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Integrable matrix nonlinear Schrödinger equations with reduced Lax pairs of AKNS type

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ABSTRACT

A specific class of Ablowitz–Kaup–Newell–Segur (AKNS) matrix spectral problems is reduced using pairs of similarity transformations. The corresponding integrable hierarchies are derived from the reduced Lax pairs, extending the standard matrix AKNS integrable hierarchies. A few illustrative examples are provided to showcase the diversity of matrix integrable nonlinear Schrödinger equations.

1. Introduction

Lax pairs of matrix spectral problems are used to generate integrable models [1]. The process begins by selecting a matrix spectral problem. Lax pairs then facilitate the solution of initial value problems for integrable models through the inverse scattering transform [2].

The nonlinear Schrödinger (NLS) equation and the modified Korteweg–de Vries (mKdV) equation are two examples that can be derived from the Ablowitz–Kaup–Newell–Segur (AKNS) matrix spectral problems by applying a single similarity transformation [3]. A pair of similarity transformations can lead to even more diverse integrable models [4]. The challenge lies in maintaining the balance between the two reductions of the potentials generated by these transformations, which introduces new requirements to ensure the invariance of the associated zero-curvature equations [5].

It is known that the concept of conducting similarity transformations has also been applied to the construction of nonlocal integrable models [6]. Three types of reduced integrable nonlinear Schrödinger equations and two types of reduced integrable modified Korteweg–de Vries equations have been proposed and classified [7]. In addition, the inverse scattering transform has been developed to solve nonlocal integrable models (see, e.g., [8,9]).

There are several efficient approaches for solving reduced integrable models, particularly in formulating soliton solutions. The Hirota bilinear method, Darboux transformation, Bäcklund transform, and Riemann-Hilbert technique have proven to be powerful tools. Numerous theories have been proposed for various reduced integrable models, both local and non-local (see, for example, [10–14]).

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In this letter, we propose a pair of local similarity transformations for the AKNS matrix spectral problems to generate reduced integrable models. The novelty lies in introducing two distinct types of similarity transformations: one involving diagonal block matrices, and the other involving off-diagonal block matrices. In Section 2, we revisit the matrix AKNS spectral problems and integrable models to set the stage for the subsequent analysis. We also propose two local similarity transformations for the AKNS matrix spectral problems, which generate reduced matrix AKNS integrable hierarchies. In Section 3, we illustrate the theory with concrete examples, showcasing various reduced AKNS matrix spectral problems and integrable models, which lead to novel matrix NLS integrable models. The final section provides a summary of the results.

2. Reduced matrix AKNS integrable hierarchies

2.1. Revisiting the standard matrix AKNS integrable hierarchies

Let $m,n\geq 1$ be two arbitrarily given integers. For each fixed pair of $m,n\geq 1$, we introduce two submatrix potentials p and q:

$$p=p(x,t)=(p_{jk})_{m\times n},\ \ q=q(x,t)=(q_{kj})_{n\times m}, \tag{2.1}$$

and $u=u(p,q)$ represents the potential consisting of p and q. Then, for each $r\geq 0$, the standard matrix AKNS spectral problems can be stated as follows:

$$-i\phi_x=U\phi,\ \ -i\phi_t=V^{[r]}\phi, \tag{2.2}$$

where the Lax pair is given by

$$\begin{cases} U=\lambda\Lambda+P,\ \ \Lambda=\begin{bmatrix}\alpha_1 I_m & 0\\ 0 & \alpha_2 I_n\end{bmatrix},\ \ P=\begin{bmatrix}0 & p\\ q & 0\end{bmatrix}, \\ V^{[r]}=\lambda^r\Omega+Q^{[r]},\ \ \Omega=\begin{bmatrix}\beta_1 I_m & 0\\ 0 & \beta_2 I_n\end{bmatrix},\ \ Q^{[r]}=\displaystyle\sum_{s=0}^{r-1}\lambda^s\begin{bmatrix}a^{[r-s]} & b^{[r-s]}\\ c^{[r-s]} & d^{[r-s]}\end{bmatrix}. \end{cases} \tag{2.3}$$

In the above Lax pair, I_k is the identity matrix of size k, α_1,α_2 and β_1,β_2 are two pairs of arbitrarily given distinct constants, $Q^{[0]}$ is the zero matrix of size $m+n$, and

$$W=\sum_{s\geq 0}\lambda^{-s}W^{[s]}=\sum_{s\geq 0}\lambda^{-s}\begin{bmatrix}a^{[s]} & b^{[s]}\\ c^{[s]} & d^{[s]}\end{bmatrix} \tag{2.4}$$

solves the stationary zero-curvature equation $W_x=i[U,W]$ with the initial data $W^{[0]}=\Omega$. Such a formal series solution is a crucial tool for generating integrable hierarchies (see, e.g., [15,16]).

