Journal of Physics: Conference Series **3264** (2026) 012012 doi:10.1088/1742-6596/3264/1/012012

Integrable fourth-order nonlinear Schrödinger systems via group reductions of Lax pairs [1]

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Abstract.
This study introduces a general framework for group reductions of matrix AKNS spectral problems. These reductions are formulated to generate reduced integrable hierarchies within the zero-curvature formulation. Four pairs of real and complex group reductions are presented, successfully constructing the corresponding integrable hierarchies of reduced nonlinear Schrödinger systems. Fourth-order nonlinear Schrödinger systems are specifically derived and illustrated with concrete examples. The study particularly provides new insights into integrable systems involving reflection points.

1. Introduction

In soliton theory, Lax pairs play a central role in the study of integrable systems. A Lax pair [1] defines a linear eigenvalue problem associated with a given matrix Lie algebra, providing a systematic framework for deriving integrable equations. Solving the corresponding stationary zero-curvature condition generates hierarchies of commuting equations with remarkable integrable properties, including infinite sequences of symmetries and conserved Hamiltonian functionals [2, 3].

These Lax pairs enable the explicit construction of soliton solutions through methods such as the inverse scattering transform [4]-[6], Darboux transformations [7]-[12], the Riemann-Hilbert problem [13]-[16], dressing procedures [17, 18], and vertex operator techniques [19]-[21]. These constructions are often formulated within affine or Kac–Moody Lie algebras [22]-[24] and Hermitian symmetric spaces [25, 26].

A Lax pair consists of linear eigenvalue equations

$$\phi_x = U(u, \lambda)\phi, \ \phi_t = V(u, \lambda)\phi$$

where U, V are matrices form a loop matrix algebra and ϕ is the eigenfunction. The zero-curvature condition

$$U_t - V_x + [U, V] = 0$$

[1] This paper is contributed in celebration of Professor Metin Gürses's 60 years in Mathematical Physics.

ensures the compatibility of the spatial and temporal evolutions, producing integrable nonlinear equations for the potential u.

On the one hand, nonlocal reductions of Lax pairs have been investigated using various soliton techniques, leading to the construction of specific integrable systems (see, e.g., [27], and [28] for the inverse scattering transform and Riemann–Hilbert problems, respectively). Of particular interest, Prof. Metin Gürses and his student, Prof. Aslı Pekcan, successfully applied the Hirota bilinear method to present soliton solutions in the nonlocal setting [29, 30].

On the other hand, by appropriately choosing U and V, one can systematically construct many classical integrable systems—including the sine–Gordon and Korteweg–de Vries equations. The Lax pair formulation provides a unifying framework for generating Liouville-integrable Hamiltonian hierarchies associated with various Lie algebras, including special linear (see, e.g., [31]-[36]), special orthogonal (see, e.g., [37, 38]), and non-semisimple algebras (see, e.g., [39]-[41]). These hierarchies form the foundation of modern integrable systems theory, enabling the systematic study of soliton dynamics, Hamiltonian structures, and conservation laws.

This paper introduces a general framework for integrable reductions of matrix spectral problems associated with the general linear Lie algebra $\mathrm{gl}(m+n)$. Group reductions of Lax pairs are identified to construct the corresponding matrix Liouville integrable hierarchies through the zero-curvature formulation. Possible group reductions are classified into four categories, each containing real and complex forms. The proposed framework is applied to fourth-order AKNS nonlinear Schrödinger systems, with illustrative examples of reduced integrable systems provided. The paper concludes with a summary of the main findings and final remarks.

2. On the matrix gl$(m+n)$ integrable hierarchy

Let $m, n \in \mathbb{N}$, and consider the general linear Lie algebra

$$\mathrm{gl}(m+n,\mathbb{C}) = \left\{ \left[\begin{array}{cc} A & B \\ C & D \end{array} \right] \middle| \, A \in \mathbb{C}^{m\times m},\ D \in \mathbb{C}^{n\times n} \right\}. \tag{2.1}$$

Its associated loop algebra $\widetilde{\mathrm{gl}}(m+n)$ consists of Laurent series in the spectral parameter λ with coefficients in $\mathrm{gl}(m+n)$.

Based on this loop algebra, we define the matrix eigenvalue problem

$$\phi_x = U\phi = U(u,\lambda)\phi,\ U = \lambda\Lambda + P, \tag{2.2}$$

with

$$\Lambda = \left[\begin{array}{cc} I_m & 0 \\ 0 & -I_n \end{array} \right],\ P = \left[\begin{array}{cc} 0 & p \\ q & 0 \end{array} \right], \tag{2.3}$$

where I_k denotes the $k \times k$ identity matrix, λ is the eigenvalue parameter, and the potential u is composed of blocks $p \in \mathbb{C}^{m\times n}$ and $q \in \mathbb{C}^{n\times m}$,

$$u = (p^T, q^T)^T = \left[\begin{array}{c} p \\ q \end{array} \right]. \tag{2.4}$$

Note that the matrix Λ lies in the Cartan subalgebra of $\widetilde{\mathrm{gl}}(m+n)$. When $m = n = 1$, the spectral problem reduces to the standard scalar AKNS eigenvalue problem [31], providing a matrix generalization of the AKNS system.

