

THE ALGEBRAIC STRUCTURE RELATED TO L-A-B TRIAD REPRESENTATIONS OF INTEGRABLE SYSTEMS*

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Evolution equations integrable by the inverse scattering transform are the compatibility condition of two linear eigenvalue problems

$$\begin{cases} L\psi = \lambda\psi, \lambda \text{ spectral parameter,} \\ \psi_t = A\psi. \end{cases}$$

Among these are the well-known KdV equation, AKNS equation in 1+1 dimensions and KP equation, Davey-Stewartson equation in 1+2 dimensions^[1,2]. Manakov^[3] has presented the L-A-B triad representation of the compatibility condition

$$\frac{\partial L}{\partial t} = [A, L] + BL, \quad (1)$$

which is more appropriate for the two-dimensional case. It has been demonstrated that many integrable systems possess this kind of L-A-B triad representations^[3-8]. An illustrative example is the following integrable system^[7]

$$\begin{cases} r_t = \Delta r + \alpha(r^2)_\xi - \beta((\partial_\xi^{-1} r_\eta)^2)_\xi + 2\alpha\partial_\eta^{-1} s_{\xi\xi} - 2\beta s_\eta, \\ s_t = -\Delta s + 2\alpha(rs)_\xi - 2\beta(s\partial_\xi^{-1} r_\eta)_\eta, \end{cases} \quad (2)$$

where $\partial_\eta = \partial_x + \partial_y$, $\partial_\xi = \partial_x - \partial_y$, $\Delta = -\alpha\partial_\xi^2 + \beta\partial_\eta^2$, $f_\xi = \frac{\partial f}{\partial \xi}$, $f_\eta = \frac{\partial f}{\partial \eta}$, α and β are arbitrary constants. Three corresponding operators of its L-A-B triad representation are as follows:

$$\begin{cases} L = \partial_\eta \partial_\xi + r\partial_\eta + s, \\ A = -\alpha\partial_\xi^2 - \beta\partial_\eta^2 - 2\beta(\partial_\xi^{-1} r_\eta)\partial_\eta - 2\alpha\partial_\eta^{-1} s_\xi, \\ B = 2\alpha r_\xi - 2\beta\partial_\xi^{-1} r_{\eta\eta} = -2\partial_\xi^{-1} \Delta r. \end{cases} \quad (3)$$

When $\alpha=1$ and $\beta=0$, the integrable system (2) is reduced to the two-dimensional dispersive longwave equations^[8]

$$r_t = -r_{\xi\xi} + (r^2)_\xi + 2\partial_\eta^{-1} s_{\xi\xi}, \quad s_t = s_{\xi\xi} + 2(rs)_\xi. \quad (4)$$

In this report, we shall construct the algebraic structure related to general

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$((p+1)$ -dimensional) L - A - B triad representations displayed in (1). A similar algebraic structure corresponding to the Lax representations has been discussed in Ref. [9].

Let $x \in \mathbb{R}^p$, $u = u(x, t) \in S^q(\mathbb{R}^p, \mathbb{R})^{[9]}$. By $\mathcal{B}$ we denote all complex (or real) functions $P[u] = P(x, t, u)$ which are C^∞ -differentiable with respect to x, t and C^∞ -Gateaux differentiable with respect to $u = u(x)^{[9]}$, and set $\mathcal{B}^r = \{(P_1, \dots, P_r)^T | P_i \in \mathcal{B}\}$. By $\mathcal{V}^r$ we denote all linear operators $\Phi = \Phi(x, t, u): \mathcal{B}^r \rightarrow \mathcal{B}^r$ which are C^∞ -differentiable with respect to x, t and C^∞ -Gateaux differentiable with respect to $u = u(x)$, and by $\mathcal{V}_0^r$ all matrix differential operators $L = L_{(x,t,u)}: \mathcal{B}^r \rightarrow \mathcal{B}^r$ with the following form^[9]

$$L = (L_{ij})_{r \times r}, L_{ij} = \sum_{|\alpha| \leq \alpha(i,j)} p_{\alpha}^{ij}[u] D^{\alpha}, p_{\alpha}^{ij}[u] \in \mathcal{B}. \quad (5)$$

For $\Phi \in \mathcal{V}^r$, we use Φ' to stand for the Gateaux derivative operator of Φ , namely

$$\Phi'[X] = \left. \frac{\partial}{\partial \varepsilon} \Phi(u + \varepsilon X) \right|_{\varepsilon=0}, X \in \mathcal{B}^q. \quad (6)$$

In this report, we always assume that $L = L(x, u) \in \mathcal{V}_0^r$ and that $L': \mathcal{B}^q \rightarrow \mathcal{V}_0^r$ is an injective mapping. Evidently, if for $X \in \mathcal{B}^q$ there exists a pair of operators $A, B \in \mathcal{V}^r$ such that

$$[A, L] = L'[X] - BL, \quad (7)$$

then the evolution equation $u_t = X$ possesses an L - A - B triad representation (1).

