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RESEARCH ARTICLE

Trilinear equations, Bell polynomials, and resonant solutions

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Abstract A class of trilinear differential operators is introduced through a technique of assigning signs to derivatives and used to create trilinear differential equations. The resulting trilinear differential operators and equations are characterized by the Bell polynomials, and the superposition principle is applied to the construction of resonant solutions of exponential waves. Two illustrative examples are made by an algorithm using weights of dependent variables.

Keywords Trilinear differential equation, Bell polynomial, superposition principle
MSC 35Q51, 37K40

1 Introduction

Bilinear differential equations are significant for their applications in physical and engineering sciences and the advancement of mathematics itself [9,12]. The Hirota bilinear D-operators [10] provide an amazingly powerful and beautiful tool for dealing with nonlinear differential equations solvable by the inverse scattering transform [8,12].

It is known that under $u = 2(\ln f)_{xx}$, the Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1.1}$$

is transformed into the Hirota bilinear form [10]:

$$(D_xD_t + D_x^4)f \cdot f = 0, \tag{1.2}$$

i.e.,

$$f_{xt}f - f_xf_t + f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2 = 0;$$

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under $u = 6(\ln f)_{xx}$, the Boussinesq equation

$$u_{tt} + (u^2)_{xx} + u_{xxxx} = 0 \tag{1.3}$$

is put into the Hirota bilinear form [23]:

$$(D_t^2 + D_x^4) f \cdot f = 0, \tag{1.4}$$

i.e.,

$$f f_{tt} - f_t^2 + f f_{xxxx} - 4 f_x f_{xxx} + 3 f_{xx}^2 = 0;$$

and under $u = 2(\ln f)_x$, the (3 + 1)-dimensional generalized Kadomtsev-Petviashvili (KP) equation [21]

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0 \tag{1.5}$$

becomes the Hirota bilinear equation

$$(D_x^3 D_y + D_t D_x + D_t D_y - D_z^2) f \cdot f = 0, \tag{1.6}$$

i.e.,

$$(f_{xxxy} + f_{tx} + f_{ty} - f_{zz}) f - 3 f_{xxy} f_x + 3 f_{xy} f_{xx} - f_y f_{xxx} - f_t f_x - f_t f_y + f_z^2 = 0.$$

Through the Hirota bilinear form, Wronskian, Grammian, and Pfaffian solutions [11,13,21,24], including solitons, positons, and complexitons [17,24], are systematically presented for integrable equations by the Hirota perturbation and Pfaffian techniques [12]. The Hirota bilinear D-operators [10] are defined to be

$$\begin{aligned} D_t^m D_x^n f \cdot g &= (\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^n f(x,t) g(x',t')|_{x'=x,t'=t} \\ &= \partial_{t'}^m \partial_{x'}^n f(x+x', t+t') g(x-x', t-t')|_{x'=t'=0}. \end{aligned} \tag{1.7}$$

For example, we have

$$\begin{cases} D_x f \cdot g = f_x g - f g_x, \\ D_x D_t f \cdot g = f_{xt} g - f_t g_x - f_x g_t + f g_{xt}, \\ D_x^3 f \cdot g = f_{xxx} g - 3 f_{xx} g_x + 3 f_x g_{xx} - f g_{xxx}. \end{cases}$$

Note that in definition (1.7), the Hirota bilinear D-operators take the positive sign for all derivatives of f and even-order derivatives of g, but the negative sign for odd-order derivatives of g. Recently, such a rule was generalized to a more general situation to introduce new types of bilinear differential equations [19].

On the other hand, Matsukidaira et al. [29] presented a class of trilinear equations which have Wronskian solutions. In particular, it is shown that the Broer-Kaup system [3,14]

$$h_t = (h_x + 2hu)_x, \quad u_t = (u^2 + 2hu_x)_x \tag{1.8}$$

is transformed into the trilinear form

$$
\begin{aligned}
& f f_{xx}(f_{4x} - f_{tt}) + f_{xx}(f_t - f_{xx})(f_t + f_{xx}) + f(f_{xt} - f_{3x})(f_{xt} + f_{3x}) \\
& \quad - f_x(f_{xt} - f_{3x})(f_t + f_{xx}) - f_x(f_t - f_{xx})(f_{xt} + f_{3x}) - f_x^2(f_{4x} - f_{tt}) = 0, \quad (1.9)
\end{aligned}
$$

through the dependent variable transformation

$$
h = (\ln f)_{xx}, \quad uh = -\frac{1}{2}(\ln f)_{xt} - \frac{1}{2}(\ln f)_{xxx}. \tag{1.10}
$$

Interestingly, the trilinear form was also used to present higher-order version of integrable lattice equations and fully discrete equations [26–28]. Moreover, the trilinear form appeared in the construction of multiple soliton solutions to the Landau-Lifshitz equation [2], and the multilinear forms were applied to discussion of both singularity of solutions and integrability of the underlying equations [7].

In this paper, we will introduce a class of trilinear differential operators and analyze the corresponding trilinear equations. More importantly, we will establish links between the multivariate Bell exponential polynomials and the presented trilinear operators and equations. We will also characterize resonant solutions of exponential waves to the trilinear equations, thereby producing linear subspaces of their solutions.

2 Trilinear differential operators and equations

2.1 Trilinear $D_{\overline{p}}$-operators

Let

$$
\overline{p} = \langle p, p', p'' \rangle,
$$

where p, p', and p'' are natural numbers. Motivated by the generalized bilinear differential operators [19], we introduce a class of trilinear differential operators by assigning different signs to derivatives as follows:

$$
\begin{aligned}
& (D^n_{\overline{p},x} f \cdot g \cdot h)(x) \\
& \quad = (D^n_{\langle p,p',p''\rangle x} f \cdot g \cdot h)(x) \\
& \quad = (\alpha_p \partial_x + \alpha_{p'} \partial_{x'} + \alpha_{p''} \partial_{x''})^n f(x) g(x') h(x'')\big|_{x''=x'=x} \\
& \quad = \sum_{i+j+k=n,\, i,j,k \geqslant 0} \frac{n!}{i!\, j!\, k!} \alpha_p^i \alpha_{p'}^j \alpha_{p''}^k (\partial_x^i f)(x)(\partial_x^j g)(x)(\partial_x^k h)(x), \quad n \geqslant 1, \quad (2.1)
\end{aligned}
$$

where the powers of α_s $(s \geqslant 1)$ are just the signs determined by

$$
\alpha_s^m = (-1)^{r_s(m)}, \tag{2.2}
$$

where

$$
m \equiv r_s(m) \pmod{s}, \quad 0 \leqslant r_s(m) < s, \; m \geqslant 0.
$$

Obviously, the case of $p = p' = p'' = 1$ gives the normal derivatives. We can also easily find that the powers α_s^m have the following patterns:

