}\frac{\mathrm{d}\rho^3}{\mathrm{d}\xi^3}=0, \tag{10}$$

which can be integrated twice with respect to ξ and take the integration constant as zero to give

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}\xi}\right)^2=\left(u_0^2+\frac{4}{k}c_0\right)\rho^2-u_0\rho^3-\frac{11}{12}\rho^4. \tag{11}$$

Setting $\rho=\frac{1}{\varphi}$ in Eq. (11) generates

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right)^2=-\frac{11}{12}-u_0\varphi+\left(u_0^2+\frac{4}{k}c_0\right)\varphi^2. \tag{12}$$

With the constraint $u_0^2+\frac{4}{k}c_0>0$, the second-order polynomial in the right-hand side of Eq. (12) has two real roots, one positive (φ_2) and the other negative (φ_1). Because the right-hand side of Eq. (12) has to be positive and $\lim_{\varphi\to+\infty}\rho=0$, the region of interest on φ-axis is $\varphi\in(\varphi_2,+\infty)$. Writing Eq. (12) as

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}=\sqrt{(u_0^2+\frac{4}{k}c_0)(\varphi-\varphi_2)(\varphi-\varphi_1)}, \tag{13}$$

we can solve Eq. (13) and have

$$\varphi=\frac{k\,u_0}{2\,k\,u_0^2+8\,c_0}+\frac{\sqrt{42\,u_0^2\,k^2+132\,k\,c_0}}{6\,k\,u_0^2+24\,c_0}\times\mathrm{Cosh}\left[\sqrt{u_0^2+\frac{4}{k}c_0}\;(\xi+\xi_0)\right], \tag{14}$$

and then the solution of Eq. (11) is obtained as

$$\rho=\frac{1}{\varphi}=\frac{1}{\dfrac{k\,u_0}{2\,k\,u_0^2+8\,c_0}+\dfrac{\sqrt{42\,u_0^2\,k^2+132\,k\,c_0}}{6\,k\,u_0^2+24\,c_0}\mathrm{Cosh}\left[\sqrt{u_0^2+\frac{4}{k}c_0}\;(\xi+\xi_0)\right]}, \tag{15}$$

and

$$v=\frac{\mathrm{d}\Theta}{\mathrm{d}\xi}=\frac{k}{2}\left(u_0+\frac{1}{\dfrac{k\,u_0}{4\,c_0+k\,u_0^2}+\dfrac{\sqrt{132\,k\,c_0+42\,u_0^2\,k^2}}{12\,c_0+3\,k\,u_0^2}\mathrm{Cosh}\left[\sqrt{u_0^2+\frac{4}{k}c_0}\;(\xi+\xi_0)\right]}\right), \tag{16}$$

$$\begin{aligned}\Theta(x,t)&=\frac{k}{2}u_0\,\xi+\frac{k}{4}\int\rho\,\mathrm{d}\xi-2\,c_0\,t-\Theta_0=\frac{k}{2}u_0\,x-\left(\frac{k}{2}u_0^2+2\,c_0\right)t\\&\quad-k\sqrt{\frac{3}{11}}\mathrm{ArcTan}\left[\left(3\,u_0\sqrt{\frac{k}{132\,c_0+33\,k\,u_0^2}}-\sqrt{\frac{132\,c_0+42\,k\,u_0^2}{132\,c_0+33\,k\,u_0^2}}\right)\right.\\&\quad\left.\times\mathrm{Tanh}\left(\frac{1}{2}\sqrt{u_0^2+\frac{4}{k}c_0}\,x-\frac{u_0}{2}\sqrt{u_0^2+\frac{4}{k}c_0}\,t+\frac{\xi_0}{2}\right)\right]-\Theta_0,\end{aligned} \tag{17}$$

where ξ_0 and Θ_0 are integration constants.

Note that the solitary wave solution of the stationary Gardner equation [i.e., Eq. (10)] possesses a *bright-soliton* profile [see Expression (15) above]. Finally, the *bright-soliton*-type envelope solution of the Gerdjikov–Ivanov model can be expressed as $\Psi(x,t) = \sqrt{\rho(x,t)}\, e^{\frac{i}{k}\Theta(x,t)}$ with $\rho(x,t)$ and $\Theta(x,t)$ given by Eqs. (15) and (17), respectively.

Case II: For the case of $\lim_{\xi\to\pm\infty}\rho(\xi) \neq 0$, we denote $\rho(\xi) = \rho_0 + \rho_1(\xi)$ with $\lim_{\xi\to\pm\infty}\rho_1(\xi) = 0$ and $\rho_0 = \text{constant} > 0$, which determines that $v = \frac{k}{2}\left[u_0 + \frac{1}{2}(\rho_0 + \rho_1(\xi)) + \frac{A_0}{\rho_0 + \rho_1(\xi)}\right]$.

