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MATRIX MODIFIED KORTEWEG-DE VRIES INTEGRABLE MODELS VIA TWO SIMILARITY TRANSFORMATIONS

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Abstract This paper explores matrix modified Korteweg-de Vries integrable models through similarity transformations. The Lax pair formulation serves as the foundation, and pairs of similarity transformations are made for the Lax pairs of the Ablowitz-Kaup-Newell-Segur matrix spectral problems. Three illustrative cases are presented to derive reduced matrix-modified Korteweg-de Vries integrable models.

Keywords integrable hierarchy; Lax pair; AKNS matrix spectral problem; zero-curvature equation; group reduction

MSC2020 37K10; 35Q58; 37K40

1 Introduction

It is fundamental to formulate integrable models in soliton theory. The Lax pairs of matrix spectral problems serve as the starting point for presenting these models [1]. The key lies in formulating appropriate matrix spectral problems [2]. These models possess infinitely many symmetries and conservation laws and can be solved through the inverse scattering transform [3, 4].

Typical examples of integrable models include the nonlinear Schrödinger (NLS) equation and the modified Korteweg-de Vries (mKdV) equation. The starting point is the matrix Ablowitz-Kaup-Newell-Segur (AKNS) spectral problems, from which a single similarity transformation or group reduction is applied [5]. Moreover, applying a pair of group reductions or similarity transformations can produce a variety of integrable models [6]. The main challenge

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lies in balancing the reductions applied to the potentials generated by the two transformations, ensuring the invariance of the associated zero-curvature equations [7].

It is known that similarity transformations or group reductions have been extensively used in the construction of nonlocal integrable models involving reflection points [8]. A complete classification of lower-order integrable models associated with AKNS matrix spectral problems has identified three types of nonlocal NLS equations and two types of nonlocal mKdV equations [9]. On the other hand, various practical methods have been established to formulate and study reduced integrable models, particularly for deriving soliton solutions.

The inverse scattering transform remains a powerful method for solving initial value problems in nonlocal integrable models [10, 11]. In addition, techniques such as the Hirota bilinear method, Darboux transformation, Bäcklund transforms, and the Riemann-Hilbert method have been proven effective. Moreover, several innovative mathematical frameworks have been proposed to study nonlocal reduced integrable models (see, e.g., [9, 12–16]).

In this paper, we explore integrable reductions obtained through a pair of similarity transformations (or group reductions) applied to AKNS matrix spectral problems and their associated integrable models. A central step in our approach is the construction of two mutually consistent similarity transformations. In Section 2, we recall the matrix AKNS spectral problems together with their corresponding integrable mKdV models, which provide the foundation for the subsequent developments. We then introduce the pair of similarity transformations (or group reductions), leading to reduced matrix mKdV integrable models. Section 3 presents three illustrative examples, each employing distinct block matrices to construct the pair of similarity transformations, thereby demonstrating the variety of reduced AKNS matrix integrable models. The final section concludes with a summary of our results and additional remarks.

2 Matrix integrable mKdV models via similarity transformations

2.1 Revisiting the matrix AKNS integrable hierarchies

Let m, n be two arbitrary natural numbers. We define two matrix potentials, p and q,

$$p = p(x,t) = (p_{jk})_{m\times n},\ q = q(x,t) = (q_{kj})_{n\times m}, \tag{2.1}$$

and use $u = u(p,q)$ to denote the dependent variable, which is a vector-valued function of p and q. Then, for all $r \geq 0$, the standard matrix AKNS spectral problems are expressed as follows:

$$-\mathrm{i}\phi_x = U\phi,\ -\mathrm{i}\phi_t = V^{[r]}\phi, \tag{2.2}$$

where the Lax pairs are defined by

$$U = U(u,\lambda) = \lambda\Lambda + P,\ \Lambda = \begin{bmatrix} \alpha_1 I_m & 0 \\ 0 & \alpha_2 I_n \end{bmatrix},\ P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{2.3}$$

$$V^{[r]} = V^{[r]}(u,\lambda) = \lambda^r\Omega + Q^{[r]},\ \Omega = \begin{bmatrix} \beta_1 I_m & 0 \\ 0 & \beta_2 I_n \end{bmatrix},\ Q^{[r]} = \sum_{s=0}^{r-1}\lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}. \tag{2.4}$$