$$s = 2k\ (k \in \mathbb{N}):\ +,\ -,\ +,\ -,\ \ldots, \tag{2.3}$$

$$s = 1:\ +,\ +,\ +,\ +,\ \ldots, \tag{2.4}$$

$$s = 3:\ +,\ -,\ +,\ +,\ -,\ +,\ \ldots, \tag{2.5}$$

$$s = 5:\ +,\ -,\ +,\ -,\ +,\ +,\ -,\ +,\ -,\ +,\ \ldots, \tag{2.6}$$

$$s = 7:\ +,\ -,\ +,\ -,\ +,\ -,\ +,\ +,\ -,\ +,\ -,\ +,\ -,\ +,\ \ldots, \tag{2.7}$$

for $m = 0, 1, 2, \ldots$; and thus, we have

$$D_{\langle 1,2,3\rangle x} f \cdot g \cdot h = f_x gh - f g_x h - f g h_x,$$

$$D^2_{\langle 1,2,3\rangle x} f \cdot g \cdot h = f_{2x} gh - 2 f_x g_x h - 2 f_x g h_x + f g_{2x} h + 2 f g_x h_x + f g h_{2x},$$

$$\begin{aligned} D^3_{\langle 1,2,3\rangle x} f \cdot g \cdot h = {} & f_{3x} gh - 3 f_{2x} g_x h - 3 f_{2x} g h_x + 3 f_x g_{2x} h + 6 f_x g_x h_x \\ & + 3 f_x g h_{2x} - f g_{3x} h - 3 f g_{2x} h_x - 3 f g_x h_{2x} + f g h_{3x}. \end{aligned}$$

A common feature that the $D_{\overline{p}}$-operators share is the Taylor expansion

$$f(x + \alpha_p \delta) g(x + \alpha_{p'} \delta) h(x + \alpha_{p''} \delta) = \sum_{i=0}^{\infty} \frac{1}{i!} \, (D^i_{\overline{p},x} f \cdot g \cdot h) \delta^i, \tag{2.8}$$

if we define

$$e(x + \alpha_s \delta) = \sum_{i=0}^{\infty} \frac{(\partial_x^i e)(x)}{i!} \, \alpha_s^i \delta^i, \tag{2.9}$$

where the powers α_s^m obey the rule in (2.2).

Trilinear operators in the multivariate case can be similarly defined as follows:

$$\begin{aligned} & (D^{n_1}_{\overline{p}_1, x_1} \cdots D^{n_l}_{\overline{p}_l, x_l} f \cdot g \cdot h)(x_1, \ldots, x_l) \\ & \quad = (\alpha_{p_1} \partial_{x_1} + \alpha_{p'_1} \partial_{x'_1} + \alpha_{p''_1} \partial_{x''_1})^{n_1} \cdots (\alpha_{p_l} \partial_{x_l} + \alpha_{p'_l} \partial_{x'_l} + \alpha_{p''_l} \partial_{x''_l})^{n_l} \\ & \qquad \times f(x_1, \ldots, x_l) g(x'_1, \ldots, x'_l) h(x''_1, \ldots, x''_l)|_{x''_i = x'_i = x_i}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n_1, \ldots, n_l \geqslant 1, \end{aligned} \tag{2.10}$$

where

$$\overline{p}_i = \langle p_i, p'_i, p''_i \rangle, \quad 1 \leqslant i \leqslant l.$$

2.2 Trilinear equations

Given a multivariate polynomial

$$\mathscr{F} = \mathscr{F}(x_{1,1}, \ldots, x_{m,1}; \ldots; x_{1,l}, \ldots, x_{m,l}),$$

through replacing the variables $x_{i,j}$ with the trilinear operators, we can define a corresponding trilinear differential equation to be

$$\mathscr{F}(D_{\overline{p}_1,x_1},\ldots,D_{\overline{p}_m,x_1};\ldots;D_{\overline{p}_1,x_l},\ldots,D_{\overline{p}_m,x_l})f\cdot f\cdot f=0. \tag{2.11}$$

Considering a simple case $\overline{p}=\langle 1,2,3\rangle$, we particularly have the trilinear KdV-like equation

$$\begin{aligned}(D_{\langle 1,2,3\rangle x}&D_{\langle 1,2,3\rangle t}+D^4_{\langle 1,2,3\rangle x})f\cdot f\cdot f\\ &=3f_{xt}f^2-2f_xf_tf-8f_{3x}f_xf+18f_{2x}^2f-12f_{2x}f_x^2+f_{4x}f^2\\ &=0,\end{aligned} \tag{2.12}$$

the trilinear Boussinesq-like equation

$$\begin{aligned}(D^2_{\langle 1,2,3\rangle t}&+D^4_{\langle 1,2,3\rangle x})f\cdot f\cdot f\\ &=3f_{2t}f^2-2f_t^2f-8f_{3x}f_xf+18f_{2x}^2f-12f_{2x}f_x^2+f_{4x}f^2\\ &=0,\end{aligned} \tag{2.13}$$

and the trilinear KP-like equation

$$\begin{aligned}(D_{5,t}&D_{\langle 1,2,3\rangle x}+D^4_{\langle 1,2,3\rangle x}+D^2_{\langle 1,2,3\rangle y})f\cdot f\\ &=3f_{xt}f^2-2f_xf_tf-8f_{3x}f_xf+18f_{2x}^2f-12f_{2x}f_x^2+f_{4x}f^2+3f_{2y}f^2-2f_y^2f\\ &=0.\end{aligned} \tag{2.14}$$

Such trilinear equations are defined through the nice mathematical differential operators, the trilinear $D_{\overline{p}}$-operators. Naturally, there are two basic questions on those trilinear equations.

- How can one characterize the trilinear equations presented in (2.11)?
- What kind of exact solutions are there to the trilinear equations presented in (2.11)?

In this paper, we would like to answer those two basic questions through applying the Bell exponential polynomials in combinatorics and the superposition principle in systems theory.

3 Characterization by Bell polynomials

3.1 Bell polynomials

We recall the Bell polynomials and discuss some of their basic properties. Let y be a C^∞ function of x and introduce

$$y_r=y_{rx}=\partial_x^r y,\quad r\geqslant 1. \tag{3.1}$$

The Bell polynomials [1] in combinatorial mathematics are defined by

$$Y_{nx}(y)=Y_n(y_1,\ldots,y_n)=\mathrm{e}^{-y}\partial_x^n\mathrm{e}^y,\quad n\geqslant 0. \tag{3.2}$$

The first few Bell polynomials read

$$
\begin{gathered}
Y_0 = 1, \quad Y_1 = y_1, \quad Y_2 = y_1^2 + y_2, \quad Y_3 = y_1^3 + 3y_1y_2 + y_3, \\
Y_4 = y_1^4 + 6y_1^2y_2 + 4y_1y_3 + 3y_2^2 + y_4, \\
Y_5 = y_1^5 + 10y_1^3y_2 + 10y_1^2y_3 + 15y_1y_2^2 + 5y_1y_4 + 10y_2y_3 + y_5, \\
\begin{aligned}
Y_6 = {} & y_1^6 + 15y_1^4y_2 + 20y_1^3y_3 + 45y_1^2y_2^2 + 15y_1^2y_4 \\
& + 60y_1y_2y_3 + 15y_2^3 + 6y_1y_5 + 15y_2y_4 + 10y_3^2 + y_6.
\end{aligned}
\end{gathered}
$$

