

Article

A Real Four-Component Integrable Extension of the Standard Kaup–Newell Hierarchy with Two Diagonal Blocks

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Abstract

This paper aims to introduce a real four-component integrable extension of the complex Kaup–Newell soliton hierarchy. Following a general idea for extending the standard Kaup–Newell spectral matrix, we propose a specific matrix eigenvalue problem involving four real potentials and construct the corresponding integrable Hamiltonian hierarchy via the zero-curvature formulation. A recursion operator and a bi-Hamiltonian structure are presented to demonstrate the Liouville integrability of the resulting hierarchy. As an illustrative example, we derive an integrable system of four real derivative nonlinear Schrödinger equations, each containing two linear dispersion terms and generalizing the standard complex derivative nonlinear Schrödinger equations.

Keywords: matrix eigenvalue problem; zero-curvature equation; integrable hierarchy; recursion operator; derivative nonlinear Schrödinger equations

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1. Introduction

Integrable models typically arise in hierarchies equipped with hereditary recursion operators [1] and are associated with Lax pairs formulated as matrix eigenvalue problems [2]. Such matrix eigenvalue problems also underpin the inverse scattering transform, which provides a powerful method for solving Cauchy problems of nonlinear integrable models, as well as Hamiltonian structures that link symmetries with conserved quantities. Owing to these rich mathematical structures, integrable models find wide-ranging applications in the physical sciences and engineering, including nonlinear optics, fluid dynamics, and quantum mechanics.

Among the well-known examples of integrable hierarchies is the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy [3] and its various hierarchies of integrable couplings [4]. Lax pairs provide a fundamental framework for studying integrable models via the zero-curvature formulation [4–6]. Classical soliton hierarchies are typically associated with Lax pairs constructed from semisimple Lie algebras, whereas integrable couplings arise from Lax pairs based on non-semisimple Lie algebras. It is therefore of considerable interest to explore what kinds of Lax pairs can generate integrable models. In this paper, by analyzing possible integrable extensions of Kaup–Newell-type Lax pairs, we propose a novel 4×4 matrix eigenvalue problem and construct the corresponding integrable hierarchy of generalized Kaup–Newell models.



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The zero-curvature formulation provides a systematic framework for constructing integrable models (see [6] for details). As usual, we denote the column vector of potentials by $u = (u_1, \dots, u_r)^T$ and the spectral parameter by λ . Let

$$\tilde{g} = g \otimes \mathbb{C}[\lambda, \lambda^{-1}] \quad (1)$$

be a loop matrix algebra associated with the matrix Lie algebra g and the loop parameter λ . A matrix $E_0 \in \tilde{g}$ is said to be pseudo-regular if it satisfies

$$[\text{Ker ad}_{E_0}, \text{Ker ad}_{E_0}] = 0, \text{Im ad}_{E_0} \oplus \text{Ker ad}_{E_0} = \tilde{g}, \quad (2)$$

where ad_{E_0} denotes the adjoint action of E_0 on $\tilde{g}$, and Ker ad_{E_0} and Im ad_{E_0} are the kernel and image of the operator ad_{E_0} , respectively. We fix one pseudo-regular matrix E_0 and choose r linearly independent matrices $E_1, \dots, E_r$ in $\tilde{g}$ to construct the spatial spectral matrix:

$$\mathcal{M} = \mathcal{M}(u, \lambda) = E_0(\lambda) + u_1 E_1(\lambda) + \dots + u_r E_r(\lambda). \quad (3)$$

We then seek a Laurent-series solution

$$Y = \sum_{k \geq 0} \lambda^{-k} Y^{[k]}, \quad Y^{[k]} \in g, \quad (4)$$

to the stationary zero-curvature equation

$$Y_x = [\mathcal{M}, Y] \quad (5)$$

in the underlying loop algebra $\tilde{g}$. The key point is to determine the solution Y recursively from the initial data for $Y^{[0]}$.

We further proceed to construct an infinite sequence of temporal spectral matrices

$$\mathcal{N}^{[l]} = (\lambda^l Y)_+ + \Delta_l, \quad (\lambda^l Y)_+ = \sum_{k=0}^l \lambda^{l-k} Y^{[k]}, \quad l \geq 0, \quad (6)$$

where $\Delta_l \in \tilde{g}$, $l \geq 0$. These matrices serve as the temporal parts of the Lax pairs, such that the zero-curvature equations

$$\mathcal{M}_{t_l} - \mathcal{N}_x^{[l]} + [\mathcal{M}, \mathcal{N}^{[l]}] = 0, \quad l \geq 0, \quad (7)$$

generate a hierarchy of integrable models:

$$u_{t_l} = Z^{[l]} = Z^{[l]}(u), \quad l \geq 0. \quad (8)$$

The equations in (7) represent the compatibility conditions of the associated spatial and temporal matrix eigenvalue problems,

$$\varphi_x = \mathcal{M}\varphi, \quad \varphi_{t_l} = \mathcal{N}^{[l]}\varphi, \quad l \geq 0. \quad (9)$$

In practice, a trial-and-error procedure is often required to determine appropriate choices of Δ_l .

Finally, we establish a bi-Hamiltonian formulation for the resulting hierarchy (8). This is achieved by computing a recursion operator and applying the so-called trace identity

$$\frac{\delta}{\delta u} \int \text{tr} \left(Y \frac{\partial \mathcal{M}}{\partial \lambda} \right) dx = \lambda^{-\kappa} \frac{\partial}{\partial \lambda} \lambda^{\kappa} \text{tr} \left(Y \frac{\partial \mathcal{M}}{\partial u} \right), \quad (10)$$

where $\frac{\delta}{\delta u}$ denotes the variational derivative with respect to u , and κ is a constant independent of the spectral parameter λ , determined by

$$\kappa = -\frac{\lambda}{2} \frac{\partial}{\partial \lambda} \ln |\text{tr}(Y^2)|. \quad (11)$$

Consequently, a bi-Hamiltonian formulation can be generated for every equation in the hierarchy, thereby establishing its Liouville integrability (see, e.g., [6,7]).