In these Lax pairs, I_k is the identity matrix of size k, λ denotes the spectral parameter, α_1, α_2 and β_1, β_2 are two pairs of arbitrary distinct constants, and $Q^{[0]}$ is the $(m+n)$-th-order zero matrix. Moreover, assume that with the initial data $W^{[0]} = \Omega$,

$$W = \sum_{s\geq 0} \lambda^{-s} W^{[s]} = \sum_{s\geq 0} \lambda^{-s} \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix} \tag{2.5}$$

presents the unique Laurent series to the stationary zero-curvature equation $W_x = \mathrm{i}[U, W]$. Such a series solution is crucial for generating integrable hierarchies (see, e.g., [17, 18]).

The zero-curvature equations: $U_t - V_x^{[r]} + \mathrm{i}[U, V^{[r]}] = 0,\ r \geq 0$, are the compatibility conditions of the two matrix spectral problems in (2.2). Based on the specific forms of (2.3) and (2.4), these lead to the matrix AKNS integrable hierarchy:

$$p_t = \mathrm{i}\alpha b^{[r+1]},\ q_t = -\mathrm{i}\alpha c^{[r+1]},\ r \geq 0, \tag{2.6}$$

where $\alpha = \alpha_1 - \alpha_2$. The simplest case with $m = n = 1$ presents the AKNS integrable hierarchy with scalar potentials [19]. Every system within the matrix AKNS integrable hierarchy has a bi-Hamiltonian structure, along with infinitely many symmetries and conserved quantities (see, e.g., [20–22]).

When $r = 2s + 1,\ s \geq 1$, the above matrix AKNS integrable hierarchy (2.6) reduces to the matrix mKdV integrable hierarchies. The first (when $s = 1$) integrable model is the matrix mKdV equations:

$$\begin{cases} p_t = -\dfrac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_xqp), \\ q_t = -\dfrac{\beta}{\alpha^3}(q_{xxx} + 3q_xpq + 3qpq_x), \end{cases} \tag{2.7}$$

where $\beta = \beta_1 - \beta_2$. The corresponding Lax matrix $V^{[3]}$ reads

$$V^{[3]} = \lambda^3\Omega + \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + iP_x) - \frac{\beta}{\alpha^3}(i[P, P_x] + P_{xx} + 2P^3), \tag{2.8}$$

where $I_{m,n} = \mathrm{diag}(I_m, -I_n)$. We remark that many other significant examples of higher-order matrix AKNS integrable models can similarly be generated (see, e.g., [23]).

2.2 Similarity transformations

To introduce a pair of similarity transformations or group reductions, we begin by taking two constant, invertible, symmetric square matrices of order m, denoted Σ_1 and Δ_1, and two constant, invertible, symmetric square matrices of order n, denoted Σ_2 and Δ_2. We then define two invertible constant square matrices of order $m + n$ as follows, as done in [7, 24–26]:

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},\ \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}. \tag{2.9}$$

Evidently, both Σ and Δ satisfy the following similarity properties

$$\Sigma\Lambda\Sigma^{-1} = \Delta\Lambda\Delta^{-1} = \Lambda,\ \Sigma\Omega\Sigma^{-1} = \Delta\Omega\Delta^{-1} = \Omega. \tag{2.10}$$

Assume that A^T stands for the matrix transpose of a matrix A. Then, the following pair of similarity transformations or group reductions can be proposed:

$$\Sigma U(\lambda)\Sigma^{-1} = -U^T(-\lambda) = -(U(-\lambda))^T,\ \Delta U(\lambda)\Delta^{-1} = -U^T(-\lambda) = -(U(-\lambda))^T. \tag{2.11}$$

Each of the similarity transformations or group reductions preserves the invariance of the original zero-curvature equations of the mKdV equations.