The Faà di Bruno formula (see, e.g., [4]) shows that the Bell polynomials are explicitly given by

$$Y_{nx}(y) = \sum \frac{n!}{m_1! \cdots m_n!\, (1!)^{m_1} \cdots (n!)^{m_n}}\, y_1^{m_1} \cdots y_n^{m_n}, \tag{3.3}$$

where the sum is over all n-tuples of nonnegative integers $(m_1, \dots, m_n)$ satisfying the constraint

$$m_1 + 2m_2 + \cdots + nm_n = n.$$

It is also easy to see that the Bell polynomials can be computed directly from

$$\exp\left(\sum_{r=1}^{\infty} \frac{y_r}{r!}\, t^r \right) = \sum_{n=0}^{\infty} \frac{Y_n(y_1, \dots, y_n)}{n!}\, t^n, \tag{3.4}$$

or recursively from

$$Y_{n+1}(y) = \sum_{i=0}^{n} \binom{n}{i} y_{n-i+1} Y_i(y), \quad n \geqslant 0. \tag{3.5}$$

The following two properties will be used to link the trilinear $D_{\overline{p}}$-operators and equations to the Bell polynomials. First, the general representation formula (3.3) implies the homogeneous property for the Bell polynomials:

$$Y_n(\alpha_s y_1, \alpha_s^2 y_2, \dots, \alpha_s^n y_n) = \alpha_s^n Y_n(y_1, \dots, y_n), \tag{3.6}$$

whose left-hand side is computed through first substituting all $\alpha_s y_1, \alpha_s^2 y_2, \dots, \alpha_s^n y_n$ into the Bell polynomial Y_n and then collecting powers of α_s and evaluating them by rule (2.2). Second, the general Leibniz rule

$$(fgh)^{-1}\partial_x^n(fgh) = \sum_{i+j+k=n,\, i,j,k \geqslant 0} \frac{n!}{i!\, j!\, k!}\, (f^{-1}\partial_x^i f)(g^{-1}\partial_x^j g)(h^{-1}\partial_x^k h) \tag{3.7}$$

directly tells the addition formula for the Bell polynomials:

$$Y_{nx}(y + y' + y'') = \sum_{i+j+k=n,\, i,j,k \geqslant 0} \frac{n!}{i!\, j!\, k!}\, Y_{ix}(y) Y_{jx}(y') Y_{kx}(y''), \tag{3.8}$$

where y' and y'' are two additional functions of x. We will see that those two properties are very helpful in characterizing the trilinear $D_{\overline{p}}$-operators and equations.

3.2 Triple Bell polynomials

We first characterize the trilinear $D_{\overline{p}}$-operators by the Bell polynomials. For the sake of computational convenience, we assume that

$$f = \mathrm{e}^{\xi(x)}, \quad g = \mathrm{e}^{\eta(x)}, \quad h = \mathrm{e}^{\zeta(x)}. \tag{3.9}$$

Then applying the homogeneous property (3.6) and the addition formula (3.8), we can compute that

$$\begin{aligned}
&(fgh)^{-1} D^n_{\overline{p},x} f \cdot g \cdot h \\
&= \sum_{i+j+k=n,\, i,j,k\geqslant 0} \frac{n!}{i!\,j!\,k!}\, \alpha^i_p \alpha^j_{p'} \alpha^k_{p''} (f^{-1}\partial^i_x f)(g^{-1}\partial^j_x g)(h^{-1}\partial^k_x h) \\
&= \sum_{i+j+k=n,\, i,j,k\geqslant 0} \frac{n!}{i!\,j!\,k!}\, \alpha^i_p \alpha^j_{p'} \alpha^k_{p''} Y_{ix}(\xi) Y_{jx}(\eta) Y_{kx}(\zeta) \\
&= Y_n(y_1, \ldots, y_n)\big|_{y_r = \alpha^r_p \xi_{rx} + \alpha^r_{p'} \eta_{rx} + \alpha^r_{p''} \zeta_{rx}},
\end{aligned} \tag{3.10}$$

where

$$\xi_{rx} = \partial^r_x \xi, \quad \eta_{rx} = \partial^r_x \eta, \quad \zeta_{rx} = \partial^r_x \zeta, \quad r \geqslant 1,$$

as defined by (3.1).

Motivated by the binary Bell polynomials for the Hirota bilinear D-operators [16] and the generalized bilinear $D_{\overline{p}}$-operators [20], we introduce triple Bell polynomials as follows:

$$\begin{aligned}
&\mathscr{Y}^{\langle p,p',p''\rangle}_{nx}(u, v, w) \\
&= Y_n(y_1, \ldots, y_n)\big|_{y_r = \frac{1}{3}\left[\alpha^r_p (u_{rx}+v_{rx}+w_{rx}) + \alpha^r_{p'}(u_{rx}-2v_{rx}+w_{rx}) + \alpha^r_{p''}(u_{rx}+v_{rx}-2w_{rx})\right]},
\end{aligned} \tag{3.11}$$

where

$$u_{rx} = \partial^r_x u, \quad v_{rx} = \partial^r_x v, \quad w_{rx} = \partial^r_x w, \quad r \geqslant 1,$$

as defined by (3.1). For example, we have

$$\mathscr{Y}^{\langle 2,3,5\rangle}_{x}(u, v, w) = -u_x, \quad \mathscr{Y}^{\langle 2,3,5\rangle}_{2x}(u, v, w) = u^2_x + u_{2x},$$

$$\mathscr{Y}^{\langle 2,3,5\rangle}_{3x}(u, v, w) = -u^3_x - 3u_x u_{2x} - \frac{1}{3}\, u_{3x} - \frac{4}{3}\, v_{3x} + \frac{2}{3}\, w_{3x},$$

$$\begin{aligned}
\mathscr{Y}^{\langle 2,3,5\rangle}_{4x}(u, v, w) = u^4_x + 6u^2_x u_{2x} + \frac{4}{3}\, u_x u_{3x} + \frac{16}{3}\, u_x v_{3x} - \frac{8}{3}\, u_x w_{3x} \\
+ 3u^2_{2x} + \frac{1}{3}\, u_{4x} + \frac{4}{3}\, v_{4x} - \frac{2}{3}\, w_{4x}.
\end{aligned}$$

This way, upon setting that

$$u=\xi+\eta+\zeta, \quad v=\xi-\eta, \quad w=\xi-\zeta, \tag{3.12}$$

we have, from (3.10), a combinatorial formula for the trilinear $D_{\overline{p}}$-operators:

$$(fgh)^{-1}D^n_{\langle p,p',p''\rangle x} f\cdot g\cdot h=\mathscr{Y}^{\langle p,p',p''\rangle}_{nx}\Big(u=\ln fgh,\ v=\ln\frac{f}{g},\ w=\ln\frac{f}{h}\Big). \tag{3.13}$$

