International Journal of Geometric Methods in Modern Physics
Vol. 23, No. 10 (2026) 2550250 (15 pages)

DOI: 10.1142/S0219887825502500

Reduced matrix mKdV integrable modes and their symmetry hierarchies

Wen-Xiu Ma

Department of Mathematics, Zhejiang Normal University
Jinhua 321004, P. R. China

Department of Mathematics, King Abdulaziz University
Jeddah 21589, Saudi Arabia

Department of Mathematics and Statistics
University of South Florida
Tampa, FL 33620-5700, USA

Material Science Innovation and Modelling
North-West University, Mafikeng Campus
Mmabatho 2735, South Africa
mawx@cas.usf.edu

Received 16 May 2025
Accepted 7 June 2025
Published 15 July 2025

This paper investigates reduced matrix modified Korteweg–de Vries (mKdV) integrable models within the framework of the zero-curvature formulation. By utilizing the Lax pairs associated with the Ablowitz–Kaup–Newell–Segur matrix spectral problems, we derive representative examples of reduced matrix mKdV integrable models along with their symmetry hierarchies. A pair of similarity transformations serves as the central mechanism in this construction, ensuring the invariance of the zero-curvature equations.

Keywords: Lax pair; matrix spectral problem; zero-curvature equation; integrable model; similarity transformation.

Mathematics Subject Classification 2020: 37K10, 35Q58, 37K40

1. Introduction

It begins with the formulation of Lax pairs [1] to construct integrable models [2], where the spectral matrices are derived from matrix Lie algebras [3]. Infinitely many commuting symmetries and conservation laws follow from the associated Lax pairs, which are connected to bi-Hamiltonian structures [4]. The inverse scattering transform can then be applied to solve their Cauchy problems [5, 6].

The matrix Ablowitz–Kaup–Newell–Segur (AKNS) spectral problems serve as a versatile framework for a wide range of integrable models, including the nonlinear

Schrödinger (NLS) equation and the modified Korteweg–de Vries (mKdV) equation. Employing a single similarity transformation enables the reduction of Lax pairs, leading to corresponding reduced integrable models [7]. Meanwhile, the use of two successive similarity transformations allows for the construction of a richer family of reduced integrable models [8]. A key difficulty lies in appropriately coordinating the reductions on the potentials induced by the two transformations, while maintaining the invariance of the associated zero-curvature equations [9].

Similarity transformations have also been extensively employed in the construction of nonlocal integrable models featuring reflection points [10]. A comprehensive classification of lower-order integrable models associated with the matrix AKNS spectral problems has identified three distinct types of nonlocal NLS equations and two types of nonlocal mKdV equations [11]. Furthermore, a variety of effective techniques have been developed for analyzing reduced integrable models and constructing their soliton solutions.

The classical inverse scattering transform continues to serve as a robust method for addressing the Cauchy problems of nonlocal integrable models [12, 13]. In addition, diverse approaches such as the Hirota bilinear method, Darboux transformation, Bäcklund transformations, and the Riemann–Hilbert method have been widely utilized. Notably, several innovative and general frameworks have been proposed to study nonlocal reductions of integrable models (see, e.g. [11, 14–17]).

In this paper, we focus on developing reduced integrable mKdV models by employing a pair of similarity transformations within the matrix AKNS spectral problem framework. Specifically, we consider one transformation with off-diagonal block matrices and the other with diagonal block matrices. Section 2 lays the groundwork by revisiting the matrix AKNS spectral problems and establishing a general scheme for performing consistent pairs of similarity transformations, along with their associated hierarchies of integrable mKdV models. In Sec. 3, we explore four illustrative examples within this framework, each employing distinct choices of block matrices to construct the pair of transformations. These examples highlight the richness and diversity of reduced matrix AKNS integrable models. Finally, we conclude with a summary of our findings and some perspectives for future research.

2. Matrix mKdV Integrable Models Via Similarity Transformations

2.1. *Revisiting the matrix AKNS integrable hierarchies*

Let m and n be two natural numbers. As usual, we define two matrix potentials, p and q, as follows:

$$p = p(x,t) = (p_{jk})_{m\times n},\ q = q(x,t) = (q_{kj})_{n\times m}, \tag{2.1}$$

and introduce the dependent variable by $u = u(p, q)$, which is a vector-valued function of p and q. For each $r \geq 0$, we associate a pair of standard matrix AKNS spectral problems:

$$-i\phi_x = U\phi, \quad -i\phi_t = V^{[r]}\phi, \tag{2.2}$$

where the Lax pairs are defined by

$$U = U(u,\lambda) = \lambda\Lambda + P, \tag{2.3}$$

and

$$V^{[r]} = V^{[r]}(u,\lambda) = \lambda^r\Omega + Q^{[r]}, \tag{2.4}$$

with

$$\Lambda = \begin{bmatrix} \alpha_1 I_m & 0 \\ 0 & \alpha_2 I_n \end{bmatrix}, \quad \Omega = \begin{bmatrix} \beta_1 I_m & 0 \\ 0 & \beta_2 I_n \end{bmatrix}, \tag{2.5}$$

and

$$P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \quad Q^{[r]} = \sum_{s=0}^{r-1} \lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}. \tag{2.6}$$

In these expressions, I_k denotes the identity matrix of size k, λ is the spectral parameter, and $\alpha_1, \alpha_2, \beta_1, \beta_2$ are two pairs of distinct arbitrary constants. Moreover, $Q^{[0]}$ is the zero matrix of order $(m+n)$. Assume further that with the initial data

$$W^{[0]} = \Omega, \tag{2.7}$$

the following Laurent series holds:

$$W = \sum_{s\geq 0} \lambda^{-s} W^{[s]} = \sum_{s\geq 0} \lambda^{-s} \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix}. \tag{2.8}$$

This series represents the unique Laurent series solution of the stationary zero-curvature equation

$$W_x = i[U, W]. \tag{2.9}$$

Such a series solution is essential for generating hierarchies of integrable models that mutually commute (see, e.g. [18, 19]).