The two similarity transformations or group reductions yield the following relations for the potential matrix P:

$$\Sigma P \Sigma^{-1} = -P^T,\ \Delta P \Delta^{-1} = -P^T, \tag{2.12}$$

respectively, and these reductions give rise to the following pairs of constraints for the two matrix potentials p and q:

$$p^T = -\Sigma_2 q \Sigma_1^{-1},\ q^T = -\Sigma_1 p \Sigma_2^{-1}, \tag{2.13}$$

$$p^T = -\Delta_2 q \Delta_1^{-1},\ q^T = -\Delta_1 p \Delta_2^{-1}, \tag{2.14}$$

respectively. Since both Σ and Δ are symmetric, the two constraints in each of (2.13) and (2.14) are compatible. One of the following conditions:

$$\Sigma_1 p \Sigma_2^{-1} = \Delta_1 p \Delta_2^{-1}, \tag{2.15}$$

or

$$\Sigma_2 q \Sigma_1^{-1} = \Delta_2 q \Delta_1^{-1}, \tag{2.16}$$

which are equivalent, guarantees the compatibility of the above two sets of constraints.

Therefore, under the condition in (2.15) or (2.16), the two similarity transformations or group reductions in (2.11) yield the reduced AKNS matrix spectral problems:

$$-\mathrm{i}\phi_x = U\phi,\ U = \begin{bmatrix} \alpha_1 \lambda I_m & p \\ -\Sigma_2^{-1} p^T \Sigma_1 & \alpha_2 \lambda I_n \end{bmatrix}, \tag{2.17}$$

where p needs to satisfy (2.15), or the other reduced AKNS matrix spectral problems:

$$-\mathrm{i}\phi_x = U\phi,\ U = \begin{bmatrix} \alpha_1 \lambda I_m & -\Sigma_1^{-1} q^T \Sigma_2 \\ q & \alpha_2 \lambda I_n \end{bmatrix}, \tag{2.18}$$

where q needs to satisfy (2.16).

2.3 Matrix integrable mKdV models

With the initial data

$$W^{[0]} = \Omega = \begin{bmatrix} \beta_1 I_m & 0 \\ 0 & \beta_2 I_n \end{bmatrix}, \tag{2.19}$$

under the similarity transformations or group reductions given in (2.11), it follows from the uniqueness of solutions to the stationary zero-curvature equation that the solution W, determined by (2.5), satisfies

$$\Sigma W(\lambda) \Sigma^{-1} = W^T(-\lambda) = (W(-\lambda))^T,\ \Delta W(\lambda) \Delta^{-1} = W^T(-\lambda) = (W(-\lambda))^T, \tag{2.20}$$

respectively. Therefore, for all $s \geq 0$, we can compute that

$$\begin{aligned} &\Sigma V^{[2s+1]}(\lambda) \Sigma^{-1} = -V^{[2s+1]T}(-\lambda) = -(V^{[2s+1]}(-\lambda))^T, \\ &\Delta V^{[2s+1]}(\lambda) \Delta^{-1} = -V^{[2s+1]T}(-\lambda) = -(V^{[2s+1]}(-\lambda))^T. \end{aligned} \tag{2.21}$$

and consequently, we see that

$$\Sigma(U_t - V_x^{[2s+1]} + \mathrm{i}[U, V^{[2s+1]}]) = -((U_t - V_x^{[2s+1]} + \mathrm{i}[U, V^{[2s+1]}])(-\lambda))^T,$$

$$\Delta(U_t - V_x^{[2s+1} + \mathrm{i}[U, V^{[2s+1]}])(\lambda)\Delta^{-1} = -((U_t - V_x^{[2s+1]} + \mathrm{i}[U, V^{[2s+1]}])(-\lambda))^T.$$