Now, to characterize the trilinear equations, we further introduce $\mathscr{P}$-polynomials:

$$\mathscr{P}^{\langle p,p',p''\rangle}_{nx}(q)=\mathscr{Y}^{\langle p,p',p''\rangle}_{nx}(q,0,0), \tag{3.14}$$

and some examples are

$$\mathscr{P}^{\langle 2,3,5\rangle}_{x}(q)=-q_x, \quad \mathscr{P}^{\langle 2,3,5\rangle}_{2x}(q)=q_x^2+q_{2x},$$
$$\mathscr{P}^{\langle 2,3,5\rangle}_{3x}(q)=-q_x^3-3q_xq_{2x}-\frac{1}{3}\,q_{3x},$$
$$\mathscr{P}^{\langle 2,3,5\rangle}_{4x}(q)=q_x^4+6q_x^2q_{2x}+\frac{4}{3}\,q_xq_{3x}+3q_{2x}^2+\frac{1}{3}\,q_{4x}.$$

Setting

$$q=u-v-w=-\ln f+2\ln g+2\ln h, \quad v=\ln\frac{f}{g}, \quad w=\ln\frac{f}{h}, \tag{3.15}$$

the combinatorial formula (3.13) becomes

$$(fgh)^{-1}D^n_{\langle p,p',p''\rangle x} f\cdot g\cdot h=\mathscr{Y}^{\langle p,p',p''\rangle}_{nx}(q+v+w,v,w). \tag{3.16}$$

With $f=g=h$, this tells a relation between the trilinear expressions and the $\mathscr{P}$-polynomials:

$$f^{-3}D^n_{\langle p,p',p''\rangle x} f\cdot f\cdot f=\mathscr{P}^{\langle p,p',p''\rangle}_{nx}(q=3\ln f). \tag{3.17}$$

Therefore, a trilinear equation defined by

$$\mathscr{F}(D_{\overline{p}_1,x},\dots,D_{\overline{p}_m,x})f\cdot f\cdot f=0 \tag{3.18}$$

with

$$\mathscr{F}(x_{1,1},\dots,x_{m,1})=\sum_{j=1}^m\sum_{i=1}^n c_j^i x_{j,1}^i$$

is equivalent to an equation on a linear combination of the $\mathscr{P}$-polynomials in $q=3\ln f$:

$$\sum_{j=1}^m\sum_{i=1}^n c_j^i\mathscr{P}^{\overline{p}_j}_{ix}(q=3\ln f)=0, \tag{3.19}$$

where

$$\overline{p}_j = \langle p_j, p'_j, p''_j \rangle, \quad 1 \leqslant j \leqslant m,$$

and the coefficients c_j^i's are constants. This characterizes our trilinear equations in one-dimensional case.

3.3 Multivariate triple Bell polynomials

For a C^∞ function $y = y(x_1, \ldots, x_l)$, let us introduce the variables [6]

$$y_{r_1,\ldots,r_l} = y_{r_1 x_1,\ldots,r_l x_l} = \partial_{x_1}^{r_1} \cdots \partial_{x_l}^{r_l} y(x_1, \ldots, x_l), \quad r_1, \ldots, r_l \geqslant 0, \tag{3.20}$$

where $r_1 + \cdots + r_l \geqslant 1$, and the multivariate Bell polynomials

$$Y_{n_1 x_1,\ldots,n_l x_l}(y) = Y_{n_1,\ldots,n_l}(y_{r_1,\ldots,r_l}) = \mathrm{e}^{-y} \partial_{x_1}^{n_1} \cdots \partial_{x_l}^{n_l} \mathrm{e}^{y}, \quad n_1, \ldots, n_l \geqslant 0, \tag{3.21}$$

which can be evaluated through the following relation:

$$\exp\Bigg(\sum_{\substack{r_1 + \cdots + r_l \geqslant 1 \\ r_1, \ldots, r_l \geqslant 0}} \frac{y_{r_1,\ldots,r_l}}{r_1! \cdots r_l!} t_1^{r_1} \cdots t_l^{r_l} \Bigg) = \sum_{n_1,\ldots,n_l \geqslant 0} \frac{Y_{n_1,\ldots,n_l}}{n_1! \cdots n_l!} t_1^{n_1} \cdots t_l^{n_l}. \tag{3.22}$$

Four simple examples in differential polynomial form are listed below:

$$\begin{aligned}
Y_{x,t} &= y_{xt} + y_x y_t, \quad Y_{2x,t} = y_{2x,t} + y_{2x} y_t + 2y_{xt} y_x + y_x^2 y_t, \\
Y_{3x,t} &= y_{3x,t} + y_{3x} y_t + 3y_{2x,t} y_x + 3y_{2x} y_{xt} + 3y_{2x} y_x y_t + 3y_x^2 y_{xt} + y_x^3 y_t, \\
Y_{4x,t} &= y_{4x,t} + 4y_{3x} y_{2x,t} + 6y_{2x,t} y_{2x} + y_{4x} y_t + 4y_{3x,t} y_x + 3y_{2x}^2 y_t \\
&\quad + 12 y_{2x} y_{x,t} y_x + 4y_{3x} y_x y_t + 6y_{2x,t} y_x^2 + 6y_{2x} y_x^2 y_t + 4y_{x,t} y_x^3 + y_x^4 y_t.
\end{aligned}$$

Based on relation (3.22), we can readily derive the homogeneous property:

$$Y_{n_1,\ldots,n_l}(\alpha_{p_1}^{r_1} \cdots \alpha_{p_l}^{r_l} y_{r_1,\ldots,r_l}) = \alpha_{p_1}^{n_1} \cdots \alpha_{p_l}^{n_l} Y_{n_1,\ldots,n_l}(y_{r_1,\ldots,r_l}), \tag{3.23}$$

and the general Lebnitz rule

$$\begin{aligned}
&(fgh)^{-1} \partial_{x_1}^{n_1} \cdots \partial_{x_l}^{n_l} (fgh) \\
&= \Bigg(\prod_{r=1}^{l} \sum_{\substack{i_r + j_r + k_r = n_r \\ i_r, j_r, k_r \geqslant 0}} \frac{n_r!}{i_r!\, j_s!\, k_r!} \Bigg) \\
&\quad \times (f^{-1} \partial_{x_1}^{i_1} \cdots \partial_{x_l}^{i_l} f)(g^{-1} \partial_{x_1}^{j_1} \cdots \partial_{x_l}^{j_l} g)(h^{-1} \partial_{x_1}^{k_1} \cdots \partial_{x_l}^{k_l} h)
\end{aligned} \tag{3.24}$$

shows the addition formula for the multivariate Bell polynomials:

$$\begin{aligned}
&Y_{n_1 x_1,\ldots,n_l x_l}(y + y' + y'') \\
&= \Bigg(\prod_{r=1}^{l} \sum_{\substack{i_r + j_r + k_r = n_r \\ i_r, j_r, k_r \geqslant 0}} \frac{n_r!}{i_r! j_r! k_r!} \Bigg)
\end{aligned}$$

$$\times Y_{i_1x_1,\ldots,i_lx_l}(y)Y_{j_1x_1,\ldots,j_lx_l}(y')Y_{k_1x_1,\ldots,k_lx_l}(y''). \tag{3.25}$$