There exist various hierarchies of Liouville integrable models with different numbers of potentials that have been studied extensively in the literature [3–12]. One-component integrable hierarchies include the Korteweg–de Vries hierarchy, the nonlinear Schrödinger hierarchy and the modified Korteweg–de Vries hierarchy [1]. The case of two components is particularly well studied, with well-known examples including the Ablowitz–Kaup–Newell–Segur integrable hierarchy [3], the Heisenberg integrable hierarchy [13], the Kaup–Newell integrable hierarchy [14,15] and the Wadati–Konno–Ichikawa integrable hierarchy [16]. All these hierarchies are associated with 2×2 spectral matrices. In contrast, the construction of integrable hierarchies based on higher-order spectral matrices presents substantially greater challenges (see, e.g., [17]).

We consider multi-component extensions of the Kaup–Newell spectral matrix of the following block form:

$$\mathcal{M} = \begin{bmatrix} \lambda^2 \Lambda_1 & \lambda p \\ \lambda q & \lambda^2 \Lambda_2 \end{bmatrix}, \quad (12)$$

where Λ_1 and Λ_2 are two constant square matrices, and p and q are matrix-valued potentials, which are not necessarily square. In order to ensure the existence of a Laurent-series solution

$$Y = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad (13)$$

partitioned in the same manner as the spectral matrix $\mathcal{M}$, to the stationary zero-curvature equation, Equation (5), we need to impose the following conditions:

$$[A, \Lambda_1] = 0, [D, \Lambda_2] = 0, [A, pq] = 0, [D, qp] = 0, \quad (14)$$

where the last two conditions are not required in the AKNS-type framework. In what follows, we focus on the case where p and q are square and Λ_1 and Λ_2 are diagonal blocks.

There are various multi-component Kaup–Newell-type integrable nonlinear Schrödinger equations proposed in the literature. For example, in [18], a generalized matrix Chen–Lee–Liu-type system associated with the generalized Kaup–Newell spectral matrix in their Equation (2.3) is studied. Likewise, Zhou and Yu in [19] establish a generalized Kaup–Newell matrix spectral problem and derive a multi-component Kaup–Newell system together with its nonlocal integrable reductions, associated with the spectral matrix in their Equation (3). Moreover, in [20], a generalized Gerdjikov–Ivanov spectral matrix in their Equation (5) and the Lax operator M in their Equation (8) are formulated for the construction of an integrable hierarchy, although M is not completely specified since the involved constants c_n remain undetermined. It is of particular importance to develop a general and workable procedure for successfully generating Kaup–Newell-type integrable hierarchies that constitute complete sets of commuting Lie symmetries of differential-polynomial type for each nonlinear equation

in the hierarchy, while simultaneously possessing associated spectral problems satisfying the spectral theorem. More importantly, such a property requires that the eigenfunctions of the associated recursion operators, namely, the squared eigenfunctions of the corresponding matrix spectral problems, form a basis that is complete in a suitable normed function space.

In this paper, following the construction idea outlined above, we propose a real four-component integrable extension of the standard Kaup–Newell soliton hierarchy. In particular, a specific 4×4 spectral matrix with four real potentials is proposed, and a hierarchy of real four-component Liouville-integrable models is generated via the zero-curvature formulation. A recursion operator is derived from the relations between the associated vector fields, and a bi-Hamiltonian structure is established using the trace identity, thereby demonstrating the Liouville integrability of the resulting hierarchy. As an illustrative example, we obtain a class of generalized coupled integrable derivative nonlinear Schrödinger equations with four real components. Finally, concluding remarks are presented in the last section.

2. An Integrable Extension with Two Diagonal Blocks

Let α_1 and α_2 be two arbitrary real numbers satisfying

$$\alpha = \alpha_1 - \alpha_2 \neq 0, \quad (15)$$

and let $u = u(x, t) = (u_1, u_2, u_3, u_4)^T$ be a column vector consisting of four potentials. We begin with two diagonal blocks

$$\Lambda_1 = \text{diag}(\alpha_1, \alpha_1), \quad \Lambda_2 = \text{diag}(\alpha_2, \alpha_2), \quad (16)$$

and two square matrix potentials of the form

$$p = \begin{bmatrix} u_1 & u_2 \\ -u_2 & u_1 \end{bmatrix}, \quad q = \begin{bmatrix} u_3 & u_4 \\ -u_4 & u_3 \end{bmatrix}. \quad (17)$$

Therefore, the matrix eigenvalue problem takes the form

$$\varphi_x = \mathcal{M}\varphi = \mathcal{M}(u, \lambda)\varphi, \quad \mathcal{M} = \begin{bmatrix} \alpha_1\lambda^2 & 0 & \lambda u_1 & \lambda u_2 \\ 0 & \alpha_1\lambda^2 & -\lambda u_2 & \lambda u_1 \\ \lambda u_3 & \lambda u_4 & \alpha_2\lambda^2 & 0 \\ -\lambda u_4 & \lambda u_3 & 0 & \alpha_2\lambda^2 \end{bmatrix}, \quad (18)$$

where λ is the spectral parameter. This spectral matrix $\mathcal{M}$ constitutes a generalization of the standard Kaup–Newell eigenvalue problem [14]. Interestingly, starting from this eigenvalue problem, we can generate an associated integrable hierarchy of bi-Hamiltonian Kaup–Newell-type equations. All equations in the hierarchy exhibit characteristic combined structures of dispersion terms.