The zero-curvature equations:

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0, \quad r \geq 0, \tag{2.10}$$

guarantee the compatibility of the two matrix spectral problems in (2.2). By utilizing the specific forms of U and $V^{[r]}$ from (2.3) and (2.4), respectively, these equations generate the matrix AKNS hierarchy of integrable models:

$$p_t = i\alpha b^{[r+1]},\ q_t = -i\alpha c^{[r+1]},\ r \geq 0, \tag{2.11}$$

where $\alpha = \alpha_1 - \alpha_2$. In the simplest case with $m = n = 1$, this reduces to the classical AKNS integrable hierarchy with scalar potentials, p and q [20]. Each system within the matrix AKNS integrable hierarchy admits a bi-Hamiltonian structure and possesses infinitely many symmetries and conserved quantities (see, e.g. [21–23]).

When $r = 2s + 1$, $s \geq 1$, the above matrix AKNS integrable hierarchy (2.11) reduces to the matrix mKdV integrable hierarchy. Specifically, for $s = 1$, we derive the first nonlinear integrable model, which corresponds to the matrix mKdV integrable equations:

$$\begin{cases} p_t = -\dfrac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_xqp), \\ q_t = -\dfrac{\beta}{\alpha^3}(q_{xxx} + 3q_xpq + 3qpq_x), \end{cases} \tag{2.12}$$

where $\beta = \beta_1 - \beta_2$. The associated Lax matrix $V^{[3]}$ is expressed as

$$V^{[3]} = \lambda^3\Omega + \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + iP_x) - \frac{\beta}{\alpha^3}(i[P, P_x] + P_{xx} + 2P^3), \tag{2.13}$$

where $I_{m,n} = \mathrm{diag}(I_m, -I_n)$. It is worth noting that many other notable examples of higher-order matrix AKNS integrable models can be similarly derived, as demonstrated in sources such as [24].

2.2. *Pairs of similarity transformations*

To introduce a consistent pair of similarity transformations, we begin by considering

$$m = n,\ \alpha_1 = -\alpha_2 = \frac{1}{2}, \quad \beta_1 = -\beta_2 = -\frac{1}{2}, \tag{2.14}$$

which results in two square potential matrices, p and q. Following this, we take two constant, invertible, and symmetric square matrices of order n, denoted Σ_1, Σ_2, along with another pair of constant, invertible, and square matrices of order n, denoted Δ_1 and Δ_2 (see, e.g. [9, 25, 26]). We then define two invertible constant square matrices of order $2n$ as follows:

$$\Delta = \begin{bmatrix} 0 & \Delta_1 \\ \Delta_2 & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \tag{2.15}$$

which provides a consistent framework for the similarity transformations.

It is easy to verify that both Δ and Σ satisfy the following similarity properties:

$$\Delta\Lambda\Delta^{-1} = -\Sigma\Lambda\Sigma^{-1} = -\Lambda,\ \Delta\Omega\Delta^{-1} = -\Sigma\Omega\Sigma^{-1} = -\Omega, \tag{2.16}$$

where Λ and Ω are defined as in (2.5). Assuming that A^T denotes the matrix transpose of a matrix A, we propose the following pair of similarity transformations:

$$\begin{aligned} \Delta U(\lambda)\Delta^{-1} &= -U^T(\lambda) = -(U(\lambda))^T, \\ \Sigma U(\lambda)\Sigma^{-1} &= -U^T(-\lambda) = -(U(-\lambda))^T, \end{aligned} \tag{2.17}$$

which are consistent with the identities in (2.16). It will be shown later that the original zero-curvature equations of the mKdV equations remain invariant under each of these similarity transformations.

The two similarity transformations naturally lead to the following relations for the potential matrix P:

$$\Delta P \Delta^{-1} = -P^T, \quad \Sigma P \Sigma^{-1} = -P^T. \tag{2.18}$$

These transformations give rise to the following pairs of constraints for the two matrix potentials p and q:

$$p^T = -\Delta_2 p \Delta_1^{-1}, \quad q^T = -\Delta_1 q \Delta_2^{-1}, \tag{2.19}$$

and

$$p^T = -\Sigma_2 q \Sigma_1^{-1}, \quad q^T = -\Sigma_1 p \Sigma_2^{-1}, \tag{2.20}$$

respectively. Since Σ is symmetric, the two constraints in (2.20) are automatically compatible. To ensure compatibility for the constraints in (2.19), we impose the following sufficient condition:

$$\Delta_1^{-1}\Sigma_1 = \Sigma_2^{-1}\Delta_2, \tag{2.21}$$

under which the two constraints in (2.19) imply each other when the constraints in (2.20) are satisfied.