Now, it direct to see that the zero-curvature equations:

$$U_t-V^{[r]}_x+i[U,V^{[r]}]=0,\ \ r\geq 0, \tag{2.5}$$

present a matrix AKNS integrable hierarchy:

$$p_t=i\alpha b^{[r+1]},\ \ q_t=-i\alpha c^{[r+1]},\ \ r\geq 0, \tag{2.6}$$

where $\alpha=\alpha_1-\alpha_2$. This generalizes the AKNS integrable hierarchy with scalar potentials [17]. Each system in the matrix AKNS integrable hierarchy possesses a bi-Hamiltonian structure and infinitely many symmetries and conserved quantities (see, e.g., [18,19]). The first nonlinear (when $r=2$) integrable model in (2.6) gives the matrix AKNS integrable NLS equations:

$$p_t=-\frac{\beta}{\alpha^2}i(p_{xx}+2pqp),\ \ q_t=\frac{\beta}{\alpha^2}i(q_{xx}+2qpq), \tag{2.7}$$

where $\beta=\beta_1-\beta_2$, and its Lax matrix $V^{[2]}$ is given by

$$V^{[2]}=\lambda^2\Omega+\frac{\beta}{\alpha}\lambda P-\frac{\beta}{\alpha^2}I_{m,n}(P^2+iP_x), \tag{2.8}$$

where $I_{m,n}=\mathrm{diag}(I_m,-I_n)$. Other examples of higher-order matrix AKNS integrable models could be computed as well (see, e.g., [20]).

In the following, we consider a specific class of AKNS matrix spectral problems with

$$\alpha_1=-\alpha_2=1,\ \ \beta_1=-\beta_2=2,\ \ m=n, \tag{2.9}$$

where $n\in\mathbb{N}$ is arbitrary. Accordingly, we will focus on integrable reductions of the matrix AKNS hierarchies associated with this particular class of AKNS spectral problems, which involve two square matrix potentials.

2.2. reductions of the AKNS matrix spectral problems

By taking two constant invertible Hermite symmetric square matrices of order n, Σ_1, Σ_2, and two constant invertible square matrices of order n, Δ_1, Δ_2, we formulate the two invertible constant square matrices of order $2n$:

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \ \Delta = \begin{bmatrix} 0 & \Delta_1 \\ \Delta_2 & 0 \end{bmatrix}. \tag{2.10}$$

Note that Σ is a diagonal block matrix, while Δ is an off-diagonal block matrix. Obviously, we have the crucial similarity properties for Λ and Δ:

$$\Sigma\Lambda\Sigma^{-1} = -\Delta\Lambda\Delta^{-1} = \Lambda, \ \Sigma\Omega\Sigma^{-1} = -\Delta\Omega\Delta^{-1} = \Omega. \tag{2.11}$$

These allows us to introduce the two similarity transformations that will preserve the invariance of the original zero-curvature equations:

$$\Sigma U(\lambda)\Sigma^{-1} = U^{\dagger}(\lambda^*) = (U(\lambda^*))^{\dagger}, \ \Delta U(\lambda)\Delta^{-1} = -U^T(\lambda) = -(U(\lambda))^T, \tag{2.12}$$

where λ^* is the complex conjugate of λ, and $A^{\dagger}$ and A^T stand for the Hermite transpose and the matrix transpose of a matrix A, respectively. The first similarity transformation has been used in presenting reduced local integrable models (see, e.g., [3]). Noting the specific form of the spectral matrix U, we can see that these two similarity transformations equivalently generate

$$\Sigma P\Sigma^{-1} = P^{\dagger}, \ \Delta P\Delta^{-1} = -P^T, \tag{2.13}$$

respectively. Clearly, these require the following two pairs of constraints for p and q:

$$p^{\dagger} = \Sigma_2 q\Sigma_1^{-1}, \ q^{\dagger} = \Sigma_1 p\Sigma_2^{-1}, \tag{2.14}$$

and

$$p^T = -\Delta_2 p\Delta_1^{-1}, \ q^T = -\Delta_1 q\Delta_2^{-1}. \tag{2.15}$$

The two constraints in (2.14) are compatible, since Σ is Hermitian. To ensure the compatibility of the two constraints in (2.15), a sufficient condition could be

$$\Delta_1^*(\Sigma_2^*)^{-1}\Delta_2 = \eta\Sigma_1, \tag{2.16}$$

where A^* denotes the conjugate of a matrix A, and $\eta \in \mathbb{R}$ and $\eta \neq 0$.

To summarize, under the condition (2.16) on Σ and Δ, the two similarity transformations in (2.12) generate a class of reduced AKNS matrix spectral problems:

$$-i\phi_x = U\phi, \ U = \begin{bmatrix} \lambda I_n & p \\ \Sigma_2^{-1}p^{\dagger}\Sigma_1 & -\lambda I_n \end{bmatrix}, \tag{2.17}$$

where p needs to satisfy $p^T = -\Delta_2 p\Delta_1^{-1}$.

2.3. Reduced matrix AKNS integrable hierarchies

Let us consider the solution W determined by (2.4), with the initial data $W^{[0]} = \Omega$. Under the two similarity transformations in (2.12), we can find

$$\Sigma W(\lambda)\Sigma^{-1}|_{\lambda=\infty} = (W(\lambda^*))^{\dagger}|_{\lambda=\infty}, \ \Delta W(\lambda)\Delta^{-1}|_{\lambda=\infty} = -(W(\lambda))^T|_{\lambda=\infty}. \tag{2.18}$$

By the uniqueness of solutions to the stationary zero-curvature equation, these ensure that

$$\Sigma W(\lambda)\Sigma^{-1} = (W(\lambda^*))^*, \ \Delta W(\lambda)\Delta^{-1} = -(W(\lambda))^T, \tag{2.19}$$

respectively. It follows from these relations that for each $r \geq 0$, we have the following pair of similarity properties:

$$\Sigma V^{[r]}(\lambda)\Sigma^{-1} = (V^{[r]}(\lambda^*))^{\dagger}, \ \Delta V^{[r]}(\lambda)\Delta^{-1} = -(V^{[r]}(\lambda))^T. \tag{2.20}$$

Consequently, under the potential constraints (2.14) and (2.15), we have

$$\begin{cases} \Sigma(U_t - V_x^{[r]} + i[U, V^{[r]}])(\lambda)\Sigma^{-1} & = (U_t - V_x^{[r]} + i[U, V^{[r]}])^{\dagger}(\lambda^*), \\ \Delta(U_t - V_x^{[r]} + i[U, V^{[r]}])(\lambda)\Delta^{-1} & = -(U_t - V_x^{[r]} + i[U, V^{[r]}])^T(\lambda), \end{cases} \tag{2.21}$$

and thus, the matrix AKNS integrable models in (2.6) become a reduced integrable hierarchy:

$$p_t = 2ib^{[r+1]}|_{q=\Sigma_2^{-1}p^{\dagger}\Sigma_1}, \ r \geq 0, \tag{2.22}$$

where the potential p satisfies $p^T = -\Delta_2 q\Delta_1^{-1}$. The matrix spectral problems, (2.17) and

$$-i\phi_t = V^{[r]}|_{q=\Sigma_2^{-1}p^{\dagger}\Sigma_1}\phi, \ r \geq 0, \tag{2.23}$$

constitute a Lax pair for every member in the reduced integrable hierarchy (2.22).

Noting that Σ_1, Σ_2 are arbitrary invertible Hermitian matrices, and Δ_1, Δ_2 are arbitrary invertible square matrices, we can obtain diverse hierarchies of reduced matrix AKNS integrable models.

3. Exemplary cases

In this section, we apply the general theory to several specific cases, providing illustrative examples of reduced matrix AKNS spectral problems and integrable NLS equations. In the following discussion, we assume that

$$\sigma = \pm 1,\ \delta = \pm 1, \tag{3.1}$$

which covers four distinct combinations.