Similarly, for the sake of computational convenience, we assume that

$$f = \mathrm{e}^{\xi(x_1,\ldots,x_l)}, \quad g = \mathrm{e}^{\eta(x_1,\ldots,x_l)}, \quad h = \mathrm{e}^{\zeta(x_1,\ldots,x_l)}. \tag{3.26}$$

Then by the homogeneous property (3.23) and the addition formula (3.25) in the multivariate case, we can compute that

$$\begin{aligned}
&(fgh)^{-1}D^{n_1}_{\overline{p}_1,x_1}\cdots D^{n_l}_{\overline{p}_l,x_l} f\cdot g\cdot h\\
&\quad=\Bigg(\prod_{r=1}^{l}\sum_{\substack{i_r+j_r+k_r=n_r\\ i_r,j_r,k_r\geqslant 0}}\frac{n_r!}{i_r!\,j_r!\,k_r!}\,\alpha^{i_r}_{p_r}\alpha^{j_r}_{p'_r}\alpha^{k_r}_{p''_r}\Bigg)\\
&\qquad\times(\mathrm{e}^{-\xi}\partial^{i_1}_{x_1}\cdots\partial^{i_l}_{x_l}\mathrm{e}^{\xi})(\mathrm{e}^{-\eta}\partial^{j_1}_{x_1}\cdots\partial^{j_l}_{x_l}\mathrm{e}^{\eta})(\mathrm{e}^{-\zeta}\partial^{k_1}_{x_1}\cdots\partial^{k_l}_{x_l}\mathrm{e}^{\zeta})\\
&\quad=\Bigg(\prod_{r=1}^{l}\sum_{\substack{i_r+j_r+k_r=n_r\\ i_r,j_r,k_r\geqslant 0}}\frac{n_r!}{i_r!\,j_r!\,k_r!}\,\alpha^{i_r}_{p_r}\alpha^{j_r}_{p'_r}\alpha^{k_r}_{p''_r}\Bigg)\\
&\qquad\times Y_{i_1x_1,\ldots,i_lx_l}(\xi)Y_{j_1x_1,\ldots,j_lx_l}(\eta)Y_{k_1x_1,\ldots,k_lx_l}(\zeta)\\
&\quad=\Bigg(\prod_{r=1}^{l}\sum_{\substack{i_r+j_r+k_r=n_r\\ i_r,j_r,k_r\geqslant 0}}\frac{n_r!}{i_r!\,j_r!\,k_r!}\Bigg)Y_{i_1x_1,\ldots,i_lx_l}(\alpha^{r_1}_{p_1}\cdots\alpha^{r_l}_{p_l}\xi_{r_1,\ldots,r_l})\\
&\qquad\times Y_{j_1x_1,\ldots,j_lx_l}(\alpha^{r_1}_{p'_1}\cdots\alpha^{r_l}_{p'_l}\eta_{r_1,\ldots,r_l})Y_{k_1x_1,\ldots,k_lx_l}(\alpha^{r_1}_{p''_1}\cdots\alpha^{r_l}_{p''_l}\zeta_{r_1,\ldots,r_l})\\
&\quad=Y_{n_1,\ldots,n_l}(y_{r_1,\ldots,r_l}=\alpha^{r_1}_{p_1}\cdots\alpha^{r_l}_{p_l}\xi_{r_1,\ldots,r_l}\\
&\qquad+\alpha^{r_1}_{p'_1}\cdots\alpha^{r_l}_{p'_l}\eta_{r_1,\ldots,r_l}+\alpha^{r_1}_{p''_1}\cdots\alpha^{r_l}_{p''_l}\zeta_{r_1,\ldots,r_l}).
\end{aligned} \tag{3.27}$$

Let us now introduce multivariate triple Bell polynomials in differential polynomial form:

$$\begin{aligned}
\mathscr{Y}^{\overline{p}_1,\ldots,\overline{p}_l}_{n_1x_1,\ldots,n_lx_l}(u,v,w)&=Y_{n_1,\ldots,n_l}(y_{r_1,\ldots,r_l}=\alpha^{r_1}_{p_1}\cdots\alpha^{r_l}_{p_l}\xi_{r_1x_1,\ldots,r_lx_l}\\
&\quad+\alpha^{r_1}_{p'_1}\cdots\alpha^{r_l}_{p'_l}\eta_{r_1x_1,\ldots,r_lx_l}+\alpha^{r_1}_{p''_1}\cdots\alpha^{r_l}_{p''_l}\zeta_{r_1x_1,\ldots,r_lx_l}),
\end{aligned} \tag{3.28}$$

where ξ, η, and ζ are defined through the system

$$u=\xi+\eta+\zeta,\quad v=\xi-\eta,\quad w=\xi-\zeta. \tag{3.29}$$

Then from (3.27), we obtain a combinatorial formula for the trilinear $D_{\overline{p}}$-operators:

$$\begin{aligned}
&(fgh)^{-1}D^{n_1}_{\overline{p}_1,x_1}\cdots D^{n_l}_{\overline{p}_l,x_l} f\cdot g\cdot h\\
&\quad=\mathscr{Y}^{\overline{p}_1,\ldots,\overline{p}_l}_{n_1x_1,\ldots,n_lx_l}\Big(u=\ln fgh,\ v=\ln\frac{f}{g},\ w=\ln\frac{f}{h}\Big).
\end{aligned} \tag{3.30}$$

Furthermore, we introduce the multivariate $\mathscr{P}$-polynomials:

$$\mathscr{P}^{\overline{p}_1,\dots,\overline{p}_l}_{n_1x_1,\dots,n_lx_l}(q) = \mathscr{Y}^{\overline{p}_1,\dots,\overline{p}_l}_{n_1x_1,\dots,n_lx_l}(u = q,\ v = 0,\ w = 0). \tag{3.31}$$

It then follows that

$$f^{-3} D^{n_1}_{\overline{p}_1,x_1} \cdots D^{n_l}_{\overline{p}_l,x_l} f \cdot f \cdot f = \mathscr{P}^{\overline{p}_1,\dots,\overline{p}_l}_{n_1x_1,\dots,n_lx_l}(q = 3\ln f). \tag{3.32}$$

