



Short communication

Comment on the 3+1 dimensional Kadomtsev–Petviashvili equations

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ABSTRACT

We comment on traveling wave solutions and rational solutions to the 3+1 dimensional Kadomtsev–Petviashvili (KP) equations: $(u_t + 6uu_x + u_{xxx})_x \pm 3u_{yy} \pm 3u_{zz} = 0$. We also show that both of the 3+1 dimensional KP equations do not possess the three-soliton solution. This suggests that none of the 3+1 dimensional KP equations should be integrable, and partially explains why they do not pass the Painlevé test. As by-products, the one-soliton and two-soliton solutions and four classes of specific three-soliton solutions are explicitly presented.

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The 3+1 dimensional Kadomtsev–Petviashvili (KP) equations read

$$(u_t + 6uu_x + u_{xxx})_x \pm 3u_{yy} \pm 3u_{zz} = 0, \quad (1)$$

which describe three-dimensional solitons in weakly dispersive media [1], particularly in fluid dynamics and plasma physics [2,3]. Like the 2 + 1 dimensional KP equations, the 3+1 dimensional KP equations with the negative sign “−” and the positive sign “+” are called the 3+1 dimensional KP-I and KP-II equations, respectively. Several classes of exact traveling wave solutions to the 3+1 dimensional KP-II equation were presented by various authors (see, e.g., [4–6]). Very recently, traveling wave solutions and rational solutions were discussed and generated in [7,8] for the 3+1 dimensional KP-I equation, based on the homogeneous balance method and under help of a Riccati equation.

In this note, on one hand, we would like to explain that more general traveling wave and rational solutions, including the ones presented in [4–8], can be constructed under transformations of dependent and independent variables. Three exact and explicit solutions to a Riccati equation will help in generating those solutions. This also contributes to identifying and correcting one common error in finding exact solutions to nonlinear wave equations: not using sufficiently general classes of solutions to ordinary differential equations [9]. On the other hand, using the Hirota bilinear method, we would like to analyze the existence condition of the three-soliton solution, and present the one-soliton and two-soliton solutions and four classes of specific three-soliton solutions to both of the 3+1 dimensional KP equations.

It is known that many direct methods are nowadays available for constructing exact traveling wave solutions to nonlinear differential equations. The Riccati equation

$$\phi_\xi = \alpha\phi^2 + \beta \quad (\alpha \neq 0) \quad (2)$$

plays a crucial role in manipulating nonlinear equations to get exact solutions by the homogeneous balance method. This equation has the following three exact solutions [10]:

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$$\phi = -\frac{1}{\alpha\zeta + \zeta_0}, \quad \zeta_0 = \text{const.}, \quad (3)$$

when $\beta = 0$;

$$\phi = -\frac{2\varepsilon\sqrt{-\alpha\beta}}{\alpha} \frac{1}{\zeta_0 \exp(-2\varepsilon\sqrt{-\alpha\beta}\zeta) + 1} + \frac{\varepsilon\sqrt{-\alpha\beta}}{\alpha} = \begin{cases} \frac{\varepsilon\sqrt{-\alpha\beta}}{\alpha}, & \text{for } \zeta_0 = 0, \\ -\frac{\sqrt{-\alpha\beta}}{\alpha} \tanh(\sqrt{-\alpha\beta}\zeta - \frac{\varepsilon \ln \zeta_0}{2}), & \text{for } \zeta_0 > 0, (\varepsilon = \pm 1) \\ -\frac{\sqrt{-\alpha\beta}}{\alpha} \coth(\sqrt{-\alpha\beta}\zeta - \frac{\varepsilon \ln(-\zeta_0)}{2}), & \text{for } \zeta_0 < 0, \end{cases} \quad (4)$$

when $\alpha\beta < 0$; and

$$\phi = \frac{\sqrt{\alpha\beta}}{\alpha} \tan(\sqrt{\alpha\beta}\zeta + \zeta_0), \quad \zeta_0 = \text{const.}, \quad (5)$$

when $\alpha\beta > 0$. These help generate various traveling wave solutions, including periodic traveling wave solutions, in elementary functions, and the traveling wave solutions obtained in [7,8] corresponds to a sub-case of the solution (4).