Finally, the matrix AKNS integrable models in (2.6) with $r = 2s+1$ reduce to the following integrable mKdV models:

$$p_t = 2\mathrm{i}b^{[2s+2]}|_{q=-\Sigma_2^{-1}p^T\Sigma_1}, s \geq 0, \tag{2.22}$$

where p satisfies (2.15), or

$$q_t = -2\mathrm{i}c^{[2s+2]}|_{p=-\Sigma_1^{-1}q^T\Sigma_2}, \ s \geq 0, \tag{2.23}$$

where q satisfies (2.16). The matrix spectral problems (2.17) and

$$-\mathrm{i}\phi_t = V^{[2s+1]}|_{q=-\Sigma_2^{-1}p^T\Sigma_1}\phi, \ s \geq 0, \tag{2.24}$$

provide Lax pairs for the reduced integrable hierarchy (2.22), or the matrix spectral problems (2.18) and

$$-\mathrm{i}\phi_t = V^{[2s+1]}|_{p=-\Sigma_1^{-1}q^T\Sigma_2}\phi, \ s \geq 0, \tag{2.25}$$

provide Lax pairs for the reduced integrable hierarchy (2.23).

As a consequence of the the Lax operator algebras (see, e.g., [27]), the resulting reduced integrable models possess infinitely many commuting symmetries. We point out that since Σ_1, Σ_2, Δ_1 and Δ_2 are arbitrary, by choosing specific forms for these matrices, one can generate a variety of integrable mKdV models as concrete examples of the resulting reduced matrix AKNS models.

3 Three application scenarios

In this section, we explore three distinct scenarios, providing illustrative examples of reduced matrix AKNS spectral problems and integrable mKdV equations.

Example 3.1 Let us start with the case where $m = 1$ and $n = 3$, selecting the following specific values for the pairs of matrices:

$$\Sigma_1 = \sigma, \ \Sigma_2 = \begin{bmatrix} 0 & \rho_1 & \rho_2 \\ \rho_1 & 0 & \rho_1 \\ \rho_2 & \rho_1 & 0 \end{bmatrix}; \ \Delta_1 = \sigma, \ \Delta_2 = \begin{bmatrix} \rho_2 & \rho_1 & 0 \\ \rho_1 & 0 & \rho_1 \\ 0 & \rho_1 & \rho_2 \end{bmatrix}; \tag{3.1}$$

where σ, ρ_1 and ρ_2 are arbitrary non-zero constants, ensuring that all four matrices are invertible. Then, the group reductions or similarity transformations in (2.11) yield the following expression for the spectral matrix U:

$$U = U(u, \lambda) = \begin{bmatrix} \alpha_1\lambda & p \\ q & \alpha_2\lambda I_3 \end{bmatrix}, \tag{3.2}$$

where $u = (p_1, p_2)^T$, and p and q are expressed by

$$p = [p_1, p_2, p_1], \ q = \begin{bmatrix} -\dfrac{\sigma p_2}{2\rho_1} \\ -\dfrac{\sigma p_1}{\rho_1} + \dfrac{\sigma\rho_2 p_2}{2\rho_1^2} \\ -\dfrac{\sigma p_2}{2\rho_1} \end{bmatrix}. \tag{3.3}$$

Accordingly, the corresponding reduced matrix integrable mKdV equations are given by:

$$\begin{cases} p_{1,t} = -\dfrac{\beta}{\alpha^3}p_{1,xxx} + \dfrac{3\beta\sigma}{2\alpha^3\rho_1^2}\big[(6\rho_1p_1-\rho_2p_2)p_2p_{1,x}+(2\rho_1p_1-\rho_2p_2)p_1p_{2,x}\big], \\ p_{2,t} = -\dfrac{\beta}{\alpha^3}p_{2,xxx} + \dfrac{3\beta\sigma}{\alpha^3\rho_1^2}\big[\rho_1p_2^2p_{1,x}+(3\rho_1p_1-\rho_2p_2)p_2p_{2,x}\big], \end{cases} \tag{3.4}$$

where α, β, σ, ρ_1 and ρ_2 are arbitrary non-zero constants.