To construct the associated integrable hierarchy, we begin by solving the stationary zero-curvature equation, Equation (5). In order to satisfy the basic requirements in (14), we assume that the matrix solution Y takes the form

$$Y = \begin{bmatrix} a & f & b & e \\ -f & a & -e & b \\ c & g & -a & -f \\ -g & c & f & -a \end{bmatrix} = \sum_{k \geq 0} \lambda^{-k} Y^{[k]}. \quad (19)$$

Substituting this ansatz into the stationary zero-curvature equation, Equation (5), we find that it is equivalent to the following system:

$$\begin{cases} a_x = \lambda c u_1 - \lambda g u_2 - \lambda b u_3 + \lambda e u_4, \\ b_x = \alpha \lambda^2 b - 2 \lambda a u_1 + 2 \lambda f u_2, \\ c_x = -\alpha \lambda^2 c + 2 \lambda a u_3 - 2 \lambda f u_4, \end{cases} \quad (20)$$

and

$$\begin{cases} e_x = \alpha \lambda^2 e - 2 \lambda f u_1 - 2 \lambda a u_2, \\ g_x = -\alpha \lambda^2 g + 2 \lambda f u_3 + 2 \lambda a u_4, \\ f_x = \lambda g u_1 + \lambda c u_2 - \lambda e u_3 - \lambda b u_4. \end{cases} \quad (21)$$

We further assume that the entries of Y admit Laurent expansions in λ of the form

$$\begin{cases} a = \sum_{n \geq 0} \lambda^{-2k} a^{[k]}, \quad b = \sum_{k \geq 0} \lambda^{-2k-1} b^{[k]}, \quad c = \sum_{k \geq 0} \lambda^{-2k-1} c^{[k]}, \\ e = \sum_{k \geq 0} \lambda^{-2k-1} e^{[k]}, \quad f = \sum_{k \geq 0} \lambda^{-2k} f^{[k]}, \quad g = \sum_{k \geq 0} \lambda^{-2k-1} g^{[k]}. \end{cases} \quad (22)$$

From (20) and (21), one readily verifies the identities

$$\begin{cases} \alpha \lambda a_x = -u_3 b_x - u_1 c_x + u_4 e_x + u_2 g_x, \\ -\alpha \lambda f_x = u_4 b_x + u_2 c_x + u_3 e_x + u_1 g_x, \end{cases} \quad (23)$$

which play a key role in deriving the recursion relations. Comparing coefficients of equal powers of λ , we obtain the initial relations

$$\begin{cases} a_x^{[0]} = u_1 c^{[0]} - u_2 g^{[0]} - u_3 b^{[0]} + u_4 e^{[0]}, \\ f_x^{[0]} = u_1 g^{[0]} + u_2 c^{[0]} - u_3 e^{[0]} - u_4 b^{[0]}, \end{cases} \quad (24)$$

together with the recursion relations, valid for $k \geq 0$,

$$\begin{cases} a_x^{[k+1]} = \frac{1}{\alpha} (-u_3 b_x^{[k]} - u_1 c_x^{[k]} + u_4 e_x^{[k]} + u_2 g_x^{[k]}), \\ f_x^{[k+1]} = -\frac{1}{\alpha} (u_4 b_x^{[k]} + u_2 c_x^{[k]} + u_3 e_x^{[k]} + u_1 g_x^{[k]}), \end{cases} \quad (25)$$

$$\begin{cases} b^{[k+1]} = \frac{1}{\alpha} (b_x^{[k]} + 2u_1 a^{[k+1]} - 2u_2 f^{[k+1]}), \\ c^{[k+1]} = \frac{1}{\alpha} (-c_x^{[k]} + 2u_3 a^{[k+1]} - 2u_4 f^{[k+1]}), \end{cases} \quad (26)$$

and

$$\begin{cases} e^{[k+1]} = \frac{1}{\alpha} (e_x^{[k]} + 2u_1 f^{[k+1]} + 2u_2 a^{[k+1]}), \\ g^{[k+1]} = \frac{1}{\alpha} (-g_x^{[k]} + 2u_3 f^{[k+1]} + 2u_4 a^{[k+1]}). \end{cases} \quad (27)$$

These relations determine the Laurent-series solution Y and, thus, generate the associated integrable hierarchy.

To ensure the uniqueness of the Laurent-series solution, we prescribe the following combined initial data:

$$\begin{cases} b^{[0]} = \beta u_1 + \gamma u_2, \quad c^{[0]} = \beta u_3 + \gamma u_4, \\ e^{[0]} = \beta u_2 - \gamma u_1, \quad g^{[0]} = \beta u_4 - \gamma u_3, \end{cases} \quad (28)$$

where β and γ are arbitrary constants, at least one of which is nonzero. From (20) and (21), we obtain

$$\begin{cases} \alpha b^{[0]} - 2(a^{[0]}u_1 - f^{[0]}u_2) = 0, \alpha c^{[0]} - 2(a^{[0]}u_3 - f^{[0]}u_4) = 0, \\ \alpha e^{[0]} - 2(f^{[0]}u_1 + a^{[0]}u_2) = 0, \alpha g^{[0]} - 2(f^{[0]}u_3 + a^{[0]}u_4) = 0, \end{cases}$$

which yields

$$a^{[0]} = \frac{1}{2}\alpha\beta, f^{[0]} = -\frac{1}{2}\alpha\gamma. \quad (29)$$

Furthermore, the constants of integration are fixed by the normalization conditions

$$a^{[k]}|_{u=0} = 0, f^{[k]}|_{u=0} = 0, k \geq 1. \quad (30)$$

Note that the parameters β and γ play an essential role: different choices of these constants lead to different associated integrable hierarchies. Using the recursion relations derived above, one can compute the first nontrivial coefficients explicitly. In particular, we obtain

$$\begin{cases} a^{[1]} = -\frac{1}{\alpha}[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 - \beta u_4)u_2], \\ f^{[1]} = \frac{1}{\alpha}[(\gamma u_3 - \beta u_4)u_1 - (\beta u_3 + \gamma u_4)u_2], \end{cases}$$

$$\begin{cases} b^{[1]} = \frac{1}{\alpha}\{\beta u_{1,x} + \gamma u_{2,x} - \frac{2}{\alpha}[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 - \beta u_4)u_2]u_1 \\ - \frac{2}{\alpha}[(\gamma u_3 - \beta u_4)u_1 - (\beta u_3 + \gamma u_4)u_2]u_2\}, \\ c^{[1]} = \frac{1}{\alpha}\{-\beta u_{3,x} - \gamma u_{4,x} - \frac{2}{\alpha}[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 - \beta u_4)u_2]u_3 \\ - \frac{2}{\alpha}[(\gamma u_3 - \beta u_4)u_1 - (\beta u_3 + \gamma u_4)u_2]u_4\}, \end{cases}$$

and

$$\begin{cases} e^{[1]} = \frac{1}{\alpha}\{-\gamma u_{1,x} + \beta u_{2,x} + \frac{2}{\alpha}[(\gamma u_3 - \beta u_4)u_1 - (\beta u_3 + \gamma u_4)u_2]u_1 \\ - \frac{2}{\alpha}[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 - \beta u_4)u_2]u_2\}, \\ g^{[1]} = \frac{1}{\alpha}\{\gamma u_{3,x} - \beta u_{4,x} + \frac{2}{\alpha}[(\gamma u_3 - \beta u_4)u_1 - (\beta u_3 + \gamma u_4)u_2]u_3 \\ - \frac{2}{\alpha}[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 - \beta u_4)u_2]u_4\}. \end{cases}$$