Definition 1. Let $A, B \in \mathcal{V}^r$. If there exists $X \in \mathcal{B}^q$ such that (7) holds, then (A, B) is called a Manakov's pair of operators and X called an eigenvector field corresponding to the Manakov's pair (A, B) . Moreover, we denote by $\mathcal{M}_0$ all Manakov's pairs, by $E(\mathcal{M}_0)$ all eigenvector fields, and by $\mathcal{P}_0$ all triples (A, B, X) satisfying (7).

It is easy to see that every Manakov's pair has only one eigenvector field. Therefore for the linear space $\mathcal{M}_0$, we can construct the following multiplication operation.

Definition 2. Let Manakov's pairs $(A, B), (\bar{A}, \bar{B}) \in \mathcal{M}_0$ have eigenvector fields $X, \bar{X} \in E(\mathcal{M}_0)$, respectively. We define the product of (A, B) and $(\bar{A}, \bar{B})$ as

$$[(A, B), (\bar{A}, \bar{B})] = ([A, \bar{A}], [B, \bar{B}]), \quad (8)$$

where

$$[A, \bar{A}] = A'[\bar{X}] - \bar{A}'[X] + [A, \bar{A}], \quad (9)$$

$$[B, \bar{B}] = B'[\bar{X}] - \bar{B}'[X] + [B, \bar{B}] + [B, \bar{A}] - [\bar{B}, A]. \quad (10)$$

Note that the multiplication operation (8) is obviously a bilinear binary operation.

Theorem 1 Let $(A, B, X), (\bar{A}, \bar{B}, \bar{X}) \in \mathcal{P}_0$. Then $([A, \bar{A}], [B, \bar{B}], [X, \bar{X}]) \in \mathcal{P}_0$, where

$$[X, \bar{X}] = \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} (X(u + \varepsilon \bar{X}) - \bar{X}(u + \varepsilon X)).$$

Thus $\mathcal{M}_0$ constitutes an algebra under the multiplication operation (8).

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Proof. Noticing that

$$\begin{aligned} [[A, \bar{A}], L] &= -[[\bar{A}, L], A] - [[L, A], \bar{A}] \\ &= -[L'[\bar{X}], A] + [L'[X], \bar{A}] + [\bar{B}L, A] - [BL, \bar{A}] \end{aligned}$$

and that (see Ref. [9])

$$(L'[X])Y[\bar{X}] - (L'[\bar{X}])Y[X] = L'[Y], \quad Y = [X, \bar{X}],$$

we can directly calculate

$$\begin{aligned} [[A, \bar{A}], L] &= [A'[\bar{X}] - \bar{A}'[X] + [A, \bar{A}], L] \\ &= [A, L]'[\bar{X}] - [\bar{A}, L]'[X] + [\bar{B}L, A] - [BL, \bar{A}] \\ &= (L'[X])Y[\bar{X}] - (BL)'[\bar{X}] - (L'[\bar{X}])Y[X] + (\bar{B}L)'[X] \\ &\quad + [\bar{B}L, A] - [BL, \bar{A}] \\ &= L'[[X, \bar{X}]] - B'[\bar{X}]L - \bar{B}'[X]L + \bar{B}'[X]L + \bar{B}L'[X] \\ &\quad + [\bar{B}L, A] - [BL, \bar{A}] \\ &= L'[[X, \bar{X}]] - B'[\bar{X}]L + \bar{B}'[X]L - B([\bar{A}, L] + \bar{B}L) \\ &\quad + \bar{B}([A, L] + BL) + [\bar{B}L, A] - [BL, \bar{A}] \\ &= L'[[X, \bar{X}]] + (-B'[\bar{X}] + \bar{B}'[X] - [B, \bar{B}])L \\ &\quad - B[\bar{A}, L] + \bar{B}[A, L] + [\bar{B}L, A] - [BL, \bar{A}] \\ &= L'[[X, \bar{X}]] + (-B'[\bar{X}] + \bar{B}'[X] - [B, \bar{B}])L \\ &\quad + [\bar{A}, B]L - [A, \bar{B}]L \\ &= L'[[X, \bar{X}]] - [B, \bar{B}]L, \end{aligned}$$

which just shows that $([A, \bar{A}], [B, \bar{B}], [X, \bar{X}])$ belongs to the space $\mathcal{S}_0$. It follows that the linear space $\mathcal{M}_0$ constitutes an algebra with the multiplication operation (8). The proof is complete.

Corollary 1. $\langle E(\mathcal{M}_0), [\cdot, \cdot] \rangle$ forms a Lie algebra.

Corollary 1 shows that if two evolution equations $u_t = X$, $u_t = \bar{X}$ ($X, \bar{X} \in \mathcal{G}^q$) all possess L - A - B triad representations, then the evolution equation $u_t = [X, \bar{X}]$ also possesses a L - A - B triad representation.