Therefore, by imposing the symmetric condition on Σ and utilizing the condition specified in (2.21), the combined effect of the two similarity transformations in (2.17) yields the reduced AKNS matrix spectral problems:

$$-i\phi_x = U\phi, \quad U = \begin{bmatrix} \frac{1}{2}\lambda I_n & p \\ -\Sigma_2^{-1} p^T \Sigma_1 & -\frac{1}{2}\lambda I_n \end{bmatrix}, \tag{2.22}$$

where p must satisfy the first constraint in (2.19), i.e., $p^T = -\Delta_2 p \Delta_1^{-1}$, or equivalently, the other reduced AKNS matrix spectral problems:

$$-i\phi_x = U\phi, \ U = \begin{bmatrix} \frac{1}{2}\lambda I_n & -\Sigma_1^{-1} q^T \Sigma_2 \\ q & -\frac{1}{2}\lambda I_n \end{bmatrix}, \tag{2.23}$$

where q must satisfy the second constraint in (2.19), i.e., $q^T = -\Delta_1 q \Delta_2^{-1}$.

2.3. *Reduced matrix mKdV integrable models*

Note that the initial data have been appropriately specified as follows:

$$W^{[0]} = \Omega = \begin{bmatrix} -\frac{1}{2} I_n & 0 \\ 0 & \frac{1}{2} I_n \end{bmatrix}, \tag{2.24}$$

for the Laurent series solution W. Under the similarity transformations given in (2.17), and using the uniqueness of solutions to the stationary zero-curvature equation, we find that the solution W, determined by (2.8), satisfies

$$\begin{aligned} \Delta W(\lambda)\Delta^{-1} &= -W^T(\lambda) = -(W(\lambda))^T, \\ \Sigma W(\lambda)\Sigma^{-1} &= W^T(-\lambda) = (W(-\lambda))^T. \end{aligned} \tag{2.25}$$

This is because the two solutions within each pair share the same initial values:

$$\begin{aligned} \Delta W(\lambda)\Delta^{-1}|_{\lambda=\infty} &= -(W(\lambda))^T|_{\lambda=\infty} = -\Omega, \\ \Sigma W(\lambda)\Sigma^{-1}|_{\lambda=\infty} &= (W(-\lambda))^T|_{\lambda=\infty} = \Omega. \end{aligned} \tag{2.26}$$

Therefore, for all $s \geq 0$, we have

$$\begin{cases} \Delta V^{[2s+1]}(\lambda)\Delta^{-1} = -V^{[2s+1]T}(\lambda) = -(V^{[2s+1]}(\lambda))^T, \\ \Sigma V^{[2s+1]}(\lambda)\Sigma^{-1} = -V^{[2s+1]T}(-\lambda) = -(V^{[2s+1]}(-\lambda))^T. \end{cases} \tag{2.27}$$

Thus, we obtain

$$\Delta(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(\lambda)\Delta^{-1} = -((U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(\lambda))^T,$$

and

$$\Sigma(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(\lambda)\Sigma^{-1} = -((U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(-\lambda))^T.$$

Consequently, the matrix AKNS integrable models in (2.11) with $r = 2s+1$ reduce to the following integrable mKdV models:

$$p_t = 2ib^{[2s+2]}|_{q=-\Sigma_2^{-1}p^T\Sigma_1}, s \geq 0, \tag{2.28}$$

where p satisfies $p^T = -\Delta_2 p \Delta_1^{-1}$, or

$$q_t = -2ic^{[2s+2]}|_{p=-\Sigma_1^{-1}q^T\Sigma_2},\ s \geq 0, \tag{2.29}$$

where q satisfies $q^T = -\Delta_1 q \Delta_2^{-1}$. In our construction, only the symmetric condition of Σ and the condition in (2.21) are required.

Moreover, the matrix spectral problems, (2.22) and

$$-i\phi_t = V^{[2s+1]}|_{q=-\Sigma_2^{-1}p^T\Sigma_1}\phi, \quad s \geq 0, \tag{2.30}$$

provide Lax pairs for the reduced integrable mKdV hierarchy (2.28), or the matrix spectral problems, (2.23) and

$$-i\phi_t = V^{[2s+1]}|_{p=-\Sigma_1^{-1}q^T\Sigma_2}\phi,\ s \geq 0, \tag{2.31}$$

provide Lax pairs for the reduced integrable mKdV hierarchy (2.29).

Owing to the underlying structure of the Lax operator algebras (see, e.g. [27]), the resulting reduced integrable models admit infinitely many commuting symmetries. Notably, since Δ_1, Δ_2 and Σ_1 remain arbitrary, by suitably choosing these matrices, one can construct a rich variety of integrable mKdV models. These models illustrate concrete instances of the broader class of reduced matrix AKNS models. However, in the specific case where $r = 2s,\ s \geq 0$, the similarity properties given in (2.27) fail to apply.

3. Four Application Scenarios

In this section, we explore four distinct scenarios by selecting four sets of pairs of similarity transformations. Each scenario provides illustrative examples of reduced matrix AKNS spectral problems and mKdV integrable models. We focus on the cases where $m = n = 2$ or $m = n = 3$, with the spectral matrix given by

$$U = U(u, \lambda) = \begin{bmatrix} \frac{1}{2}\lambda I_2 & p \\ q & -\frac{1}{2}\lambda I_2 \end{bmatrix}, \tag{3.1}$$

or

$$U = U(u, \lambda) = \begin{bmatrix} \frac{1}{2}\lambda I_3 & p \\ q & -\frac{1}{2}\lambda I_3 \end{bmatrix}, \tag{3.2}$$

where p satisfied the first constraint in (2.19) and q is given by the second formula in (2.20). Note that once the three matrices Δ_1, Δ_2 and Σ_1 are chosen in formulating the two pairs of similarity transformations, the matrix Σ_2 can be determined using the condition in (2.21).