These recursion relations enable us to introduce the temporal matrix eigenvalue problems:

$$\varphi_{t_l} = \mathcal{N}^{[l]}\varphi = \mathcal{N}^{[l]}(u, \lambda)\varphi, \mathcal{N}^{[l]} = \lambda(\lambda^{2l+1}Y)_+, l \geq 0, \quad (31)$$

where the subscript $+$ denotes the polynomial part of λ . The compatibility conditions between the spatial and temporal matrix eigenvalue problems (18) and (31) are exactly the zero-curvature equations given in (7). As a consequence, these zero-curvature equations generate a hierarchy of integrable models with four potentials,

$$u_{t_l} = Z^{[l]} = Z^{[l]}(u) = (b_x^{[l]}, e_x^{[l]}, c_x^{[l]}, g_x^{[l]})^T, l \geq 0, \quad (32)$$

or, equivalently,

$$u_{1,t_l} = b_x^{[l]}, u_{2,t_l} = e_x^{[l]}, u_{3,t_l} = c_x^{[l]}, u_{4,t_l} = g_x^{[l]}, l \geq 0. \quad (33)$$

The first nontrivial nonlinear member of this hierarchy corresponds to $l = 1$ and yields a system of combined integrable derivative nonlinear Schrödinger equations:

$$\left\{ \begin{array}{l} u_{1,t_1} = \frac{1}{\alpha}(\beta u_{1,xx} + \gamma u_{2,xx}) - \frac{2}{\alpha^2} \{[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 - \beta u_4)u_2]u_1\}_x \\ \quad - \frac{2}{\alpha^2} \{[(\gamma u_3 - \beta u_4)u_1 - (\beta u_3 + \gamma u_4)u_2]u_2\}_x, \\ u_{2,t_1} = -\frac{1}{\alpha}(\gamma u_{1,xx} - \beta u_{2,xx}) + \frac{2}{\alpha^2} \{[(\gamma u_3 - \beta u_4)u_1 - (\beta u_3 + \gamma u_4)u_2]u_1\}_x \\ \quad - \frac{2}{\alpha^2} \{[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 - \beta u_4)u_2]u_2\}_x, \\ u_{3,t_1} = -\frac{1}{\alpha}(\beta u_{3,xx} + \gamma u_{4,xx}) - \frac{2}{\alpha^2} \{[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 - \beta u_4)u_2]u_3\}_x \\ \quad - \frac{2}{\alpha^2} \{[(\gamma u_3 - \beta u_4)u_1 - (\beta u_3 + \gamma u_4)u_2]u_4\}_x, \\ u_{4,t_1} = \frac{1}{\alpha}(\gamma u_{3,xx} - \beta u_{4,xx}) + \frac{2}{\alpha^2} \{[(\gamma u_3 - \beta u_4)u_1 - (\beta u_3 + \gamma u_4)u_2]u_3\}_x \\ \quad - \frac{2}{\alpha^2} \{[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 - \beta u_4)u_2]u_4\}_x. \end{array} \right. \quad (34)$$

This system defines a combined coupled integrable model with four potentials, enlarging the class of coupled integrable derivative nonlinear Schrödinger equations (see, e.g., [21–26]). A distinctive feature is that each equation involves a linear combination of two second-order dispersion terms, motivating the term combined models.

Two special reductions of the hierarchy, corresponding to $\beta = 0$ or $\gamma = 0$, are of particular interest. These choices lead to reduced hierarchies of uncombined, yet still coupled, integrable models.

For instance, setting $\alpha = \beta = 1$ and $\gamma = 0$ in (34) yields the coupled integrable derivative nonlinear Schrödinger system

$$\left\{ \begin{array}{l} u_{1,t_1} = u_{1,xx} - 2[(u_1 u_3 - u_2 u_4)u_1 - (u_1 u_4 + u_2 u_3)u_2]_x, \\ u_{2,t_1} = u_{2,xx} - 2[(u_1 u_4 + u_2 u_3)u_1 + (u_1 u_3 - u_2 u_4)u_2]_x, \\ u_{3,t_1} = -u_{3,xx} - 2[(u_1 u_3 - u_2 u_4)u_3 - (u_1 u_4 + u_2 u_3)u_4]_x, \\ u_{4,t_1} = -u_{4,xx} - 2[(u_1 u_4 + u_2 u_3)u_3 + (u_1 u_3 - u_2 u_4)u_4]_x. \end{array} \right. \quad (35)$$

Alternatively, choosing $\alpha = \gamma = 1$ and $\beta = 0$ produces another coupled integrable derivative nonlinear Schrödinger model,

$$\left\{ \begin{array}{l} u_{1,t_1} = u_{2,xx} - 2[(u_1 u_4 + u_2 u_3)u_1 + (u_1 u_3 - u_2 u_4)u_2]_x, \\ u_{2,t_1} = -u_{1,xx} + 2[(u_1 u_3 - u_2 u_4)u_1 - (u_1 u_4 + u_2 u_3)u_2]_x, \\ u_{3,t_1} = -u_{4,xx} - 2[(u_1 u_4 + u_2 u_3)u_3 + (u_1 u_3 - u_2 u_4)u_4]_x, \\ u_{4,t_1} = u_{3,xx} + 2[(u_1 u_3 - u_2 u_4)u_3 - (u_1 u_4 + u_2 u_3)u_4]_x. \end{array} \right. \quad (36)$$

Moreover, Equation (35) becomes Equation (36) after the replacement

$$(u_{1,t_1}, u_{2,t_1}, u_{3,t_1}, u_{4,t_1}) \rightarrow (-u_{2,t_1}, u_{1,t_1}, -u_{4,t_1}, u_{3,t_1}),$$

which corresponds to a permutation combined with sign changes of the time derivatives.