Substituting $\rho(\xi) = \rho_0 + \rho_1(\xi)$ into Eq. (9), we have

$$\left(\frac{k}{2}u_0^2 - \frac{k}{2}A_0 + 2c_0 - \frac{3}{2}k\,u_0\,\rho_0 - \frac{11}{4}k\,\rho_0^2\right)\frac{d\rho_1}{d\xi} - \left(\frac{3}{2}k\,u_0 + \frac{11}{2}k\,\rho_0\right)\rho_1\frac{d\rho_1}{d\xi} - \frac{11}{4}k\,\rho_1^2\frac{d\rho_1}{d\xi} - \frac{k}{2}\frac{d\rho_1^3}{d\xi^3} = 0, \tag{18}$$

which is also a generalized stationary Gardner equation and can be similarly discussed as that in Case I.

Attention should be paid to the sign of ρ_1, which (can be negative or positive) is different from the sign of ρ in Case I. This inspires us to study other types of solutions.

Integrating Eq. (18) twice with respect to ξ, taking the integration constants as zero and setting $\rho_1 = \frac{1}{\varphi_1}$, we get

$$\left(\frac{d\varphi_1}{d\xi}\right)^2 = \left(u_0^2 - A_0 + \frac{4}{k}c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\rho_0^2\right)\varphi_1^2 - \left(u_0 + \frac{11}{3}\rho_0\right)\varphi_1 - \frac{11}{12}. \tag{19}$$

With the constraint $u_0^2 - A_0 + \frac{4}{k}c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\rho_0^2 > 0$, the second-order polynomial in the right-hand side of Eq. (19) has two real roots, as follows,

$$\alpha = \frac{22\,k\,\rho_0 + 6\,k\,u_0 + \sqrt{528\,k\,c_0 - 132\,A_0\,k^2 - 242\,\rho_0^2\,k^2 - 132\,\rho_0\,u_0\,k^2 + 168\,u_0^2\,k^2}}{48\,c_0 - 12\,A_0\,k - 66\,k\,\rho_0^2 - 36\,k\,u_0\,\rho_0 + 12\,k\,u_0^2} > 0,$$

$$\beta = \frac{22\,k\,\rho_0 + 6\,k\,u_0 - \sqrt{528\,k\,c_0 - 132\,A_0\,k^2 - 242\,\rho_0^2\,k^2 - 132\,\rho_0\,u_0\,k^2 + 168\,u_0^2\,k^2}}{48\,c_0 - 12\,A_0\,k - 66\,k\,\rho_0^2 - 36\,k\,u_0\,\rho_0 + 12\,k\,u_0^2} < 0.$$

Case II-I: With the assumption $\rho_1 > 0$, we take the solution of Eq. (19) as

$$\varphi_1 = \frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\,\mathrm{Cosh}\left[\sqrt{u_0^2 - A_0 + \frac{4}{k}c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\rho_0^2}\,(\xi - \xi_{01})\right] \geq \alpha > 0.$$

Therefore,

$$\rho_1 = \frac{1}{\varphi_1} = \frac{1}{\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\,\mathrm{Cosh}\left[\sqrt{u_0^2 - A_0 + \frac{4}{k}c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\rho_0^2}\,(\xi - \xi_{01})\right]}, \tag{20}$$

$$\rho = \rho_0 + \frac{1}{\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\,\mathrm{Cosh}\left[\sqrt{u_0^2 - A_0 + \frac{4}{k}c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\rho_0^2}\,(\xi - \xi_{01})\right]}, \tag{21}$$

$$
\begin{aligned}
\Theta(x,t) = {} & \left(\frac{k}{2}\,u_0 + \frac{1}{4}\,\rho_0 + \frac{k\,A_0}{2\,\rho_0}\right) x - \left(\frac{k}{2}\,u_0^2 + \frac{1}{4}\,\rho_0\,u_0 + 2\,c_0\right) t - \frac{k\,A_0\,\xi_{01}}{2\,\rho_0} - \Theta_{01} \\
& + \frac{\mathrm{ArcTanh}\left[\sqrt{\frac{\beta}{\alpha}}\,\mathrm{Tanh}\left[\frac{1}{2}\sqrt{u_0^2 - A_0 + \frac{4}{k}\,c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\,\rho_0^2}\,(x - u_0\,t - \xi_{01})\right]\right]}{2\sqrt{\alpha\,\beta\,(u_0^2 - A_0 + \frac{4}{k}\,c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\,\rho_0^2)}} \\
& - \frac{k\,A_0\,\mathrm{ArcTanh}\left[\sqrt{\frac{1+\rho_0\,\beta}{1+\rho_0\,\alpha}}\,\mathrm{Tanh}\left[\frac{1}{2}\sqrt{u_0^2 - A_0 + \frac{4}{k}\,c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\,\rho_0^2}\,(x - u_0\,t - \xi_{01})\right]\right]}{\rho_0\sqrt{(1+\rho_0\,\alpha)(1+\rho_0\,\beta)(u_0^2 - A_0 + \frac{4}{k}\,c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\,\rho_0^2)}}, \\
\Psi(x,t) = {} & \sqrt{\rho(x,t)}\,e^{\frac{i}{k}\,\Theta(x,t)},
\end{aligned}
\tag{22}
$$

where Θ_{01} and ξ_{01} are two integration constants.