Finally, assuming that a multivariate polynomial is given by

$$\mathscr{F}(x_{1,1},\dots,x_{m,1};\dots;x_{1,l},\dots,x_{m,l}) = \sum_{j_1,\dots,j_l=1}^{m} \sum_{i_1,\dots,i_l=1}^{n} c^{i_1,\dots,i_l}_{j_1,\dots,j_l} x^{i_1}_{j_1,1} \cdots x^{i_l}_{j_l,l}, \tag{3.33}$$

where the coefficients $c^{i_1,\dots,i_l}_{j_1,\dots,j_l}$'s are constants, we see that a trilinear equation defined by

$$\mathscr{F}(D_{\overline{p}_1,x_1},\dots,D_{\overline{p}_m,x_1};\dots;D_{\overline{p}_1,x_l},\dots,D_{\overline{p}_m,x_l}) f \cdot f \cdot f = 0 \tag{3.34}$$

is equivalent to an equation on a linear combination of the multivariate $\mathscr{P}$-polynomials in $q = 3\ln f$:

$$\sum_{j_1,\dots,j_l=1}^{m} \sum_{i_1,\dots,i_l=1}^{n} c^{i_1,\dots,i_l}_{j_1,\dots,j_l} \mathscr{P}^{\overline{p}_{j_1},\dots,\overline{p}_{j_l}}_{i_1x_1,\dots,i_lx_l}(q = 3\ln f) = 0. \tag{3.35}$$

This characterizes our trilinear equations in the multivariate case, in terms of the $\mathscr{P}$-polynomials.

4 Resonant solutions

4.1 Superposition principle

Let $\mathscr{F}(x_{1,1},\dots,x_{m,1};\dots;x_{1,l},\dots,x_{m,l})$ be a multivariate polynomial and consider a trilinear equation defined by

$$\mathscr{F}(D_{\overline{p}_1,x_1},\dots,D_{\overline{p}_m,x_1};\dots;D_{\overline{p}_1,x_l},\dots,D_{\overline{p}_m,x_l}) f \cdot f \cdot f = 0, \tag{4.1}$$

where $\overline{p}_i = \langle p_i, p'_i, p''_i \rangle$, $1 \leqslant i \leqslant m$. Define a set of N wave variables

$$\theta_i = k_{1,i}x_1 + \cdots + k_{l,i}x_l, \quad 1 \leqslant i \leqslant N, \tag{4.2}$$

where the $k_{j,i}$'s are constants, and use the superposition principle to form a resonant solution of N exponential waves

$$f = \sum_{i=1}^{N} \varepsilon_i \mathrm{e}^{\theta_i} = \sum_{i=1}^{N} \varepsilon_i \mathrm{e}^{k_{1,i}x_1+\cdots+k_{l,i}x_l}, \tag{4.3}$$

where all ε_i's are arbitrary constants. If this function with arbitrary constants ε_i's solves the trilinear equation (4.1), then all exponential wave solutions are resonant, and (4.3) is called a resonant solution of exponential waves to the trilinear equation (4.1).

Note that we readily have the trilinear identities:

$$\begin{aligned}&\mathscr{F}(D_{\overline{p}_1,x_1},\dots,D_{\overline{p}_m,x_1};\dots;D_{\overline{p}_1,x_l},\dots,D_{\overline{p}_m,x_l})\mathrm{e}^{\theta_i}\cdot\mathrm{e}^{\theta_{i'}}\cdot\mathrm{e}^{\theta_{i''}}\\&\quad=\mathscr{F}(\beta_{1,1},\dots,\beta_{1,m};\dots;\beta_{l,1},\dots,\beta_{l,m})\mathrm{e}^{\theta_i+\theta_{i'}+\theta_{i''}}\\&\quad=\mathscr{F}(\dots,\beta_{r,s}(i,i',i''),\dots)\mathrm{e}^{\theta_i+\theta_{i'}+\theta_{i''}},\quad 1\leqslant i,i',i''\leqslant N,\end{aligned}\tag{4.4}$$

where

$$\begin{aligned}\beta_{r,s}=\beta_{r,s}(i,i',i'')=\alpha_{p_s}k_{r,i}+\alpha_{p_s'}k_{r,i'}+\alpha_{p_s''}k_{r,i''},\\ 1\leqslant r\leqslant l,\ 1\leqslant s\leqslant m,\end{aligned}\tag{4.5}$$

and the powers α_s^m obey rule (2.2). It then follows that the following criterion holds for the resonant solution defined by (4.3) (see, e.g., [19,22] for the case of bilinear equations).

Theorem 4.1 *Let $N\geqslant 1$ be an integer. A linear combination of N exponential waves defined by* (4.3) *presents a resonant solution to the trilinear equation* (4.1) *if and only if the constants $k_{j,i}$'s satisfy*

$$\sum_{(j,j',j'')\in S(i,i',i'')}\mathscr{F}(\dots,\beta_{r,s}(j,j',j''),\dots)=0,\quad 1\leqslant i\leqslant i'\leqslant i''\leqslant N,\tag{4.6}$$

where $\beta_{r,s}(i,i',i'')$'s are defined by (4.5) *and*

$$\begin{aligned}S(i,i',i'')=\{&(i,i',i''),\ (i,i'',i'),\ (i',i,i''),\ (i',i'',i),\\&(i'',i,i'),\ (i'',i',i)\},\quad 1\leqslant i,i',i''\leqslant N.\end{aligned}\tag{4.7}$$

Assume that a multivariate polynomial $\mathscr{F}$ is given by (3.33). Note that the combinatorial formula (3.30) yields

$$\begin{aligned}&D^{i_1}_{\overline{s}_{j_1},x_1}\cdots D^{i_l}_{\overline{s}_{j_l},x_l}\mathrm{e}^{\theta_i}\cdot\mathrm{e}^{\theta_{i'}}\cdot\mathrm{e}^{\theta_{i''}}\\&=\mathscr{Y}^{\overline{s}_{j_1},\dots,\overline{s}_{j_l}}_{i_1x_1,\dots,i_lx_l}(\theta_i+\theta_{i'}+\theta_{i''},\theta_i-\theta_{i'},\theta_i-\theta_{i''})\mathrm{e}^{\theta_i+\theta_{i'}+\theta_{i''}},\ 1\leqslant i,i',i''\leqslant N,\end{aligned}\tag{4.8}$$

where $\mathscr{Y}^{\overline{s}_{j_1},\dots,\overline{s}_{j_l}}_{i_1x_1,\dots,i_lx_l}$'s are the triple Bell polynomials defined by (3.28). Therefore, we obtain an equivalent theorem on the resonant solution of N exponential waves, defined by (4.3).