Example 3.1. Let us begin by introducing a pair of similarity transformations. We consider the following specific pairs of matrices:

$$\Delta_1 = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} -\delta_1 & 0 \\ 0 & -\delta_2 \end{bmatrix}; \quad \Sigma_1 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix},$$
$$\Sigma_2 = \begin{bmatrix} -\frac{\delta_1^2}{\sigma_1} & 0 \\ 0 & -\frac{\delta_2^2}{\sigma_2} \end{bmatrix}, \tag{3.3}$$

where $\delta_1, \delta_2, \sigma_1$ and σ_2 are arbitrary non-zero constants. In this manner, the similarity transformations in (2.17) generate the expressions for p and q:

$$p = \begin{bmatrix} p_1 & p_3 \\ \frac{\delta_1}{\delta_2}p_3 & p_2 \end{bmatrix}, \quad q = \begin{bmatrix} \frac{\sigma_1^2}{\delta_1^2}p_1 & \frac{\sigma_1\sigma_2}{\delta_1\delta_2}p_3 \\ \frac{\sigma_1\sigma_2}{\delta_2^2}p_3 & \frac{\sigma_2^2}{\delta_2^2}p_2 \end{bmatrix}. \tag{3.4}$$

It is now straightforward to observe that the corresponding reduced matrix mKdV integrable model, with $u = (p_1, p_2, p_3)^T$, is given by

$$\begin{cases} p_{1,t} = p_{1,xxx} + \dfrac{6}{\delta_1^2\delta_2^3}\big[(\delta_2^3\sigma_1^2 p_1^2 + \delta_1^2\delta_2\sigma_1\sigma_2 p_3^2)p_{1,x} \\ \qquad\qquad +\delta_1^2\sigma_2(\delta_2\sigma_1 p_1 + \delta_1\sigma_2 p_2)p_3 p_{3,x}\big], \\ p_{2,t} = p_{2,xxx} + \dfrac{6}{\delta_1\delta_2^2}\big[\delta_1\sigma_2(\sigma_2 p_2^2 + \sigma_1 p_3^2)p_{2,x} \\ \qquad\qquad +\sigma_1(\delta_2\sigma_1 p_1 + \delta_1\sigma_2 p_2)p_3 p_{3,x}\big], \\ p_{3,t} = p_{3,xxx} + \dfrac{3}{\delta_1^2\delta_2^2}\big[(\delta_2\sigma_1 p_1 + \delta_1\sigma_2 p_2)p_3(\delta_2\sigma_1 p_{1,x} + \delta_1\sigma_2 p_{2,x}) \\ \qquad\qquad +(\delta_2^2\sigma_1^2 p_1^2 + \delta_1^2\sigma_2^2 p_2^2 + 2\delta_1^2\sigma_1\sigma_2 p_3^2)p_{3,x}\big], \end{cases} \tag{3.5}$$

where $\sigma_1, \sigma_2, \delta_1$ and δ_2 are arbitrary non-zero constants.

When taking

$$\sigma_1 = \sigma_2 = 1, \quad \delta_1 = -\rho, \quad \delta_2 = -1, \tag{3.6}$$

the equations further reduce to the following mKdV integrable model:

$$\begin{cases} p_{1,t} = p_{1,xxx} + 6(p_1^2 + p_3^2)p_{1,x} + 6(p_1 + \rho p_2)p_3 p_{3,x}, \\ p_{2,t} = p_{2,xxx} + 6(p_2^2 + p_3^2)p_{2,x} + 6(\rho p_1 + p_2)p_3 p_{3,x}, \\ p_{3,t} = p_{3,xxx} + 3p_3[p_1 p_{1,x} + \rho(p_1 p_2)_x + p_2 p_{2,x}] \\ \qquad\qquad +3(p_1^2 + p_2^2 + 2p_3^2)p_{3,x}, \end{cases} \tag{3.7}$$

where $\rho = \pm 1$.

Example 3.2. Next, we explore the second scenario by choosing the following specific pairs of matrices:

$$\Delta_1 = \begin{bmatrix} 0 & \delta_1 \\ \delta_2 & 0 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} 0 & -\delta_2 \\ -\delta_1 & 0 \end{bmatrix}; \quad \Sigma_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix},$$

$$\Sigma_2 = \begin{bmatrix} 0 & -\dfrac{\delta_1\delta_2}{\sigma_1} \\ -\dfrac{\delta_1\delta_2}{\sigma_1} & 0 \end{bmatrix}, \tag{3.8}$$

where δ_1, δ_2 and σ_1 are arbitrary non-zero constants. Note that we have used off-diagonal matrices for Δ_1, Δ_2 and Σ_1, which differs from the choices made in the previous Example 3.1. Under these choices, the similarity transformations in (2.17) result in the expressions for p and q:

$$p = \begin{bmatrix} p_3 & p_1 \\ p_2 & \dfrac{\delta_1}{\delta_2} p_3 \end{bmatrix}, \quad q = \begin{bmatrix} \dfrac{\sigma_1^2}{\delta_2^2} p_3 & \dfrac{\sigma_1^2}{\delta_1 \delta_2} p_1 \\ \dfrac{\sigma_1^2}{\delta_1 \delta_2} p_2 & \dfrac{\sigma_1^2}{\delta_1 \delta_2} p_3 \end{bmatrix}. \tag{3.9}$$