3. Bi-Hamiltonian Formulation

The spatial matrix eigenvalue problem (18) provides the foundation for a Hamiltonian formulation of the soliton hierarchy (33). In particular, the Liouville integrability of the hierarchy can be established by means of the trace identity (10).

Using the Laurent-series representation of the stationary zero-curvature solution Y given in (19), one readily computes

$$\begin{cases} \operatorname{tr}(Y \frac{\partial \mathcal{M}}{\partial \lambda}) = 2(2\alpha\lambda a + bu_3 + cu_1 - eu_4 - gu_2), \\ \operatorname{tr}(Y \frac{\partial \mathcal{M}}{\partial u}) = 2(\lambda c, -\lambda g, \lambda b, -\lambda e)^T. \end{cases} \quad (37)$$

An application of the trace identity yields

$$\begin{aligned} & \frac{\delta}{\delta u} \int \lambda^{-2k-1} (2\alpha a^{[k+1]} + u_3 b^{[k]} + u_1 c^{[k]} - u_4 e^{[k]} - u_2 g^{[k]}) dx \\ &= \lambda^{-\kappa} \frac{\partial}{\partial \lambda} \lambda^{\kappa-2k} (c^{[k]}, -g^{[k]}, b^{[k]}, -e^{[k]})^T, \quad k \geq 0. \end{aligned} \quad (38)$$

Using the expressions for $a^{[1]}, b^{[0]}, c^{[0]}, e^{[0]}$, and $g^{[0]}$, we have

$$2\alpha a^{[1]} + u_3 b^{[0]} + u_1 c^{[0]} - u_4 e^{[0]} - u_2 g^{[0]} = 0.$$

Hence, the above identity implies that $\kappa c^{[0]} = \kappa b^{[0]} = \kappa e^{[0]} = \kappa g^{[0]} = 0$, which in turn yields $\kappa = 0$. Consequently, we obtain

$$\frac{\delta}{\delta u} \mathcal{H}^{[k]} = (c^{[k]}, -g^{[k]}, b^{[k]}, -e^{[k]})^T, \quad k \geq 0, \quad (39)$$

where the Hamiltonian functionals are defined by

$$\begin{cases} \mathcal{H}^{[0]} = \int [\beta(u_1 u_3 - u_2 u_4) + \gamma(u_1 u_4 + u_2 u_3)] dx, \\ \mathcal{H}^{[k]} = - \int \frac{1}{2k} (2\alpha a^{[k+1]} + u_3 b^{[k]} + u_1 c^{[k]} - u_4 e^{[k]} - u_2 g^{[k]}) dx, \quad k \geq 1. \end{cases} \quad (40)$$

As a result, the hierarchy (33) admits the Hamiltonian formulation

$$u_{t_l} = Z^{[l]} = J_1 \frac{\delta \mathcal{H}^{[l]}}{\delta u}, \quad l \geq 0, \quad (41)$$

where the Hamiltonian operator J_1 is given by

$$J_1 = \left[\begin{array}{cc|cc} 0 & & \partial & 0 \\ & & 0 & -\partial \\ \hline \partial & 0 & & \\ 0 & -\partial & & 0 \end{array} \right]. \quad (42)$$

Therefore, for each flow in the hierarchy, there exists a one-to-one correspondence

$$Z = J_1 \frac{\delta \mathcal{H}}{\delta u}$$

between a symmetry Z and a conserved functional $\mathcal{H}$.

Moreover, the vector fields $Z^{[k]}$ satisfy the characteristic commutativity property

$$[[Z^{[k_1]}, Z^{[k_2]}]] = Z^{[k_1]'}(u)[Z^{[k_2]}] - Z^{[k_2]'}(u)[Z^{[k_1]}] = 0, \quad k_1, k_2 \geq 0, \quad (43)$$

which follows from the algebraic relations among the temporal Lax operators,

$$[[\mathcal{N}^{[k_1]}, \mathcal{N}^{[k_2]}]] = \mathcal{N}^{[k_1]'}(u)[Z^{[k_2]}] - \mathcal{N}^{[k_2]'}(u)[Z^{[k_1]}] + [\mathcal{N}^{[k_1]}, \mathcal{N}^{[k_2]}] = 0, \quad k_1, k_2 \geq 0. \quad (44)$$

These identities can be verified directly by examining the relations between the isospectral zero-curvature equations; see, for example, Ref. [27] for further details.

On the other hand, starting from the recursion relation

$$Z^{[l+1]} = \Phi Z^{[l]},$$

we can explicitly construct a hereditary recursion operator $\Phi = (\Phi_{jk})_{4 \times 4}$ [28]. Its entries are given by

$$\begin{cases} \Phi_{11} = \frac{1}{\alpha} \partial_x - \frac{2}{\alpha^2} (\partial u_1 \partial^{-1} u_3 - \partial u_2 \partial^{-1} u_4), & \Phi_{12} = \frac{2}{\alpha^2} (\partial u_1 \partial^{-1} u_4 + \partial u_2 \partial^{-1} u_3), \\ \Phi_{13} = -\frac{2}{\alpha^2} (\partial u_1 \partial^{-1} u_1 - \partial u_2 \partial^{-1} u_2), & \Phi_{14} = \frac{2}{\alpha^2} (\partial u_1 \partial^{-1} u_2 + \partial u_2 \partial^{-1} u_1); \end{cases} \quad (45)$$

$$\begin{cases} \Phi_{21} = -\frac{2}{\alpha^2} (\partial u_1 \partial^{-1} u_4 + \partial u_2 \partial^{-1} u_3), & \Phi_{22} = \frac{1}{\alpha} \partial_x - \frac{2}{\alpha^2} (\partial u_1 \partial^{-1} u_3 - \partial u_2 \partial^{-1} u_4), \\ \Phi_{23} = -\frac{2}{\alpha^2} (\partial u_1 \partial^{-1} u_2 + \partial u_2 \partial^{-1} u_1), & \Phi_{24} = -\frac{2}{\alpha^2} (\partial u_1 \partial^{-1} u_1 - \partial u_2 \partial^{-1} u_2); \end{cases} \quad (46)$$