Theorem 4.2 *Let $\mathscr{F}$ be defined by* (3.33) *and $N\geqslant 1$ be an integer. An arbitrary linear combination of N exponential waves defined by* (4.3) *solves the trilinear equation* (4.1) *if and only if the wave variables θ_i's satisfy*

$$\begin{aligned}&\sum_{j_1,\dots,j_l=1}^{m}\sum_{i_1,\dots,i_l=1}^{n}c^{i_1,\dots,i_l}_{j_1,\dots,j_l}\sum_{(j,j',j'')\in S(i,i',i'')}\mathscr{Y}^{\overline{p}_{j_1},\dots,\overline{p}_{j_l}}_{i_1x_1,\dots,i_lx_l}(\theta_j+\theta_{j'}+\theta_{j''},\theta_j-\theta_{j'},\theta_j-\theta_{j''})\\&\quad=0,\quad 1\leqslant i\leqslant i'\leqslant i''\leqslant N,\end{aligned}\tag{4.9}$$

where $S(i,i',i'')$ *is defined by* (4.7) *and* $\mathscr{Y}^{\overline{s}_{j_1},\ldots,\overline{s}_{j_l}}_{i_1x_1,\ldots,i_lx_l}$*'s are the triple Bell polynomials defined by* (3.28)*.*

Theorem 4.2 has an advantage that the wave variables θ_i's can be nonlinear functions of dependent variables $x_1,\ldots,x_l$, but Theorem 4.1 only allows linear wave variables θ_i's.

Now, given a multivariate polynomial $\mathscr{F}$, we state one way of solving system (4.6) or (4.9) for $k_{j,i}$ and $c^{i_1,\ldots,i_l}_{j_1,\ldots,j_l}$ below, to obtain trilinear equations and their resonant solutions (see, e.g., [22,25] for the cases of bilinear equations). We adopt a kind of parametrization for wave numbers and frequencies to introduce free parameters, and list the procedure as follows.

Step 1 Introduce weights for the independent variables

$$(w(x_{j,1}),\ldots,w(x_{j,l}))=(w_1,\ldots,w_l),\quad 1\leqslant j\leqslant m, \tag{4.10}$$

where the weights w_i's can be both positive and negative.

Step 2 Form a homogeneous multivariate polynomial $\mathscr{F}$, defined by (3.33), in some weight.

Step 3 Parameterize $k_{1,i},\ldots,k_{l,i}$ using a parameter k_i:

$$k_{j,i}=b_jk_i^{w_j},\quad 1\leqslant j\leqslant l, \tag{4.11}$$

and then determine the proportional constants b_j's and the coefficients $c^{i_1,\ldots,i_l}_{j_1,\ldots,j_l}$'s by solving system (4.6) or (4.9).

4.2 Illustrative examples

To present illustrative examples, we consider the $(3+1)$-dimensional case with the dependent variables: x,y,z,t, and introduce

$$(w(x),w(y),w(z),w(t))=(w_x,w_y,w_z,w_t), \tag{4.12}$$

and

$$\left.\begin{aligned}&\theta_i=k_ix+l_iy+m_iz-\omega_it,\\ &l_i=b_1k_i^{w_y},\quad m_i=b_2k_i^{w_z},\quad \omega_i=-b_3k_i^{w_t},\end{aligned}\right\}\quad 1\leqslant i\leqslant N. \tag{4.13}$$

Then, upon forming a homogeneous multivariate polynomial in some weight

$$\mathscr{F}=\sum_{j_1,j_2,j_3,j_4=1}^{m}\ \sum_{i_1,i_2,i_3,i_4=1}^{n}c^{i_1,i_2,i_3,i_4}_{j_1,j_2,j_3,j_4}x^{i_1}_{j_1}y^{i_2}_{j_2}z^{i_3}_{j_3}t^{i_4}_{j_4}, \tag{4.14}$$

we solve system (4.6) or (4.9) for the proportional constants b_1,b_2,b_3 and the coefficients $c^{i_1,i_2,i_3,i_4}_{j_1,j_2,j_3,j_4}$'s, to determine the corresponding trilinear equation and its associated resonant solution generated by a linear superposition of exponential waves. We are now ready to present two concrete illustrative examples by applying this general scheme below.

Example 1 Example with positive weights.

Let us set the weights of independent variables

$$(w(x), w(y), w(z), w(t)) = (1, 2, 3, 4), \tag{4.15}$$

and consider a general polynomial being homogeneous in weight 6:

$$\mathscr{F} = c_1x^6 + c_2x^4y + c_3x^3z + c_4x^2t + c_5x^2y^2 + c_6yt + c_7y^3 + c_8xyz + c_9z^2. \tag{4.16}$$

Following the parametrization of wave numbers and frequencies in (4.13), we set the wave variables

$$\theta_i = k_ix + b_1k_i^2y + b_2k_i^3z + b_3k_i^3t, \quad 1 \leqslant i \leqslant N, \tag{4.17}$$

where the k_i's are arbitrary constants but the proportional constants b_1, b_2, and b_3 are to be determined by (4.6) or (4.9).

Now, a direct computation shows that the trilinear equation

$$\mathscr{F}(D_{\langle 1,2,3\rangle x}, D_{\langle 1,2,3\rangle y}, D_{\langle 1,2,3\rangle z}, D_{\langle 1,2,3\rangle t})f \cdot f \cdot f = 0 \tag{4.18}$$

possesses a resonant solution of N exponential waves, defined by

$$f = \sum_{i=1}^{N} \varepsilon_i \mathrm{e}^{\theta_i} = \sum_{i=1}^{N} \varepsilon_i \mathrm{e}^{k_ix + b_1k_i^2y + b_2k_i^3z + b_3k_i^4t}, \tag{4.19}$$

where the ε_i's and k_i's are arbitrary constants, if and only if the coefficients c_i's are determined by

$$\begin{aligned} c_1 &= -\frac{27}{368}\,c_7b_1^3 + \frac{3}{736}\,c_9b_2^2, \quad c_2 = 0, \quad c_3 = \frac{73c_7b_1^3}{184b_2} - \frac{49}{368}\,c_9b_2, \\ c_4 &= \frac{122c_7b_1^3 - 17c_9b_2^2}{184b_3}, \quad c_5 = -\frac{111}{368}\,c_7b_1 + \frac{135c_9b_2^2}{736b_1^2}, \\ c_6 &= -\frac{2c_7b_1^3 + 97c_9b_2^2}{92b_1b_3}, \quad c_8 = -\frac{542c_7b_1^3 - 71c_9b_2^2}{368b_1b_2}, \end{aligned} \tag{4.20}$$

where the coefficients c_7, c_9, and the proportional constants b_i's are arbitrary.

Example 2 Example with positive and negative weights.

Let us set the weights of independent variables

$$(w(x), w(y), w(z), w(t)) = (1, -1, 2, 3), \tag{4.21}$$

and consider a polynomial being homogeneous in weight 4:

$$\mathscr{F} = c_1x^4 + c_2x^5y + c_3x^2z + c_4xt + c_5yzt + c_6z^2 + c_7xyz^2 + c_8x^3yz. \tag{4.22}$$

Following the parametrization of wave numbers and frequencies in (4.13), we set the wave variables

$$\theta_i = k_ix + b_1k_i^{-1}y + b_2k_i^2z + b_3k_i^3t, \quad 1 \leqslant i \leqslant N, \tag{4.23}$$

where the k_i's are arbitrary constants but the proportional constants b_1, b_2, and b_3 are to be determined by (4.6) or (4.9).