In the Cartesian product space $\mathcal{V}' \times \mathcal{V}'$, we define the following equivalent relation $\sim$:

$$(A, B) \sim (\bar{A}, \bar{B}) \iff [A, L] + BL = [\bar{A}, L] + \bar{B}L, \quad (A, B), (\bar{A}, \bar{B}) \in \mathcal{V}' \times \mathcal{V}'. \quad \text{Set } K_0(L) = \{(A, B) \in \mathcal{V}' \times \mathcal{V}' \mid [A, L] + BL = 0\}. \quad \text{Obviously } K_0(L) \text{ is a subalgebra of } \langle \mathcal{M}_0, [\cdot, \cdot] \rangle.$$

Theorem 2. The subalgebra $\langle K_0(L), [\cdot, \cdot] \rangle$ is an ideal subalgebra of $\langle \mathcal{M}_0, [\cdot, \cdot] \rangle$.

Proof. For any $(A, B, X) \in \mathcal{S}_0$, $(\bar{A}, \bar{B}) \in K_0(L)$, it follows from Theorem 1 that

$$\begin{aligned} [[A, \bar{A}], L] + [[B, \bar{B}]]L &= L'[[X, 0]] = 0, \\ [[\bar{A}, A], L] + [[\bar{B}, B]]L &= L'[[0, X]] = 0. \end{aligned}$$

These show that $[(A, B), (\bar{A}, \bar{B})], [(\bar{A}, \bar{B}), (A, B)]$ all belong to $K_v(L)$. Therefore the statement in the theorem is really true. The proof is complete.

Let $(A, B) \in \mathcal{V} \times \mathcal{V}'$. We use $CL(A, B)$ to stand for the equivalent class to which (A, B) belongs. Set $CL(\mathcal{H}_v) = \{CL(A, B) | (A, B) \in \mathcal{H}_v\}$. By Theorem 2 we see that $CL(\mathcal{H}_v) = \mathcal{H}_v/K_v(L)$ is a quotient algebra, whose multiplication operation is as follows:

$$[CL(A, B), CL(\bar{A}, \bar{B})] = CL([(A, B), (\bar{A}, \bar{B})]), (A, B), (\bar{A}, \bar{B}) \in \mathcal{H}_v. \quad (11)$$

Theorem 3. *The quotient algebra $\langle CL(\mathcal{H}_v), [\cdot, \cdot] \rangle$ is a Lie algebra and isomorphic to Lie algebra $\langle E(\mathcal{H}_v), [\cdot, \cdot] \rangle$.*

Proof. For any $(A_1, B_1, X), (A_2, B_2, Y), (A_3, B_3, Z) \in \mathcal{H}_v$, by Theorem 1 we have

$$\begin{aligned} & [([A_1, A_2], A_3) + \text{cycle}(A_1, A_2, A_3), L] \\ &= L'([X, Y], Z) + \text{cycle}(X, Y, Z) - ([[B_1, B_2], B_3] + \text{cycle}(B_1, B_2, B_3))L \\ &= -([[B_1, B_2], B_3] + \text{cycle}(B_1, B_2, B_3))L. \end{aligned}$$

This shows by (11) that

$$[[CL(A_1, B_1), CL(A_2, B_2)], CL(A_3, B_3)] + \text{cycle}(CL(A_1, B_1), CL(A_2, B_2), CL(A_3, B_3)) = 0.$$

Thus $\langle CL(\mathcal{H}_v), [\cdot, \cdot] \rangle$ constitutes a Lie algebra.

Now let us prove that two Lie algebras are isomorphic to each other. Make $\rho: CL(\mathcal{H}_v) \rightarrow E(\mathcal{H}_v)$, $CL(A, B) \mapsto X((A, B, X) \in \mathcal{H}_v)$. Obviously ρ is a linear mapping. Note that for $(A, B, X), (\bar{A}, \bar{B}, \bar{X}) \in \mathcal{H}_v$, we have

$$\begin{aligned} \rho([CL(A, B), CL(\bar{A}, \bar{B})]) &= \rho(CL([(A, B), (\bar{A}, \bar{B})])) \\ &= [X, \bar{X}] = [\rho(CL(A, B)), \rho(CL(\bar{A}, \bar{B}))]. \end{aligned}$$

Hence ρ is an isomorphism of Lie algebras, which implies that Lie algebras $\langle CL(\mathcal{H}_v), [\cdot, \cdot] \rangle$ and $\langle E(\mathcal{H}_v), [\cdot, \cdot] \rangle$ are isomorphic. The proof is complete.

Through the above theorem we easily see that when $L = L(x, u) \in \mathcal{V}_0'$ is fixed and L' is injective, every evolution equation $u_t = X(X \in E(\mathcal{H}_v))$ has just a set of Manakov's pairs $CL(A, B)$ in L - A - B triad representations and no more. However, there exists an open problem in this kind of representations: How do we construct a corresponding Manakov's pair of operators for a given evolution equation $u_t = X(X \in E(\mathcal{H}_v))$? In addition, we have not known yet what relations there exist between two sorts of Manakov's pairs of operators corresponding to different spectral operators $L_1, L_2 (L_i = L_i(x, u) \in \mathcal{V}_0', i=1, 2)$. These need a further investigation.

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