It is difficult and even impossible in most cases to determine the solution set of a nonlinear differential equation, and so, the concept of general solutions is introduced only in the field of linear differential equations. Nonlinear differential equations possess diverse solutions, indeed [11]. For example, there exist soliton solutions, positon solutions and complexiton solutions to many integrable equations (see, e.g., [12] for the KdV case). For a generalized differential equation

$$(u_t + K(u, u_x, \dots))_x + au_{yy} + bu_{zz} = 0, \quad a, b = \text{const.},$$

a kind of traveling wave solutions

$$u = f\left(kx + ly + mz + \omega t - \frac{al^2 + bm^2}{k}t\right), \quad (6)$$

where l and m are arbitrary constants, can be generated [13] from a known traveling wave solution $u = f(kx + \omega t)$ ($k \neq 0$) to the original differential equation

$$u_t + K(u, u_x, \dots) = 0,$$

which is assumed to be invariant under the translation of independent variables. The traveling wave solutions presented for the 3+1 dimensional KP-I equation in [7] and for the 3+1 dimensional KP-II equation in [4–6] are among such a class of exact solutions. A general practical algorithm for generating traveling wave solutions was given in [14], which unifies many existing methods such as the tanh-function method, the homogeneous balance method and the exp-function method.

Upon taking the transformation $r = x + cz + dt$, where c and d are constants, the 3+1 dimensional KP-II equation by (1) can be reduced to the good Boussinesq equation

$$(6uu_r + u_{rrr})_r + (d + 3c^2)u_{rr} + 3u_{yy} = 0.$$

This equation can be transformed into the standard good Boussinesq equation

$$(v^2)_r + v_{rrr} + v_{yy} = 0, \quad (7)$$

under a transformation

$$u(r, y) = -\frac{d + 3c^2}{6} + \frac{1}{3}v\left(r, \frac{\sqrt{3}}{3}y\right).$$

A Wronskian formulation to get rational solutions to the standard good Boussinesq equation was given in [15], and it can be used to present rational solutions to the 3+1 dimensional KP-II equation defined by (1). An interesting open question for us is: Is there any other rational solution to (7) not in the Wronskian form presented in [15]?

Let us now analyze the existence condition of the three-soliton solution. It is direct to see that under the transformation of dependent variables

$$u = 2(\ln f)_{xx}, \quad (8)$$

the 3+1 dimensional KP equations defined by (1) can be cast into

$$P_{\pm}(D_x, D_y, D_z, D_t)f \cdot f = 0, \quad (9)$$

where D_x, D_y, D_z and D_t are Hirota's differential operators [16] and two polynomials $P_{\pm}$ are defined by

$$P_{\pm}(x, y, z, t) = x^4 + xt \pm 3y^2 \pm 3z^2. \quad (10)$$

Those bilinear equations in (9) exactly give

$$f_{xxxx}f - 4f_x f_{xxx} + 3f_{xx}^2 + ff_{xt} - f_x f_t \pm 3(ff_{yy} - f_y^2 + ff_{zz} - f_z^2) = 0,$$

for which a sufficient condition is

$$\begin{cases} f_{xxxx} + f_{xt} \pm 3f_{yy} \pm 3f_{zz} = 0, \\ -4f_x f_{xxx} + 3f_{xx}^2 - f_x f_t \mp 3f_y^2 \mp 3f_z^2 = 0. \end{cases} \quad (11)$$

In attempting to find the three-soliton solution, we always introduce three wave variables:

$$\eta_i = k_i x + l_i y + m_i z + \omega_i t + \eta_{i,0}, \quad 1 \leq i \leq 3, \quad (12)$$

where $k_i, l_i, m_i, \omega_i, 1 \leq i \leq 3$, are constants to be determined, and $\eta_{i,0}, 1 \leq i \leq 3$, are arbitrary constant shifts; and define a set of prominent constants:

$$A_{ij} = -\frac{P_{\pm}(k_i - k_j, l_i - l_j, m_i - m_j, \omega_i - \omega_j)}{P_{\pm}(k_i + k_j, l_i + l_j, m_i + m_j, \omega_i + \omega_j)}, \quad 1 \leq i, j \leq 3. \quad (13)$$

Following Hirota's bilinear theory [16], we know that under the dispersion relations

$$k_i^4 + k_i \omega_i \pm 3l_i^2 \pm 3m_i^2 = 0, \quad 1 \leq i \leq 3, \quad (14)$$

the 3+1 dimensional KP equations by (1) have the one-soliton and two-soliton solutions:

$$f = 1 + \varepsilon e^{\eta_1}, f = 1 + \varepsilon(e^{\eta_1} + e^{\eta_2}) + \varepsilon^2 A_{12} e^{\eta_1 + \eta_2}, \quad (15)$$

where ε is an arbitrary perturbation parameter. Moreover, under (14), the 3+1 dimensional KP equations by (1) have the three-soliton solution