Similarly, a direct computation shows that the corresponding trilinear equation

$$\mathscr{F}(D_{\langle 1,2,3\rangle x}, D_{\langle 1,2,3\rangle y}, D_{\langle 1,2,3\rangle z}, D_{\langle 1,2,3\rangle t}) f \cdot f \cdot f = 0 \tag{4.24}$$

possesses a resonant solution of N exponential waves, defined by

$$f = \sum_{i=1}^{N} \varepsilon_i \mathrm{e}^{\theta_i} = \sum_{i=1}^{N} \varepsilon_i \mathrm{e}^{k_i x + b_1 k_i^{-1} y + b_2 k_i^2 z + b_3 k_i^3 t}, \tag{4.25}$$

where the ε_i's and k_i's are arbitrary, if and only if the coefficients c_i's are determined by

$$\begin{aligned} c_1 &= \frac{1}{7} c_6 b_2^2, \quad c_2 = 0, \quad c_3 = \frac{2}{7} c_6 b_2, \\ c_4 &= -\frac{8 c_6 b_2^2}{7 b_3}, \quad c_5 = c_7 = c_8 = 0, \end{aligned} \tag{4.26}$$

where the coefficient c_6 and the proportional constants b_i's are arbitrary.

5 Conclusion and remarks

We created a class of trilinear differential $D_{\overline{p}}$-operators, explored their links with the Bell polynomials, and applied the linear superposition principle to the corresponding trilinear equations to generate resonant solutions. Two illustrative examples were made to shed light on the general theory.

We remark that the trilinear equations defined by (4.1) is a different kind of trilinear equations from (1.9), but they are characterized by the beautiful Bell polynomials. This will also bring convenience for introducing multilinear generalizations through similar differential operators. On the other hand, there are many basic questions, which need further investigation. We list a few of them below, closely related to our research interests.

(i) Parameterizations achieved by multiple parameters.

We can adopt parametrizations of $k_{1,i}, \ldots, k_{l,i}$ using multiple parameters, for example, two parameters k_i and l_i:

$$k_{j,i} = \sum_{r=0}^{w_i} b_{j,r} k_i^r l_i^{w_i - r}, \quad 1 \leqslant j \leqslant l.$$

What kind of spaces can exist for the proportional constants $b_{j,r}$ which will solve the system

$$\sum_{(j,j',j'') \in S(i,i',i'')} \mathscr{F}(\ldots, \beta_{r,s}(j,j',j''), \ldots) = 0,$$

where $\beta_{r,s}(j,j',j'')$'s are defined by (4.5) and $S(i,i',i'')$ is given by (4.7)?

(ii) Wronskian, Grammian, and Pfaffian solutions.

There are Wronskian, Grammian, and Pfaffian solutions to Hirota bilinear equations [12,13,21,24]. Do there exist the same type solutions to the trilinear equations defined by (4.1)? How can one characterize such solutions, more generally, the Plücker relation and the Jacobi identity, in terms of the Bell polynomials, even in the Hirota bilinear case?

(iii) Basic geometries related to multivariate polynomials.

What kind of geometries of a multivariate polynomial $\mathscr{F}$ does the following equation define?

$$\sum_{(j,j',j'')\in S(i,i',i'')} \mathscr{F}(\ldots,\mu_{r,s}(j,j',j''),\ldots)=0,$$

where

$$\mu_{r,s}(i,i',i'')=\alpha_{p_s}k_r+\alpha_{p'_s}k'_r+\alpha_{p''_s}k''_r,$$
$$\mu_{r,s}(i',i'',i)=\alpha_{p_s}k'_r+\alpha_{p'_s}k''_r+\alpha_{p''_s}k_r, \quad \ldots.$$

It determines an affine geometry [25] of $\mathscr{F}$ when $p_i=1$ and $p'_i=2n_i$, $n_i\in\mathbb{N}$, $1\leqslant i\leqslant l$, in the Hirota bilinear case. The corresponding trilinear equations could also describe certain differential-geometrical properties of resonant solitons [30].

(iv) Bilinear Bäcklund transformations and Lax pairs.

In the case of the Hirota bilinear D-operators, the binary Bell polynomials are used to build bilinear Bäcklund transformations for soliton equations [15]. Is there any similar theory in the cases of the general bilinear and trilinear $D_{\overline{p}}$-operators? This case could be more sophisticated than the Hirota bilinear case, since the sign function $(-1)^{r_s(i)}$ in definition (2.2) does not satisfy a decomposition property

$$(-1)^{r_s(i+j)}=(-1)^{r_s(i)+r_s(j)}, \quad i,j\geqslant 0,$$

when $s>1$ is odd, while it is true when s is one or even [20]. It is obvious that the above property holds when s is one or even; but it does not hold because we have

$$(-1)^{r_s(s)}\neq(-1)^{r_s(s-1)+r_s(1)},$$

due to

$$(-1)^{r_s(s)}=1, \quad (-1)^{r_s(s-1)}=1, \quad (-1)^{r_s(1)}=-1,$$

when $s>1$ is odd. The above property is also crucial in deriving Lax pairs of differential operators from bilinear Bäcklund transformations in the Hirota bilinear case (see, e.g., [6,16]). How about the Lax pairs in the general case of the bilinear and trilinear $D_{\overline{p}}$-operators?

(v) Hamiltonian formulations.

It is known that Hamiltonian structures of soliton equations can be furnished by applying both the variational identities [18] and the Lie algebra splittings

[31]. A natural question is how one can establish Hamiltonian structures by utilizing the Bell polynomials. There should be certain approaches like the variational identities and the Lie algebra splittings, which can be adopted to generate Poisson brackets and Hamiltonian functionals in the Hamiltonian formulations.

(vi) Criterion for multivariate polynomials with one zero.

While we used multivariate polynomials to generate Hirota bilinear equations with given resonant solutions, we came up with a fundamental and interesting question [25]: how can one determine if a multivariate polynomial $\mathscr{F}(x_1,\ldots,x_l)$ over the real field has one and only one zero in $\mathbb{R}^l$? The following are two examples of such multivariate polynomials in the case of $l=3$:

$$x^2+y^2+z^2-2x+2y-4z+6,$$
$$6x^2+35y^2+6z^2-24xy+8xz-28yz+24x-34y+8z+29,$$

which have unique zeros

$$(x,y,z)=(1,-1,2),\quad (x,y,z)=(-2,1,3),$$

respectively. Hilbert's 17th problem states that all nonnegative multivariate polynomials are sums of squares of rational functions [5]. Our problem above is more restrictive than Hilbert's 17th problem, since we can readily check that all multivariate polynomials with one zero must be either non-positive or non-negative. We expect that there would be a definitive answer to this question in the near future.

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