Example 3.1. Let us first consider $n = 2$ and choose

$$\Sigma_1 = I_2,\ \Sigma_2 = \sigma I_2,\ \Delta_1 = \begin{bmatrix} 0 & 1 \\ \delta & 0 \end{bmatrix},\ \Delta_2 = -\delta \Delta_1. \tag{3.2}$$

Then, the two similarity transformations in (2.12) yield

$$U = U(u,\lambda) = \begin{bmatrix} \lambda I_2 & p \\ \sigma p^\dagger & -\lambda I_2 \end{bmatrix},\ p = \begin{bmatrix} p_2 & p_1 \\ p_3 & \delta p_2 \end{bmatrix}, \tag{3.3}$$

where $u = (p_1, p_2, p_3)^T$, and the reduced matrix integrable NLS equations are formulated as

$$\begin{cases} ip_1 = p_{1,xx} + 2\sigma[(|p_1|^2 + 2|p_2|^2)p_1 + \delta p_2^2 p_3^*], \\ ip_2 = p_{2,xx} + 2\sigma[(|p_1|^2 + |p_2|^2 + |p_3|^2)p_2 + \delta p_1 p_2^* p_3], \\ ip_3 = p_{3,xx} + 2\sigma[(2|p_2|^2 + |p_3|^2)p_3 + \delta p_1^* p_2^2]. \end{cases} \tag{3.4}$$

Example 3.2. Next, let us consider $n = 2$ and take

$$\Sigma_1 = I_2,\ \Sigma_2 = \sigma I_2,\ \Delta_1 = \begin{bmatrix} -\delta & 1 \\ 1 & 0 \end{bmatrix},\ \Delta_2 = -\Delta_1. \tag{3.5}$$

Then, the two similarity transformations in (2.12) determine

$$U = U(u,\lambda) = \begin{bmatrix} \lambda I_2 & p \\ \sigma p^\dagger & -\lambda I_2 \end{bmatrix},\ p = \begin{bmatrix} p_2 & p_1 \\ p_3 & \delta p_1 + p_2 \end{bmatrix}, \tag{3.6}$$

where $u = (p_1, p_2, p_3)^T$, and the reduced matrix integrable NLS equations are given by

$$\begin{cases} ip_1 = p_{1,xx} + 2\sigma[(2|p_1|^2 + \delta p_1 p_2^*)p_1 + \delta p_1(p_1^* + 2\delta p_2^* + p_3^*)p_2 + p_2^2 p_3^*], \\ ip_2 = p_{2,xx} + 2\sigma[(|p_1|^2 + |p_2|^2)p_2 + \delta p_1(p_1^* + \delta p_2^*)p_3 + p_2|p_3|^2], \\ ip_3 = p_{3,xx} + 2\sigma[(2|p_2|^2 + |p_3|^2)p_3 + (p_1^* p_2 + \delta|p_1|^2)(p_2 + \delta p_3) + \delta p_1 p_2^* p_3]. \end{cases} \tag{3.7}$$

Example 3.3. For $n = 3$, we take the matrix choices

$$\Sigma_1 = I_3,\ \Sigma_2 = \sigma I_3,\ \Delta_1 = \Delta_2 = \begin{bmatrix} 1 & 0 & \delta \\ 0 & \delta & 0 \\ \delta & 0 & 0 \end{bmatrix}. \tag{3.8}$$

Then, the two similarity transformations in (2.12) lead to

$$U = U(u,\lambda) = \begin{bmatrix} \lambda I_3 & p \\ \sigma p^\dagger & -\lambda I_3 \end{bmatrix},\ p = \begin{bmatrix} p_2 & p_1 & 0 \\ p_3 & 0 & -p_1 \\ -\delta p_2 & -\delta p_1 - p_3 & -p_2 \end{bmatrix}, \tag{3.9}$$

where $u = (p_1, p_2, p_3)^T$. The resulting system of equations involves multiple interactions between three components, showing various terms like nonlinear interactions and differential terms. In this case, the associated reduced matrix integrable NLS equations are as follows:

$$\begin{cases} ip_1 = p_{1,xx} + 2\sigma[(2|p_1|^2 + 2|p_2|^2 + |p_3|^2)p_1 + \delta p_1^2 p_3^* + \delta(|p_1|^2 + |p_2|^2)p_3], \\ ip_2 = p_{2,xx} + 2\sigma[(2|p_1|^2 + 2|p_2|^2 + |p_3|^2)p_2 + \delta p_1 p_2 p_3^*], \\ ip_3 = p_{3,xx} + 2\sigma[(|p_1|^2 + 2|p_2|^2 + |p_3|^2)p_3 - \delta p_1 |p_2|^2]. \end{cases} \tag{3.10}$$