$$\begin{cases} \Phi_{31} = -\frac{2}{\alpha^2} (\partial u_3 \partial^{-1} u_3 - \partial u_4 \partial^{-1} u_4), & \Phi_{32} = \frac{2}{\alpha^2} (\partial u_3 \partial^{-1} u_4 + \partial u_4 \partial^{-1} u_3), \\ \Phi_{33} = -\frac{1}{\alpha} \partial_x - \frac{2}{\alpha^2} (\partial u_3 \partial^{-1} u_1 - \partial u_4 \partial^{-1} u_2), & \Phi_{34} = \frac{2}{\alpha^2} (\partial u_3 \partial^{-1} u_2 + \partial u_4 \partial^{-1} u_1); \end{cases} \quad (47)$$

$$\begin{cases} \Phi_{41} = -\frac{2}{\alpha^2} (\partial u_3 \partial^{-1} u_4 + \partial u_4 \partial^{-1} u_3), & \Phi_{42} = -\frac{2}{\alpha^2} (\partial u_3 \partial^{-1} u_3 - \partial u_4 \partial^{-1} u_4), \\ \Phi_{43} = -\frac{2}{\alpha^2} (\partial u_3 \partial^{-1} u_2 + \partial u_4 \partial^{-1} u_1), & \Phi_{44} = -\frac{1}{\alpha} \partial_x - \frac{2}{\alpha^2} (\partial u_3 \partial^{-1} u_1 - \partial u_4 \partial^{-1} u_2). \end{cases} \quad (48)$$

A direct but lengthy calculation (see, e.g., [29] for an illustrative example of such proofs for Hamiltonian operators) can be carried out to show that the Hamiltonian operators J_1 and $J_2 = \Phi J_1$ form a Hamiltonian pair; that is, any linear combination of J_1 and J_2 is again a Hamiltonian operator. Note that the recursion operator Φ and the second Hamiltonian operator J_2 are well defined on the spaces spanned by $Z^{[l]}$, $l \geq 0$, and $\frac{\delta \mathcal{H}^{[l]}}{\delta u}$, $l \geq 0$, respectively. Consequently, the hierarchy (33) admits a bi-Hamiltonian formulation in the sense of Magri [30]:

$$u_{t_l} = Z^{[l]} = J_1 \frac{\delta \mathcal{H}^{[l]}}{\delta u} = J_2 \frac{\delta \mathcal{H}^{[l-1]}}{\delta u}, \quad l \geq 1. \quad (49)$$

As a result, the Hamiltonian functionals $\mathcal{H}^{[l]}$ commute with respect to both Poisson brackets associated with J_1 and J_2 [6]:

$$\{\mathcal{H}^{[k_1]}, \mathcal{H}^{[k_2]}\}_{J_1} = \int \left(\frac{\delta \mathcal{H}^{[k_1]}}{\delta u} \right)^T J_1 \frac{\delta \mathcal{H}^{[k_2]}}{\delta u} dx = 0, \quad k_1, k_2 \geq 0, \quad (50)$$

and

$$\{\mathcal{H}^{[k_1]}, \mathcal{H}^{[k_2]}\}_{J_2} = \int \left(\frac{\delta \mathcal{H}^{[k_1]}}{\delta u} \right)^T J_2 \frac{\delta \mathcal{H}^{[k_2]}}{\delta u} dx = 0, \quad k_1, k_2 \geq 0. \quad (51)$$

The bi-Hamiltonian structure further implies the hereditary property of the recursion operator Φ , which, therefore, serves as a common recursion operator for the entire hierarchy (33).

In conclusion, each model in the hierarchy (33) is Liouville integrable and possesses infinitely many commuting symmetries $\{Z^{[k]}\}_{k=0}^{\infty}$ and conserved Hamiltonian functionals $\{\mathcal{H}^{[k]}\}_{k=0}^{\infty}$. In particular, the coupled system (34) provides a concrete example of a non-linear combined Liouville-integrable Hamiltonian Kaup–Newell-type model with four components, enriching the existing class of such integrable systems (see, e.g., [31]).

4. Concluding Remarks

By analyzing possible integrable extensions of the standard Kaup–Newell matrix eigenvalue problem, we propose a specific 4×4 matrix eigenvalue problem and construct a hierarchy of real four-component Liouville-integrable models via the zero-curvature formulation. A key step is to construct a particular Laurent-series solution of the corresponding stationary zero-curvature equation with combined initial data. Using the trace identity, we further show that the resulting integrable hierarchy possesses bi-Hamiltonian

structures and a hereditary recursion operator. These results provide an integrable real four-component extension of the standard Kaup–Newell soliton hierarchy.

It is worth noting that the Kaup–Newell-type spectral matrix considered in (18) differs from those in [21,31] in two respects. First, the coefficient matrix of λ^2 in (18) possesses two repeated real eigenvalues, α_1 and α_2 , and involves two negative signs. In contrast, the corresponding coefficient matrix in [21] generally has four purely imaginary eigenvalues, $\pm\alpha_1 i$ and $\pm\alpha_2 i$. Second, the spectral matrix in [31] does not involve any negative signs. Moreover, the spectral matrices with four components in [22,26] are of AKNS type, and are therefore fundamentally different. It is also interesting to note that the Lax pairs in (18) and (31) could be derived from the 2×2 Kaup–Newell Lax pair through a collective transformation

$$\psi_1 = \phi_1 + i\phi_2, \psi_2 = \phi_3 + i\phi_4, q = u_1 - iu_2, r = u_3 - iu_4, t \rightarrow \frac{\beta - i\gamma}{\beta^2 + \gamma^2} t,$$

where $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ is the eigenfunction of the spectral problem (18), and $\psi = (\psi_1, \psi_2)^T$ is the eigenfunction of the 2×2 Kaup–Newell spectral problem involving the two potentials q and r . In particular, the four-component integrable system in (34) can be generated from the Kaup–Newell system

$$q_t = \frac{1}{\alpha} q_{xx} - \frac{2}{\alpha^2} (q^2 r)_x, r_t = -\frac{1}{\alpha} r_{xx} - \frac{2}{\alpha^2} (qr^2)_x.$$

More specifically, the reduced system (35) is obtained without any change of the time variable t , whereas the reduced system (36) is derived through the rotation transformation $t \rightarrow -it$. However, such a transformation does not necessarily preserve integrability properties. One illustrative example is the backward heat equation $u_t = -u_{xx}$. Its Cauchy problem on the real line is not well posed, although it is formally generated from the heat equation $u_t = u_{xx}$ through the time reversal $t \rightarrow -t$.