It is clear that in Expression (20), the solution ρ_1 enjoys a *bright-soliton* profile. Based on Expression (21), we can discuss different cases of solutions for Eq. (9), correspondingly for the original Eq. (5).

Under the condition $3\,u_0 + 11\,\rho_0 = 0$, Expression (21) reduces to

Correspondingly, via Expression (23), i.e., different forms and types of the solitary waves for stationary Gardner equation, we can investigate the associated different types of envelope solitons for the original Gerdjikov–Ivanov model, i.e., Eq. (5) by means of Expression (22) (details ignored here).

$$
\rho = \rho_0\left\{1 + \varepsilon\,\mathrm{Sech}\left[\sqrt{u_0^2 - A_0 + \frac{4}{k}\,c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\,\rho_0^2}\,(x - u_0\,t - \xi_{01})\right]\right\}
\tag{23}
$$

with $\varepsilon = \frac{288\,c_0\,k - 72\,A_0\,k^2 - 396\,k^2\,\rho_0^2 - 216\,k^2\,u_0\,\rho_0 + 72\,k^2\,u_0^2}{\rho_0\sqrt{528\,c_0\,k - 132\,A_0\,k^2 - 242\,\rho_0^2\,k^2 - 132\,\rho_0\,u_0\,k^2 + 168\,u_0^2\,k^2}}$.

- When $0 < \varepsilon < 1$, Expression (23) represents a *up-shifted bright soliton*, whose maximum amplitude is $\rho_0(1+\varepsilon)$ and up-shifted by the quantity ρ_0.
- When $\varepsilon = 1$, Expression (23) represents a *upper-shifted bright soliton*, whose maximum amplitude is $2\,\rho_0$ and up-shifted by the quantity ρ_0.

Case II-II: With the assumption $\rho_1 < 0$, we take the solution of Eq. (19) as

$$
\varphi_1 = \frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\,\mathrm{Cosh}\left[\sqrt{u_0^2 - A_0 + \frac{4}{k}\,c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\,\rho_0^2}\;(\xi - \xi_{02})\right] \leq \beta < 0.
$$

Thus,

$$
\rho_1 = \frac{1}{\varphi_1} = \frac{1}{\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\,\mathrm{Cosh}\left[\sqrt{u_0^2 - A_0 + \frac{4}{k}\,c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\,\rho_0^2}\;(\xi - \xi_{02})\right]},
\tag{24}
$$

$$
\rho = \rho_0 + \frac{1}{\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\,\mathrm{Cosh}\left[\sqrt{u_0^2 - A_0 + \frac{4}{k}\,c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\,\rho_0^2}\;(\xi - \xi_{02})\right]},
\tag{25}
$$

$$\Theta(x,t) = \left(\frac{k}{2}u_0 + \frac{1}{4}\rho_0 + \frac{k\,A_0}{2\,\rho_0}\right)x - \left(\frac{k}{2}u_0^2 + \frac{1}{4}\rho_0\,u_0 + 2\,c_0\right)t - \frac{k\,A_0\,\xi_{02}}{2\,\rho_0} - \Theta_{02}$$
$$+\frac{\mathrm{ArcTanh}\left[\sqrt{\frac{\alpha}{\beta}}\,\mathrm{Tanh}\left[\frac{1}{2}\sqrt{u_0^2 - A_0 + \frac{4}{k}c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\rho_0^2}\,(x - u_0\,t - \xi_{02})\right]\right]}{2\sqrt{\alpha\,\beta\,(u_0^2 - A_0 + \frac{4}{k}c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\rho_0^2)}}$$
$$-\frac{k\,A_0\,\mathrm{ArcTanh}\left[\sqrt{\frac{1+\rho_0\,\alpha}{1+\rho_0\,\beta}}\,\mathrm{Tanh}\left[\frac{1}{2}\sqrt{u_0^2 - A_0 + \frac{4}{k}c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\rho_0^2}\,(x - u_0\,t - \xi_{02})\right]\right]}{\rho_0\sqrt{(1+\rho_0\,\alpha)(1+\rho_0\,\beta)(u_0^2 - A_0 + \frac{4}{k}c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\rho_0^2)}},$$
$$\Psi(x,t) = \sqrt{\rho(x,t)}\,e^{\frac{i}{k}\Theta(x,t)}, \tag{26}$$

where Θ_{02} and ξ_{02} are two integration constants.