Consequently, the corresponding reduced matrix mKdV integrable model, with $u = (p_1, p_2, p_3)^T$, is given by

$$\begin{cases} p_{1,t} = p_{1,xxx} + \dfrac{6\sigma_1^2}{\delta_1 \delta_2^2} \big[(\delta_2 p_1 p_2 + \delta_1 p_3^2) p_{1,x} + 2\delta_1 p_1 p_3 p_{3,x} \big], \\ p_{2,t} = p_{2,xxx} + \dfrac{6\sigma_1^2}{\delta_1 \delta_2^2} \big[(\delta_2 p_1 p_2 + \delta_1 p_3^2) p_{2,x} + 2\delta_1 p_2 p_3 p_{3,x} \big], \\ p_{3,t} = p_{3,xxx} + \dfrac{6\sigma_1^2}{\delta_1 \delta_2^2} \big[\delta_2 p_3 (p_1 p_2)_x + (\delta_2 p_1 p_2 + \delta_1 p_3^2) p_{3,x} \big], \end{cases} \tag{3.10}$$

where the constants δ_1, δ_2 and σ_1 are arbitrary but non-zero.

By taking

$$\delta_1 = -\rho, \quad \delta_2 = -1, \quad \sigma_1 = 1, \tag{3.11}$$

the equations simplify to the following mKdV integrable model:

$$\begin{cases} p_{1,t} = p_{1,xxx} + 6[(\rho p_1 p_2 + p_3^2) p_{1,x} + 2 p_1 p_3 p_{3,x}], \\ p_{2,t} = p_{2,xxx} + 6[(\rho p_1 p_2 + p_3^2) p_{2,x} + 2 p_2 p_3 p_{3,x}], \\ p_{3,t} = p_{3,xxx} + 6[\rho p_3 (p_1 p_2)_x + (\rho p_1 p_2 + p_3^2) p_{3,x}], \end{cases} \tag{3.12}$$

where $\rho = \pm 1$.

Example 3.3. Now, we examine the third scenario by selecting the following specific pairs of matrices:

$$\Delta_1 = \Delta_2 = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_1 \end{bmatrix}; \quad \Sigma_1 = \begin{bmatrix} 0 & 0 & \sigma_1 \\ 0 & \sigma_2 & 0 \\ \sigma_1 & 0 & 0 \end{bmatrix},$$

$$\Sigma_2 = \begin{bmatrix} 0 & 0 & \dfrac{\delta_1^2}{\sigma_1} \\ 0 & \dfrac{\delta_2^2}{\sigma_2} & 0 \\ \dfrac{\delta_1^2}{\sigma_1} & 0 & 0 \end{bmatrix}, \tag{3.13}$$

where $\delta_1, \delta_2, \sigma_1$ and σ_2 are arbitrary non-zero constants. Once these matrices have been set, the similarity transformations described in (2.17) lead to the explicit expressions for p and q:

$$p = \begin{bmatrix} 0 & p_2 & p_1 \\ -\dfrac{\delta_1}{\delta_2}p_2 & 0 & p_3 \\ -p_1 & -\dfrac{\delta_2}{\delta_1}p_3 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 0 & -\dfrac{\sigma_1\sigma_2}{\delta_1^2}p_3 & -\dfrac{\sigma_1^2}{\delta_1^2}p_1 \\ \dfrac{\sigma_1\sigma_2}{\delta_1\delta_2}p_3 & 0 & -\dfrac{\sigma_1\sigma_2}{\delta_2^2}p_2 \\ \dfrac{\sigma_1^2}{\delta_1^2}p_1 & \dfrac{\sigma_1\sigma_2}{\delta_1\delta_2}p_2 & 0 \end{bmatrix}. \tag{3.14}$$

Then, it is direct to see that the corresponding reduced matrix mKdV integrable model, with $u = (p_1, p_2, p_3)^T$, takes the following form:

$$\begin{cases} p_{1,t} = p_{1,xxx} + \dfrac{3\sigma_1}{\delta_1^2\delta_2}\big[2(\delta_2\sigma_1 p_1^2 + \delta_1\sigma_2 p_2p_3)p_{1,x} + \delta_1\sigma_2 p_1(p_2p_3)_x\big], \\ p_{2,t} = p_{2,xxx} + \dfrac{3\sigma_1}{\delta_1^2\delta_2}\big[\delta_2\sigma_1 p_1p_2p_{1,x} + (\delta_2\sigma_1 p_1^2 + 3\delta_1\sigma_2 p_2p_3)p_{2,x} + \delta_1\sigma_2 p_2^2 p_{3,x}\big], \\ p_{3,t} = p_{3,xxx} + \dfrac{3\sigma_1}{\delta_1^2\delta_2}\big[\delta_2\sigma_1 p_1p_3p_{1,x} + \delta_1\sigma_2 p_3^2 p_{2,x} + (\delta_2\sigma_1 p_1^2 + 3\delta_1\sigma_2 p_2p_3)p_{3,x}\big], \end{cases} \tag{3.15}$$

where $\delta_1, \delta_2, \sigma_1$ and σ_2 are arbitrary non-zero constants.