It is also of particular interest to understand what mathematical structures of soliton solutions may exist for the integrable models presented here. Various powerful methods are available for this purpose, including the Riemann–Hilbert technique [32], the Zakharov–Shabat dressing method [33], the Darboux transformation [34–36], and determinant-based approaches [37]. In addition to solitons, other coherent structures such as lumps, kinks, breathers, and rogue waves, as well as especially their interaction solutions, are also of significant interest (see, e.g., [38–44]). Often, these solutions can be obtained by applying suitable reductions to multi-soliton solutions.

Recently, nonlocal integrable reductions have attracted considerable attention. Such models arise from nonlocal symmetry reductions or similarity transformations of Lax pairs, and their soliton solutions deserve special study due to their novel mathematical structures and physical significance (see, e.g., [45]). Nonlocality leads to a rich variety of new linear and nonlinear phenomena in solutions.

Integrable models are of broad interest because they provide deep connections between mathematics and real-world phenomena. They are closely related to diverse areas of mathematics, including algebraic geometry, Lie theory, and the theory of special functions. The study of integrable models not only offers insight into universal dynamical behaviors of physical systems, but also underpins the understanding of complex wave phenomena in the mathematical, physical and engineering sciences (see, e.g., [46]).

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References

1. Ablowitz, M.J.; Segur, H. *Solitons and the Inverse Scattering Transform*; SIAM: Philadelphia, PA, USA, 1981. [\[CrossRef\]](#)
2. Lax, P.D. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.* **1968**, *21*, 467–490. [\[CrossRef\]](#)
3. Ablowitz, M.J.; Kaup, D.J.; Newell, A.C.; Segur, H. The inverse scattering transform–Fourier analysis for nonlinear problems. *Stud. Appl. Math.* **1974**, *53*, 249–315. [\[CrossRef\]](#)
4. Ma, W.X. Integrable couplings and matrix loop algebras. *AIP Conf. Proc.* **2013**, *1562*, 105–122. [\[CrossRef\]](#)
5. Drinfel’d, V.; Sokolov, V.V. Lie algebras and equations of Korteweg–de Vries type. *Sov. J. Math.* **1985**, *30*, 1975–2036. [\[CrossRef\]](#)
6. Tu, G.Z. On Liouville integrability of zero-curvature equations and the Yang hierarchy. *J. Phys. A Math. Gen.* **1989**, *22*, 2375–2392. [\[CrossRef\]](#)
7. Liu, C.S. How many first integrals imply integrability in infinite-dimensional Hamilton system. *Rep. Math. Phys.* **2011**, *67*, 109–123. [\[CrossRef\]](#)
8. Antonowicz, M.; Fordy, A.P. Coupled KdV equations with multi-Hamiltonian structures. *Phys. D* **1987**, *28*, 345–357. [\[CrossRef\]](#)
9. Xia, T.C.; Yu, F.J.; Zhang, Y. The multi-component coupled Burgers hierarchy of soliton equations and its multi-component integrable couplings system with two arbitrary functions. *Phys. A* **2004**, *343*, 238–246. [\[CrossRef\]](#)
10. Wang, H.F.; Zhang, Y.F. Application of Riemann–Hilbert method to an extended coupled nonlinear Schrödinger equations. *J. Comput. Appl. Math.* **2023**, *420*, 114812. [\[CrossRef\]](#)
11. Gerdjikov, V.S. Nonlinear evolution equations related to Kac–Moody algebras $A_r^{(1)}$: spectral aspects. *Turkish J. Math.* **2022**, *46*, 1828–1844. [\[CrossRef\]](#)
12. Ma, W.X. A matrix integrable Hamiltonian hierarchy and its two integrable reductions. *Int. J. Geom. Methods Mod. Phys.* **2024**, *21*, 2450293. [\[CrossRef\]](#)
13. Takhtajan, L.A. Integration of the continuous Heisenberg spin chain through the inverse scattering method. *Phys. Lett. A* **1977**, *64*, 235–237. [\[CrossRef\]](#)
14. Kaup, D.J.; Newell, A.C. An exact solution for a derivative nonlinear Schrödinger equation. *J. Math. Phys.* **1978**, *19*, 798–801. [\[CrossRef\]](#)
15. Smirnov, A.O.; Filimonova, E.G.; Matveev, V.B. The spectral curve method for the Kaup–Newell hierarchy. *IOP Conf. Ser. Mater. Sci. Eng.* **2020**, *919*, 052051. [\[CrossRef\]](#)
16. Wadati, M.; Konno, K.; Ichikawa, Y.H. New integrable nonlinear evolution equations. *J. Phys. Soc. Jpn.* **1979**, *47*, 1698–1700. [\[CrossRef\]](#)
17. Hu, X.B. A powerful approach to generate new integrable systems. *J. Phys. A Math. Gen.* **1994**, *27*, 2497–2514. [\[CrossRef\]](#)
18. Tsuchida, T.; Wadati, M. New integrable systems of derivative nonlinear Schrödinger equations with multiple components. *Phys. Lett. A* **1999**, *257*, 53–64. [\[CrossRef\]](#)
19. Zhou, R.G.; Yu, Z.L. Integrable reductions of the multi-component Kaup–Newell equations. *Phys. D* **2024**, *458*, 134011. [\[CrossRef\]](#)
20. Smirnov, A.O.; Frolov, E.A.; Dmitrieva, L.L. On a hierarchy of vector derivative nonlinear Schrödinger equations. *Symmetry* **2024**, *16*, 60. [\[CrossRef\]](#)
21. Ma, W.X. A combined derivative nonlinear Schrödinger soliton hierarchy. *Rep. Math. Phys.* **2024**, *93*, 313–325. [\[CrossRef\]](#)
22. Ma, W.X. A combined Liouville integrable hierarchy associated with a fourth-order matrix spectral problem. *Commun. Theoret. Phys.* **2024**, *76*, 075001. [\[CrossRef\]](#)
23. Zhang, Y.F. A few expanding integrable models, Hamiltonian structures and constrained flows. *Commun. Theor. Phys.* **2011**, *55*, 273–290. [\[CrossRef\]](#)
24. Zhaqilao. A generalized AKNS hierarchy, bi-Hamiltonian structure, and Darboux transformation. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 2319–2332. [\[CrossRef\]](#)
25. Ma, W.X. Novel Liouville integrable Hamiltonian models with six components and three signs. *Chin. J. Phys.* **2023**, *86*, 292–299. [\[CrossRef\]](#)
26. Ma, W.X. Four-component combined integrable equations possessing bi-Hamiltonian formulations. *Mod. Phys. Lett. B* **2024**, *38*, 2450319. [\[CrossRef\]](#)
27. Ma, W.X. The algebraic structure of zero-curvature representations and application to coupled KdV systems. *J. Phys. A Math. Gen.* **1993**, *26*, 2573–2582. [\[CrossRef\]](#)