Notice that in Expression (24), the solution ρ_1 enjoys a *non-bright-soliton* profile. Based on Expression (25), we can discuss different cases of solution for Eq. (9), correspondingly for the original Eq. (5).

Under the following condition

$$\begin{cases} 3\,u_0 + 11\,\rho_0 = 0, \\ \rho_0 > \dfrac{48\,c_0 - 12\,A_0\,k - 66\,k\,\rho_0^2 - 36\,k\,u_0\,\rho_0 + 12\,k\,u_0^2}{-22\,k\,\rho_0 - 6\,k\,u_0 + \sqrt{528\,k\,c_0 - 132\,A_0\,k^2 - 242\,\rho_0^2\,k^2 - 132\,\rho_0\,u_0\,k^2 + 168\,u_0^2\,k^2}}, \end{cases} \tag{27}$$

Expression (25) reduces to

$$\rho = \rho_0\left\{1 - \delta\,\mathrm{Sech}\left[\sqrt{u_0^2 - A_0 + \frac{4}{k}c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\rho_0^2}\,(x - u_0\,t - \xi_{01})\right]\right\} \tag{28}$$

with $\delta = \frac{288\,c_0\,k - 72\,A_0\,k^2 - 396\,k^2\,\rho_0^2 - 216\,k^2\,u_0\,\rho_0 + 72\,k^2\,u_0^2}{\rho_0\sqrt{528\,c_0\,k - 132\,A_0\,k^2 - 242\,\rho_0^2\,k^2 - 132\,\rho_0\,u_0\,k^2 + 168\,u_0^2\,k^2}}$.

- When $0 < \delta < 1$, Expression (28) denotes a *gray soliton*, whose minimum amplitude is $\rho_0(1-\delta)$ and reaches asymptotically the upper limit ρ_0.
- When $\delta = 1$, Expression (28) denotes a *black soliton*, whose minimum amplitude is zero and reaches asymptotically the upper limit ρ_0.

Correspondingly, via Expression (28), i.e., different forms and types of the solitary waves for the stationary Gardner equation, we can investigate the associated different types of envelope solitons for the original Gerdjikov–Ivanov model, i.e., Eq. (5) by means of Expression (26) (details ignored here).

5 Conclusions

With the application of Madelung fluid description finding solutions for nonlinear Schrödinger equations, fruitful results have been obtained. It is proven that the Madelung fluid description is a remarkably useful approach. In particular, on the basis of the present theory, we have derived *bright*- and *dark*-type (including *gray* and *black*) soliton solutions for the Gerdjikov–Ivanov model, by solving the corresponding stationary

Table 1 Solitary wave types

Solitary wave types for stationary Gardner Eq. (9)	Parametric constraints
bright-soliton-type	$u_0^2 + \frac{4}{k}\,c_0 > 0$; $A_0 = 0$;
bright-soliton-type	$\rho_0 > 0$; $u_0^2 - A_0 + \frac{4}{k}\,c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\,\rho_0^2 > 0$;
up-shifted-bright-type	$\rho_0 > 0$; $0 < \varepsilon < 1$; $3\,u_0 + 11\,\rho_0 = 0$; $u_0^2 - A_0 + \frac{4}{k}\,c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\,\rho_0^2 > 0$;
upper-shifted-bright-type	$\rho_0 > 0$; $\varepsilon = 1$; $3\,u_0 + 11\,\rho_0 = 0$; $u_0^2 - A_0 + \frac{4}{k}\,c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\,\rho_0^2 > 0$;
gray-soliton-type	$0 < \delta < 1$; Expression (27); $u_0^2 - A_0 + \frac{4}{k}\,c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\,\rho_0^2 > 0$;
black-soliton-type	$\delta = 1$; Expression (27); $u_0^2 - A_0 + \frac{4}{k}\,c_0 - 3\,u_0\,\rho_0 - \frac{11}{2}\,\rho_0^2 > 0$;

Gardner equations. Attention should be emphasized on: (**a**) We can only consider the motion with the stationary profile current velocity case, and the motion with the constant current velocity case is excluded [see Eq. (8)]; (**b**) Our results are derived under suitable assumptions for the current velocity associated with corresponding boundary conditions of ρ, and under corresponding parametric constraints. See Table 1 for the types of the solitary waves with associated parametric constraints. In our future works, we will consider employing the Madelung fluid description approach to study other nonlinear evolution equations [39–43].

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