Example 3.2 Next, let us consider the case where $m=1$ and $n=4$. We choose the following specific values for the two pairs of matrices:

$$\Sigma_1=\sigma,\ \Sigma_2=\begin{bmatrix} 0 & \rho_1 & 0 & \rho_2 \\ \rho_1 & 0 & \rho_1 & 0 \\ 0 & \rho_1 & 0 & \rho_1 \\ \rho_2 & 0 & \rho_1 & 0 \end{bmatrix};\ \Delta_1=\sigma,\ \Delta_2=\begin{bmatrix} \rho_2 & \rho_1 & 0 & 0 \\ \rho_1 & 0 & \rho_1 & 0 \\ 0 & \rho_1 & 0 & \rho_1 \\ 0 & 0 & \rho_1 & \rho_2 \end{bmatrix}, \tag{3.5}$$

where σ and ρ_1 are arbitrary non-zero constants, while ρ_2 is arbitrary but distinct from $\pm\rho_1$. This guarantees that all four matrices are invertible, since $\det\Sigma_1=\rho_1^2(\rho_1-\rho_2)^2$, $\det\Delta_2=\rho_1^2(\rho_1^2-\rho_2^2)$. Then, the group reductions or similarity transformations in (2.11) generate

$$U=U(u,\lambda)=\begin{bmatrix} \alpha_1\lambda & p \\ q & \alpha_2\lambda I_4 \end{bmatrix}, \tag{3.6}$$

where $u=(p_1,p_2,p_3)^T$, and p and q are given by

$$p=(p_1,p_2,p_3,p_1+p_2-p_3),\ q=\begin{bmatrix} \dfrac{\sigma(p_1-p_3)}{\rho_1-\rho_2} \\ -\dfrac{\sigma(\rho_1p_1-\rho_2p_3)}{\rho_1(\rho_1-\rho_2)} \\ -\dfrac{\sigma(p_1-p_3)}{\rho_1-\rho_2}-\dfrac{\sigma p_2}{\rho_1} \\ \dfrac{\sigma(p_1-p_3)}{\rho_1-\rho_2} \end{bmatrix}. \tag{3.7}$$

Consequently, the corresponding reduced matrix integrable mKdV equations are given by

$$\begin{cases} p_{1,t} = -\dfrac{\beta}{\alpha^3}p_{1,xxx} + \dfrac{3\beta\sigma}{\alpha^3\rho_1(\rho_1-\rho_2)}\big\{\big[-4\rho_1p_1^2+6\rho_1p_1p_3+2(\rho_1-\rho_2)p_2p_3-2\rho_1p_3^2\big]p_{1,x} \\ \qquad +(\rho_1-\rho_2)p_1p_3p_{2,x}+\big[2\rho_1(p_1-p_3)p_1+(\rho_1-\rho_2)p_1p_2\big]p_{3,x}\big\}, \\ p_{2,t} = -\dfrac{\beta}{\alpha^3}p_{2,xxx} - \dfrac{3\beta\sigma}{\alpha^3\rho_1(\rho_1-\rho_2)}\big\{2\rho_1(p_1-p_3)p_2p_{1,x} \\ \qquad +\big[2\rho_1(p_1-p_3)^2-3(\rho_1-\rho_2)p_2p_3\big]p_{2,x}-\big[(\rho_1-\rho_2)p_2+2\rho_1(p_1-p_3)\big]p_{3,x}\big\}, \\ p_{3,t} = -\dfrac{\beta}{\alpha^3}p_{3,xxx} + \dfrac{3\beta\sigma}{\alpha^3\rho_1(\rho_1-\rho_2)}\big\{-2\rho_1(p_1-p_3)p_3p_{1,x} \\ \qquad +(\rho_1-\rho_2)p_3^2p_{2,x}-\big[2\rho_1p_1^2-6\rho_1p_1p_3-3(\rho_1-\rho_2)p_2p_3+4\rho_1p_3^2\big]p_{3,x}\big\}, \end{cases} \tag{3.8}$$

where α,β,σ, and ρ_1 denote arbitrary nonzero constants, while ρ_2 is arbitrary but required to be distinct from $\pm\rho_1$.