28. Fuchssteiner, B.; Fokas, A.S. Symplectic structures, their Bäcklund transformations and hereditary symmetries. *Phys. D* **1981**, *4*, 47–66. [\[CrossRef\]](#)
29. Ma, W.X.; Zhou, R.G. A coupled AKNS–Kaup–Newell soliton hierarchy. *J. Math. Phys.* **1999**, *40*, 4419–4428. [\[CrossRef\]](#)
30. Magri, F. A simple model of the integrable Hamiltonian equation. *J. Math. Phys.* **1978**, *19*, 1156–1162. [\[CrossRef\]](#)
31. Ma, W.X. A combined Kaup–Newell type integrable hierarchy with four potentials and its bi-Hamiltonian formulation. *Rev. Math. Phys.* **2025**, *37*, 2450049. [\[CrossRef\]](#)
32. Novikov, S.P.; Manakov, S.V.; Pitaevskii, L.P.; Zakharov, V.E. *Theory of Solitons: the Inverse Scattering Method*; Consultants Bureau: New York, NY, USA, 1984.
33. Doktorov, E.V.; Leble, S.B. *A Dressing Method in Mathematical Physics*; Springer: Dordrecht, The Netherlands, 2007. [\[CrossRef\]](#)
34. Matveev, V.B.; Salle, M.A. *Darboux Transformations and Solitons*; Springer: Berlin/Heidelberg, Germany, 1991. [\[CrossRef\]](#)
35. Geng, X.G.; Li, R.M.; Xue, B. A vector general nonlinear Schrödinger equation with $(m + n)$ components. *J. Nonlinear Sci.* **2020**, *30*, 991–1013. [\[CrossRef\]](#)
36. Ye, R.S.; Zhang, Y. A vectorial Darboux transformation for the Fokas–Lenells system. *Chaos Solitons Fractals* **2023**, *169*, 113233. [\[CrossRef\]](#)
37. Aktosun, T.; Busse, T.; Demontis, F.; van der Mee, C. Symmetries for exact solutions to the nonlinear Schrödinger equation. *J. Phys. A Math. Theoret.* **2010**, *43*, 025202. [\[CrossRef\]](#)
38. Sulaiman, T.A.; Yusuf, A.; Abdeljabbar, A.; Alquran, M. Dynamics of lump collision phenomena to the (3+1)-dimensional nonlinear evolution equation. *J. Geom. Phys.* **2021**, *169*, 104347. [\[CrossRef\]](#)
39. Ma, W.X. Lump waves and their dynamics in a generalized Kadomtsev–Petviashvili-like model. *Mod. Phys. Lett. A* **2026**, *41*, 2550215. [\[CrossRef\]](#)
40. Manukure, S.; Chowdhury, A.; Zhou, Y. Complexiton solutions to the asymmetric Nizhnik–Novikov–Veselov equation. *Int. J. Mod. Phys. B* **2019**, *33*, 1950098. [\[CrossRef\]](#)
41. Sabi’u, J.; Rezazadeh, H.; Cimpoiasu, R.; Constantinescu, R. Traveling wave solutions of the generalized Rosenau–Kawahara–RLW equation via the sine–cosine method and a generalized auxiliary equation method. *Int. J. Nonlinear Sci. Numer. Simul.* **2022**, *23*, 539–551. [\[CrossRef\]](#)
42. Ma, W.X. Dispersion-governed lump waves in a generalized Calogero–Bogoyavlenskii–Schiff-like model with spatially symmetric nonlinearity. *Axioms* **2025**, *14*, 869. [\[CrossRef\]](#)
43. Yang, S.X.; Wang, Y.F.; Zhang, X. Conservation laws, Darboux transformation and localized waves for the N -coupled nonautonomous Gross–Pitaevskii equations in the Bose–Einstein condensates. *Chaos Solitons Fractals* **2023**, *169*, 113272. [\[CrossRef\]](#)
44. Li, Y.; Yao, R.X.; Lou, S.Y. Novel nonlinear wave transitions and interactions for (2+1)-dimensional generalized fifth-order KdV equation. *Commun. Theor. Phys.* **2024**, *76*, 125003. [\[CrossRef\]](#)
45. Ma, W.X.; Huang, Y.H.; Wang, F.D.; Zhang, Y.; Ding, L.Y. Darboux transformation of vector nonlocal reverse-space nonlinear Schrödinger equations. *Int. J. Geom. Methods Mod. Phys.* **2024**, *21*, 2450182. [\[CrossRef\]](#)
46. Fakhte, S.; Taskhiri, M.M. High-gain higher-order mode THz dielectric antenna with graphene-based magnetic control for polarization switching. *Opt. Quantum Electron.* **2026**, *58*, 251. [\[CrossRef\]](#)

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