$$f = 1 + \varepsilon(e^{\eta_1} + e^{\eta_2} + e^{\eta_3}) + \varepsilon^2(A_{12} e^{\eta_1 + \eta_2} + A_{13} e^{\eta_1 + \eta_3} + A_{23} e^{\eta_2 + \eta_3}) + \varepsilon^3 A_{123} e^{\eta_1 + \eta_2 + \eta_3}, \quad (16)$$

where $A_{123} = A_{12}A_{13}A_{23}$ and ε is an arbitrary perturbation parameter, if and only if the corresponding three-soliton conditions [17,18]:

$$\sum_{\sigma_1, \sigma_2, \sigma_3 = \pm 1} P_{\pm}(\sigma_1 \bar{p}_1 + \sigma_2 \bar{p}_2 + \sigma_3 \bar{p}_3) P_{\pm}(\sigma_1 \bar{p}_1 - \sigma_2 \bar{p}_2) P_{\pm}(\sigma_2 \bar{p}_2 - \sigma_3 \bar{p}_3) P_{\pm}(\sigma_1 \bar{p}_1 - \sigma_3 \bar{p}_3) = 0 \quad (17)$$

are satisfied, where $\bar{p}_i = (k_i, l_i, m_i, \omega_i), 1 \leq i \leq 3$. More general exact solutions than the two-soliton solution can be computed by adopting an ansatz:

$$f = 1 + g(\eta_1) e^{\eta_2},$$

where g has many choices determined by (11) [19].

The existence of the three-soliton solutions usually implies the integrability [20] of the considered equations. Noting that the even property of the polynomials $P_{\pm}$, a direct computation can show that

$$\begin{aligned} & \sum_{\sigma_1, \sigma_2, \sigma_3 = \pm 1} P_{\pm}(\sigma_1 \bar{p}_1 + \sigma_2 \bar{p}_2 + \sigma_3 \bar{p}_3) P_{\pm}(\sigma_1 \bar{p}_1 - \sigma_2 \bar{p}_2) P_{\pm}(\sigma_2 \bar{p}_2 - \sigma_3 \bar{p}_3) P_{\pm}(\sigma_1 \bar{p}_1 - \sigma_3 \bar{p}_3) \\ &= 2 \sum_{(\sigma_1, \sigma_2, \sigma_3) \in S} P_{\pm}(\sigma_1 \bar{p}_1 + \sigma_2 \bar{p}_2 + \sigma_3 \bar{p}_3) P_{\pm}(\sigma_1 \bar{p}_1 - \sigma_2 \bar{p}_2) P_{\pm}(\sigma_2 \bar{p}_2 - \sigma_3 \bar{p}_3) P_{\pm}(\sigma_1 \bar{p}_1 - \sigma_3 \bar{p}_3) \\ &= 10368 k_1^2 k_2^2 k_3^2 \begin{vmatrix} k_1 & l_1 & m_1 \\ k_2 & l_2 & m_2 \\ k_3 & l_3 & m_3 \end{vmatrix}^2, \end{aligned} \quad (18)$$

where $S = \{(1, 1, 1), (1, 1, -1), (1, -1, 1), (-1, 1, 1)\}$. Since the last term cannot equal to zero automatically, the 3+1 dimensional bilinear KP equations by (9) do not have the three-soliton solution. This is in agreement with that the equations by (9) do not pass the Painlevé test [21]. For more generalized KP equations, their existence and non-existence cases for localized solitary waves were classified according to the sign of the transverse dispersion coefficients and to the nonlinearity in [22]. The non-existence of the three-soliton solution also suggests that none of the 3+1 dimensional KP equations by (1) should be integrable. Nevertheless, the formula in (18) allows us to conclude that under (14), the function f defined by (16) gives rise to an exact three-wave solution provide that one of the following two conditions holds:

- (a) one of three wave numbers $k_i, 1 \leq i \leq 3$, is zero;
- (b) any two of three vectors $(k_1, k_2, k_3), (l_1, l_2, l_3)$ and (m_1, m_2, m_3) are parallel.

The condition (a) presents one class of specific three-wave solutions, due to a cyclic characteristic, and the condition (b) presents three classes of specific three-wave solutions. The last three classes actually correspond to the three-soliton solutions of three dimensional reductions of the 3+1 dimensional KP equations.

To summarize, we have discussed traveling wave solutions, including periodic traveling wave solutions, and rational solutions to both of the 3+1 dimensional KP equations defined by (1), using transformations of independent and dependent variables and three exact solutions to the Riccati Eq. (2). We have also showed that both of the 3+1 dimensional KP equations by (1) do not pass the three-soliton solution test. This suggests that none of the 3+1 dimensional KP equations by (1) should be integrable, and partially explains why they do not possess the Painlevé property. The one-soliton and two-soliton solutions and four classes of specific three-soliton solutions are presented as by-products.

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