When choosing

$$\delta_1 = \rho, \quad \delta_2 = 1, \quad \sigma_1 = \sigma_2 = 1, \tag{3.16}$$

we obtain the simplified mKdV integrable model:

$$\begin{cases} p_{1,t} = p_{1,xxx} + 6(p_1^2 + \rho p_2p_3)p_{1,x} + 3\rho p_1(p_2p_3)_x, \\ p_{2,t} = p_{2,xxx} + 3\rho p_2(3p_3p_{2,x} + p_2p_{3,x}) + 3p_1(p_1p_2)_x, \\ p_{3,t} = p_{3,xxx} + 3\rho p_3(p_3p_{2,x} + 3p_2p_{3,x}) + 3p_1(p_1p_3)_x, \end{cases} \tag{3.17}$$

where $\rho = \pm 1$.

Example 3.4. Lastly, we explore the fourth scenario by defining the following specific matrix pairs:

$$\Delta_1 = \Delta_2 = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_1 \end{bmatrix}; \quad \Sigma_1 = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix},$$

$$\Sigma_2 = \begin{bmatrix} \dfrac{\delta_1^2}{\sigma_1} & 0 & 0 \\ 0 & \dfrac{\delta_2^2}{\sigma_2} & 0 \\ 0 & 0 & \dfrac{\delta_1^2}{\sigma_3} \end{bmatrix}, \tag{3.18}$$

where once again, we have δ_1, δ_2 and $\sigma_i,\ 1 \le i \le 3$, as arbitrary non-zero constants. After setting these matrices, the similarity transformations outlined in (2.17) provide the explicit expressions for p and q:

$$p = \begin{bmatrix} 0 & p_2 & p_1 \\ -\frac{\delta_1}{\delta_2}p_2 & 0 & p_3 \\ -p_1 & -\frac{\delta_2}{\delta_1}p_3 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 0 & \frac{\sigma_1\sigma_2}{\delta_1\delta_2}p_2 & \frac{\sigma_1\sigma_3}{\delta_1^2}p_1 \\ -\frac{\sigma_1\sigma_2}{\delta_2^2}p_2 & 0 & \frac{\sigma_2\sigma_3}{\delta_1\delta_2}p_3 \\ -\frac{\sigma_1\sigma_3}{\delta_1^2}p_1 & -\frac{\sigma_2\sigma_3}{\delta_1^2}p_3 & 0 \end{bmatrix}. \tag{3.19}$$

Thus, it is clear that the corresponding reduced matrix mKdV integrable model, with $u = (p_1, p_2, p_3)^T$, takes the following form:

$$\begin{cases} p_{1,t} = p_{1,xxx} - \frac{3}{\delta_1^2\delta_2^2}\big\{\big[2\delta_2^2\sigma_1\sigma_3p_1^2 + \sigma_2(\delta_1^2\sigma_1p_2^2 + \delta_2^2\sigma_3p_3^2)\big]p_{1,x} \\ \qquad\qquad +\sigma_2p_1(\delta_1^2\sigma_1p_2p_{2,x} + \delta_2^2\sigma_3p_3p_{3,x})\big\}, \\ p_{2,t} = p_{2,xxx} - \frac{3}{\delta_1^2\delta_2^2}\big\{\big[2\delta_1^2\sigma_1\sigma_2p_2^2 + \delta_2^2\sigma_3(\sigma_1p_1^2 + \sigma_2p_3^2)\big]p_{2,x} + \delta_2^2\sigma_3p_2 \\ \qquad\qquad \times(\sigma_1p_1p_{1,x} + \sigma_2p_3p_{3,x})\big\}, \\ p_{3,t} = p_{3,xxx} - \frac{3}{\delta_1^2\delta_2^2}\big\{\big[2\delta_2^2\sigma_2\sigma_3p_3^2 + \sigma_1(\delta_2^2\sigma_3p_1^2 + \delta_1^2\sigma_2p_2^2)\big]p_{3,x} \\ \qquad\qquad +\sigma_1p_3(\delta_2^2\sigma_3p_1p_{1,x} + \delta_1^2\sigma_2p_2p_{2,x})\big\}, \end{cases} \tag{3.20}$$

where δ_1, δ_2 and $\sigma_i, 1 \le i \le 3$, are arbitrary non-zero constants.

When choosing

$$\delta_1 = -\delta_2 = 1, \quad \sigma_1 = 1, \quad \sigma_2 = \rho, \quad \sigma_3 = 1, \tag{3.21}$$

we obtain the simplified mKdV integrable model:

$$\begin{cases} p_{1,t} = p_{1,xxx} - 3[2p_1^2 + \rho(p_2^2 + p_3^2)]p_{1,x} - 3\rho p_1(p_2p_{2,x} + p_3p_{3,x}), \\ p_{2,t} = p_{2,xxx} - 3[p_1^2 + \rho(2p_2^2 + p_3^2)]p_{2,x} - 3p_2(p_1p_{1,x} + \rho p_3p_{3,x}), \\ p_{3,t} = p_{3,xxx} - 3[p_1^2 + \rho(p_2^2 + 2p_3^2)]p_{3,x} - 3p_3(p_1p_{1,x} + \rho p_2p_{2,x}), \end{cases} \tag{3.22}$$

where ρ is an arbitrary non-zero constant.

It is also easy to compute the Lax matrix $V^{[3]}$, as defined by (2.13), for each of these four scenarios. This matrix represents the temporal component of the Lax pairs for the resulting reduced matrix mKdV integrable models.