Example 3.3 Finally, let us examine the case where $m = 1$ and $n = 5$, selecting the following specific values for the pairs of matrices:

$$\Sigma_1 = \sigma,\ \Sigma_2 = \begin{bmatrix} 0 & \rho_1 & 0 & 0 & \rho_2 \\ \rho_1 & 0 & \rho_1 & 0 & 0 \\ 0 & \rho_1 & 0 & \rho_1 & 0 \\ 0 & 0 & \rho_1 & 0 & \rho_1 \\ \rho_2 & 0 & 0 & \rho_1 & 0 \end{bmatrix};\ \Delta_1 = \sigma,\ \Delta_2 = \begin{bmatrix} \rho_2 & \rho_1 & 0 & 0 & 0 \\ \rho_1 & 0 & \rho_1 & 0 & 0 \\ 0 & \rho_1 & 0 & \rho_1 & 0 \\ 0 & 0 & \rho_1 & 0 & \rho_1 \\ 0 & 0 & 0 & \rho_1 & \rho_2 \end{bmatrix}, \tag{3.9}$$

where σ, ρ_1 and ρ_2, are arbitrary non-zero constants, ensuring that all four matrices are invertible, since $\det \Sigma_2 = \det \Delta_2 = 2\rho_1^4\rho_2$. Then, the group reductions or similarity transformations in (2.11) yield the explicit expression for U:

$$U = U(u,\lambda) = \begin{bmatrix} \alpha_1 & p \\ q & \alpha_2\lambda I_5 \end{bmatrix}, \tag{3.10}$$

where $u = (p_1, p_2, p_3, p_4)^T$, and p and q are determined by

$$p = \big[p_1, p_2, p_3, p_2, p_4\big],\ q = \begin{bmatrix} -\dfrac{\sigma}{2\rho_2}(p_1 - p_3 + p_4) \\ -\dfrac{\sigma}{2\rho_1}(p_1 + p_3 - p_4) \\ \dfrac{\sigma}{2\rho_2}(p_1 - p_3 + p_4) - \dfrac{\sigma p_2}{\rho_1} \\ \dfrac{\sigma}{2\rho_1}(p_1 - p_3 - p_4) \\ -\dfrac{\sigma}{2\rho_2}(p_1 - p_3 + p_4) \end{bmatrix}. \tag{3.11}$$

Furthermore, the corresponding reduced matrix integrable mKdV equations take the form:

$$\begin{cases} p_{1,t} = -\dfrac{\beta}{\alpha^3}p_{1,xxx} + \dfrac{3\beta\sigma}{2\alpha^3\rho_1\rho_2}\big\{\big[(2\rho_1 p_1^2 - 3\rho_1(p_3 - p_4)p_1 + \rho_1 p_3^2 + (4\rho_2 p_2 - 2\rho_1 p_4)p_3 + \rho_1 p_4^2\big]p_{1,x} \\ \qquad +2\rho_2 p_1 p_3 p_{2,x} - \big[\rho_1(p_1 - p_3 + p_4) - 2\rho_2 p_2\big]p_1 p_{3,x} + \rho_1(p_1 - p_3 + p_4)p_1 p_{4,x}\big\}, \\ p_{2,t} = -\dfrac{\beta}{\alpha^3}p_{2,xxx} + \dfrac{3\beta\sigma}{2\alpha^3\rho_1\rho_2}\big\{\rho_1(p_1 - p_3 + p_4)p_2 p_{1,x} + \big[6\rho_2 p_2 p_3 + \rho_1(p_1 - p_3 + p_4)^2\big]p_{2,x} \\ \qquad +\big[2\rho_2 p_2 - \rho_1(p_1 - p_3 + p_4)\big]p_2 p_{3,x} + \rho_1(p_1 - p_3 + p_4)p_2 p_{4,x}\big\}, \\ p_{3,t} = -\dfrac{\beta}{\alpha^3}p_{3,xxx} + \dfrac{3\beta\sigma}{2\alpha^3\rho_1\rho_2}\big\{\rho_1(p_1 - p_3 + p_4)p_3 p_{1,x} + 2\rho_2 p_3^2 p_{2,x} \\ \qquad +\big[2\rho_1 p_3^2 - 3(\rho_1 p_1 - 2\rho_2 p_2 + \rho_1 p_4)p_3 + \rho_1(p_1 + p_4)^2\big]p_{3,x} + \rho_1(p_1 - p_3 + p_4)p_3 p_{4,x}\big\}, \\ p_{4,t} = -\dfrac{\beta}{\alpha^3}p_{4,xxx} + \dfrac{3\beta\sigma}{2\alpha^3\rho_1\rho_2}\big\{\rho_1(p_1 - p_3 + p_4)p_4 p_{1,x} + 2\rho_2 p_3 p_4 p_{2,x} - (\rho_1 p_1 - 2\rho_2 p_2 - \rho_1 p_3 \\ \qquad +\rho_1 p_4)p_4 p_{3,x} + \big[\rho_1 p_1^2 - 2(\rho_1 p_1 - 2\rho_2 p_2)p_3 + 3\rho_1(p_1 - p_3)p_4 + \rho_1 p_3^2 + 2\rho_1 p_4^2\big]p_{4,x}\big\}, \end{cases} \tag{3.12}$$

where $\alpha, \beta, \sigma, \rho_1$ and ρ_2 are arbitrary non-zero constants.

In this example, incorporating the spectral matrix into the system of equations unveils intricate nonlinear interactions that shape the structure of the integrable mKdV equations. The parameters σ, ρ_1 and ρ_2 play a pivotal role in governing the system's dynamics and influencing the interplay between its components.

All the examples presented above underscore the versatility and depth of the Lax pair formulation in constructing integrable models. By applying various similarity transformations to the zero-curvature equations, a diverse range of integrable reductions can be obtained (see, e.g., [28–33]). These transformations open pathways for exploring diverse nonlinear wave phenomena, with potential applications across multiple fields. Furthermore, these examples enrich the ongoing research into integrable models associated with higher-order matrix spectral problems, as discussed in [34–39].

4 Concluding remarks

This paper explores a pair of similarity transformations, or group reductions, applied to the AKNS matrix spectral problems, resulting in reduced matrix modified Korteweg-de Vries (mKdV) integrable models. Three concrete examples of these reduced matrix integrable mKdV models are presented, along with their corresponding reduced AKNS matrix spectral problems. A key focus of this study is the identification of two suitable similarity transformations or group reductions, which give rise to novel mKdV integrable models and extend the framework established in previous research [7, 26, 40, 41].

The presented examples highlight the versatility of the Lax pair formulation in constructing integrable models, demonstrating how different group reductions or similarity transformations can produce a diverse array of integrable mKdV models, each characterized by distinct nonlinear interactions. The choice of diagonal block matrices in the similarity transformations or group reductions plays a crucial role in shaping the structure of these systems. The flexibility of the Lax pair approach not only facilitates the development of tailored models but also serves as an effective tool for both theoretical exploration and practical applications.

This research provides an introduction to the construction of integrable models within a more advanced framework. Comparing these models with other approaches may help identify the underlying algebraic and geometric structures of multi-component integrable systems. Moreover, it would be worthwhile to investigate intriguing solution phenomena, such as rogue waves, lump waves, and soliton waves (see, e.g., [42–49]), which can be derived using various soliton techniques. The integrable models presented here contribute new perspectives to the classification of multi-component integrable systems within the zero-curvature framework. We further anticipate that these models will find promising applications in the applied and engineering sciences.

Conflict of Interest The author declares that there is no conflict of interest.

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