Example 3.4. For $n = 3$, we take another set of matrix choices

$$\Sigma_1 = I_3,\ \Sigma_2 = \sigma I_3,\ \Delta_1 = \Delta_2 = \begin{bmatrix} -\delta & 0 & \delta \\ 0 & -1 & 0 \\ \delta & 0 & \delta \end{bmatrix}. \tag{3.11}$$

Then, the two similarity transformations in (2.12) yield

$$U = U(u,\lambda) = \begin{bmatrix} \lambda I_3 & p \\ \sigma p^\dagger & -\lambda I_3 \end{bmatrix},\ p = \begin{bmatrix} p_2 & p_1 & p_2 \\ p_3 & 0 & 2\delta p_1 + p_3 \\ p_2 & p_1 + \delta p_3 & -p_2 \end{bmatrix}, \tag{3.12}$$

where $u = (p_1, p_2, p_3)^T$. This shows a different set of interactions in the corresponding reduced AKNS spectral matrix and NLS equations. The form of the equations is similar to Example 3.3 but with distinct nonlinear interaction terms. In this case, the associated reduced matrix integrable NLS equations are expressed as follows:

$$\begin{cases} ip_1 = p_{1,xx} + 2\sigma[(2|p_1|^2 + 2|p_2|^2 + |p_3|^2)p_1 + \delta|p_1|^2 p_3 + \delta p_1^2 p_3^*], \\ ip_2 = p_{2,xx} + 2\sigma[(2|p_1|^2 + 2|p_2|^2 + 2|p_3|^2)p_2 + 2\delta p_1^* p_2 p_3 + \delta p_1 p_2 p_3^*], \\ ip_3 = p_{3,xx} + 2\sigma[(4|p_1|^2 + 2|p_2|^2 + 2|p_3|^2)p_3 + 2\delta p_1^* p_3^2 + 2\delta p_1 |p_3|^2]. \end{cases} \tag{3.13}$$

In these examples, the integration of the spectral matrix into the system of equations reveals deep nonlinear interactions that form the structure of the integrable NLS equations. The varying parameters σ and δ influence the interactions and lead to different types of nonlinear wave behaviors depending on the specific choice of the parameter values.

These examples showcase the flexibility and richness of the Lax pair formulation for constructing integrable models. We note that more specific integrable reductions can be derived by conducting different similarity transformations for the zero-curvature equations (see, e.g., [21–24]). The versatility and depth of such models allow them to be used across disciplines, making them invaluable tools for understanding complex systems, predicting wave interactions, and gaining insights into nonlinear dynamics.

4. Concluding remarks

Pairs of local similarity transformations have been introduced and analyzed for a specific class of AKNS matrix spectral problems. The associated reductions lead to reduced matrix AKNS integrable hierarchies. Several concrete examples of these reduced AKNS matrix spectral problems and the corresponding reduced matrix AKNS integrable NLS hierarchies have been constructed. This paper presents a novel type of pair of similarity transformations or group reductions, one involving a diagonal block matrix and the other involving an off-diagonal block matrix. These transformations differ from those discussed in the literature [5,25], where the similarity matrices are exclusively of the diagonal block matrix type.

Our examples illustrate the power of the Lax pair formulation in constructing integrable models and demonstrate how different transformations or reductions can lead to a diverse set of integrable NLS equations with varying nonlinear interactions. The choice of parameters, such as σ and δ, is critical in defining the structure of these systems, and the flexibility of the Lax pair approach allows for tailored models to describe different phenomena in both theoretical and applied contexts.

If we dive deeper into this approach, working through specific similarity transformations and exploring different forms of the zero-curvature equations can unveil even richer structures and special cases that might be of particular interest. This process paves the way for a broader class of integrable models, which could be applied to various fields such as fluid dynamics, plasma physics, and nonlinear optics.

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No data was used for the research described in the article.

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