4. Concluding Remarks

This paper studies a pair of distinct similarity transformations applied to the matrix AKNS spectral problems, which lead to reduced matrix mKdV integrable models. Four representative cases of these reduced matrix integrable mKdV models are constructed, together with their corresponding reduced matrix AKNS spectral problems. A primary emphasis of this work is the introduction of two suitably designed similarity transformations that yield novel mKdV integrable models, thereby extending the theoretical framework developed in earlier studies (see, e.g. [9, 25, 26]).

This research further develops a systematic framework for the formulation and in-depth analysis of integrable models. The examples presented herein highlight the versatility and richness of reduced Lax pairs in constructing integrable systems. Comparative analysis of these models with other existing frameworks could yield valuable insights into the underlying algebraic and geometric structures of various integrable systems.

By employing a variety of similarity transformations within the zero-curvature formulation, a broad class of integrable reductions can be realized (see, e.g. [28–31]). The selection of diagonal and off-diagonal block matrices in these transformations plays a crucial role in shaping the structure of the resulting systems. Moreover, these constructions contribute to the ongoing development of integrable models associated with higher-order matrix spectral problems, as explored in [32, 33, 33, 35–37].

Additionally, the investigation of intriguing solution phenomena — such as rogue waves, lump solutions, and soliton waves — within these newly developed integrable models promises to be a fruitful direction for future research (see, e.g. [38–46]). The use of similarity transformations not only enriches the theoretical landscape but also opens new avenues for the exploration of nonlinear wave phenomena, with meaningful applications in the physical and engineering sciences.

Overall, the integrable models introduced in this study offer fresh perspectives on the classification of multi-component integrable systems within the Lax pair framework. It is anticipated that these models will contribute to future advancements in both theoretical studies and practical applications across various fields of science and engineering.

Acknowledgments

The work was supported in part by the Ministry of Science and Technology of China (G2021016032L and G2023016011L) and the National Natural Science Foundation of China (12271488 and 11975145).

ORCID

Wen-Xiu Ma https://orcid.org/0000-0001-5309-1493

References

[1] P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.* **21** (1968) 467–490.

[2] A. Das, *Integrable Models* (World Scientific, Teaneck, NJ, 1989).

[3] G. Z. Tu, On Liouville integrability of zero-curvature equations and the Yang hierarchy, *J. Phys. A: Math. Gen.* **22** (1989) 2375–2392.

[4] F. Magri, A simple model of the integrable Hamiltonian equation, *J. Math. Phys.* **19** (1978) 1156–1162.

[5] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).

[6] P. G. Drazin and R. S. Johnson, *Solitons: An Introduction* (Cambridge University Press, Cambridge, 1983).

[7] W. X. Ma, Application of the Riemann–Hilbert approach to the multicomponent AKNS integrable hierarchies, *Nonlinear Anal.: Real World Appl.* **47** (2019) 1–17.

[8] W. X. Ma, Sasa-Satsuma type matrix integrable hierarchies and their Riemann–Hilbert problems and soliton solutions, *Phys. D* **446** (2023) 133672.

[9] W. X. Ma, Real reduced matrix mKdV integrable hierarchies under two local group reductions, *East Asian J. Appl. Math.* **14** (2024) 281–292.

[10] M. J. Ablowitz and Z. H. Musslimani, Integrable nonlocal nonlinear equations, *Stud. Appl. Math.* **139** (2017) 7–59.

[11] W. X. Ma, Nonlocal PT-symmetric integrable equations and related Riemann–Hilbert problems, *Partial Differ. Equ. Appl. Math.* **4** (2021) 100190.

[12] M. J. Ablowitz and Z. H. Musslimani, Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation, *Nonlinearity* **29** (2016) 915–946.

[13] W. X. Ma, Y. H. Huang and F. D. Wang, Inverse scattering transforms for non-local reverse-space matrix non-linear Schrödinger equations, *Eur. J. Appl. Math.* **33** (2022) 1062–1082.

[14] J. L. Ji and Z. N. Zhu, On a nonlocal modified Korteweg–de Vries equation: Integrability, Darboux transformation and soliton solutions, *Commun. Nonlinear Sci. Numer. Simul.* **42** (2017) 699–708.

[15] M. Gürses and A. Pekcan, Nonlocal nonlinear Schrödinger equations and their soliton solutions, *J. Math. Phys.* **59** (2018) 051501.

[16] C. Q. Song, D. Y. Liu and L. Y. Ma, Soliton solutions of a novel nonlocal Hirota system and a nonlocal complex modified Korteweg–de Vries equation, *Chaos Solitons Fractals* **181** (2024) 114707.

[17] X. Wang, D. L. Du and H. Wang, A nonlocal finite-dimensional integrable system related to the nonlocal mKdV equation, *Theor. Math. Phys.* **218** (2024) 370–387.

[18] J. Zhang, C. P. Zhang and Y. N. Cui, Bi-integrable and tri-integrable couplings of a soliton hierarchy associated with SO(3), *Adv. Math. Phys.* **2017** (2017) 9743475.

[19] Q. L. Zhao, Y. D. Zhong and X. Y. Li, Explicit solutions to a hierarchy of discrete coupling Korteweg–de Vries equations, *J. Appl. Anal. Comput.* **12** (2022) 1353–1370.

[20] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems, *Stud. Appl. Math.* **53** (1974) 249–315.

[21] H. F. Wang and Y. F. Zhang, A kind of generalized integrable couplings and their bi-Hamiltonian structure, *Int. J. Theor. Phys.* **60** (2021) 1797–1812.

[22] T. S. Liu and T. C. Xia, Multi-component generalized Gerdjikov–Ivanov integrable hierarchy and its Riemann–Hilbert problem, *Nonlinear Anal. Real World Appl.* **68** (2022) 103667.

[23] X. M. Zhu and J. B. Zhang, The integrability of a new fractional soliton hierarchy and its application, *Adv. Math. Phys.* **2022** (2022) 2200092.

[24] W. X. Ma, Matrix integrable fifth-order mKdV equations and their soliton solutions, *Chin. Phys. B* **32** (2022) 020201.

[25] W. X. Ma, Nonlocal integrable mKdV equations by two nonlocal reductions and their soliton solutions, *J. Geom. Phys.* **177** (2022) 104522.

[26] W. X. Ma, Matrix mKdV integrable hierarchies via two identical group reductions, Mathematics, 13 (2025) 1438.

[27] W. X. Ma, The algebraic structures of isospectral Lax operators and applications to integrable equations, *J. Phys. A: Math. Gen.* **25** (1992) 5329–5343.

[28] F. Yu and L. Li, Vector dark and bright soliton wave solutions and collisions for spin-1 Bose–Einstein condensate, *Nonlinear Dyn.* **87** (2017) 2697–2713.

[29] T. A. Sulaiman, U. Younas, A. Yusuf, M. Younis, M. Bilal and Shafqat-Ur-Rehman, Extraction of new optical solitons and MI analysis to three coupled Gross–Pitaevskii system in the spinor Bose–Einstein condensate, *Mod. Phys. Lett. B* **35** (2021) 2150109.

[30] R. S. Ye, Y. Zhang and W. X. Ma, Bound states of dark solitons in N-coupled complex modified Korteweg–de Vries equations, *Acta Appl. Math.* **178** (2022) 7.

[31] U. Younas, T. A. Sulaiman and J. Ren, Diversity of optical soliton structures in the spinor Bose–Einstein condensate modeled by three-component Gross–Pitaevskii system, *Int. J. Mod. Phys. B* **37** (2023) 2350004.

[32] W. X. Ma, A combined integrable hierarchy with four potentials and its recursion operator and bi-Hamiltonian structure, *Indian J. Phys.* **99** (2025) 1063–1069.

[33] W. X. Ma, A combined generalized Kaup–Newell soliton hierarchy and its hereditary recursion operator and bi-Hamiltonian structure, *Theor. Math. Phys.* **221** (2024) 1603–1614.

[34] X. G. Geng and X. Zeng, Algebro-geometric quasi-periodic solutions to the Satsuma–Hirota hierarchy, *Phys. D.* **448** (2023) 133738.

[35] X. G. Geng, M. X. Jia, B. Xue and Y. Y. Zhai, Application of tetragonal curves to coupled Boussinesq equations, *Lett. Math. Phys.* **114** (2024) 30.

[36] W. X. Ma, A matrix integrable Hamiltonian hierarchy and its two integrable reductions, *Int. J. Geom. Methods Mod. Phys.* **22** (2025) 2450293.

[37] W. X. Ma, A soliton hierarchy derived from a fourth-order matrix spectral problem possessing four fields, *Chaos Solitons Fractals* **195** (2025) 116309.

[38] H. C. Ma, Y. X. Bai and A. P. Deng, General M-lump, high-order breather, and localized interaction solutions to (2+1)-dimensional generalized Bogoyavlensky–Konopelchenko equation, *Front. Math. China* **17** (2022) 943–960.

[39] B. D. V. Campos, Characterization of rational solutions of a KdV-like equation, *Math. Comput. Simul.* **201** (2022) 396–416.

[40] J. J. Yang, S. F. Tian and Z. Q. Li, Riemann–Hilbert method and multi-soliton solutions of an extended modified Korteweg–de Vries equation with N distinct arbitrary-order poles, *J. Math. Anal. Appl.* **511** (2022) 126103.

[41] H. C. Ma, S. P. Yue, Y. D. Gao and A. P. Deng, Lump solution, breather soliton and more soliton solutions for a (2+1)-dimensional generalized Benjamin–Ono equation, *Qual. Theory Dyn. Syst.* **22** (2023) 72.

[42] M. Kaplan and N. Raza, Construction of complexiton-type solutions using bilinear form of Hirota-type, *Int. J. Nonlinear Sci. Numer. Simul.* **24** (2023) 349–357.

[43] W. X. Ma, Lump waves and their dynamics of a spatial symmetric generalized KP model, *Rom. Rep. Phys.* **76** (2024) 108.

[44] D. Gao, W. X. Ma and X. Lü, Wronskian solution, Bäcklund transformation and Painlevé analysis to a (2+1)-dimensional Konopelchenko–Dubrovsky equation, *Z. Naturforsch.* **79a** (2024) 887–895.

[45] J. Y. Chu, Y. Q. Liu and W. X. Ma, Integrability and multiple-rogue and multi-soliton wave solutions of the 3+1-dimensional Hirota–Satsuma–Ito equation, *Mod. Phys. Lett. B* **39** (2025) 2550060.

[46] L. Cheng, Y. Zhang and W. X. Ma, An extended (2+1)-dimensional modified Korteweg–de Vries–Calogero–Bogoyavlenskii–Schiff equation: Lax pair and Darboux transformation, *Commun. Theor. Phys.* **77** (2025) 035002.