The corresponding Liouville integrable hierarchy is constructed via a Laurent-series solution of the stationary zero-curvature equation

$$W_x = [U, W] \tag{2.5}$$

by

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \sum_{l\geq 0} \lambda^{-l} W^{[l]},\ a \in \mathbb{C}^{m\times m},\ d \in \mathbb{C}^{n\times n}, \tag{2.6}$$

with block components expanded as

$$a = \sum_{l\geq 0} \lambda^{-l} a^{[l]},\ b = \sum_{l\geq 0} \lambda^{-l} b^{[l]},\ c = \sum_{l\geq 0} \lambda^{-l} c^{[l]},\ d = \sum_{l\geq 0} \lambda^{-l} d^{[l]}. \tag{2.7}$$

The corresponding stationary zero-curvature equation yields

$$\begin{cases} a_x = pc - bq, \\ b_x = 2\lambda b + pd - ap, \\ c_x = -2\lambda c + qa - dq, \\ d_x = qb - cp. \end{cases} \tag{2.8}$$

From this system, the Laurent coefficients $\{a^{[l]}, b^{[l]}, c^{[l]}, d^{[l]}\}$ are determined recursively under the normalization

$$a^{[0]} = I_m,\ d^{[0]} = -I_n,\ \text{i.e., } W^{[0]} = \Lambda, \tag{2.9}$$

and the vanishing integration constant condition:

$$a^{[l]}|_{u=0} = d^{[l]}|_{u=0} = 0,\ l \geq 1. \tag{2.10}$$

The first few coefficients are:
$l = 1$:

$$b^{[1]} = p,\ c^{[1]} = q;\ a^{[1]} = 0,\ d^{[1]} = 0.$$

$l = 2$:

$$b^{[2]} = \frac{1}{2} p_x,\ c^{[2]} = -\frac{1}{2} q_x;\ a^{[2]} = -\frac{1}{2} pq,\ d^{[2]} = \frac{1}{2} qp.$$

$l = 3$:

$$b^{[3]} = \frac{1}{4}(p_{xx} - 2pqp),\ c^{[3]} = \frac{1}{4}(q_{xx} - 2qpq);\ a^{[3]} = \frac{1}{4}(pq_x - p_x q),\ d^{[3]} = \frac{1}{4}(qp_x - q_x p).$$

$l = 4$:

$$\begin{cases} b^{[4]} = \frac{1}{8}(p_{xxx} - 3pqp_x - 3p_x qp), \\ c^{[4]} = -\frac{1}{8}(q_{xxx} - 3q_x pq - 3qpq_x); \end{cases}$$

$$\begin{cases} a^{[4]} = -\frac{1}{8}(pq_{xx} - p_x q_x + p_{xx} q - 3pqpq), \\ d^{[4]} = \frac{1}{8}(qp_{xx} - q_x p_x + q_{xx} p - 3qpqp). \end{cases}$$

$l = 5$:

$$\begin{cases} b^{[5]} = \frac{1}{16}\big[p_{xxxx} - 4pqp_{xx} - 2(3p_x q + pq_x)p_x - 2(2p_{xx} q + p_x q_x + pq_{xx} - 3pqpq)p\big], \\ c^{[5]} = \frac{1}{16}\big[q_{xxxx} - 4q_{xx} pq - 2q_x(3pq_x + p_x q) - 2q(2pq_{xx} + p_x q_x + p_{xx} q - 3pqpq)\big]; \end{cases}$$

$$\begin{cases} a^{[5]} = -\frac{1}{16}\big[2p(q_x p - qp_x)q - 4p_x qpq + 4pqpq_x + p_{xxx} q - pq_{xxx} + p_x q_{xx} - p_{xx} q_x\big], \\ d^{[5]} = \frac{1}{16}\big[2q(pq_x - p_x q)p + 4q_x pqp - 4qpqp_x + qp_{xxx} - q_{xxx} p + q_{xx} p_x - q_x p_{xx}\big]. \end{cases}$$

The temporal part of the Lax pair is

$$\phi_{t_k} = V^{[k]}\phi = V^{[k]}(u,\lambda)\phi,\ V^{[k]} = (\lambda^k W)_+ := \sum_{l=0}^{k} \lambda^l W^{[k-l]},\ k \geq 0, \tag{2.11}$$

and the zero-curvature condition

$$U_{t_k} - V_x^{[k]} + [U, V^{[k]}] = 0,\ k \geq 0, \tag{2.12}$$

define sa hierarchy of integrable systems for $u = (p^T, q^T)^T$:

$$u_{t_k} = X^{[k]}(u) = \begin{bmatrix} 2b^{[k+1]} \\ -2c^{[k+1]} \end{bmatrix},\ \text{i.e., } p_{t_k} = 2b^{[k+1]},\ q_{t_k} = -2c^{[k+1]},\ k \geq 0. \tag{2.13}$$

Particular examples, are coupled systems of integrable NLS equations and mKdV equations, as well as their higher-order generalizations. $k = 2$ ((coupled NLS):

$$p_t = \frac{1}{2}(p_{xx} - 2pqp),\ q_{t_2} = -\frac{1}{2}(q_{xx} - 2qpq). \tag{2.14}$$

$k = 3$ (coupled mKdV):

$$p_t = \frac{1}{4}(p_{xxx} - 3pqp_x - 3p_xqp),\ q_{t_3} = \frac{1}{4}(q_{xxx} - 3q_xpq - 3qpq_x). \tag{2.15}$$

$k = 4$ (fourth-order NLS-type system):

$$\begin{cases} p_{t_4} = \dfrac{1}{8}[p_{xxxx} - 2(2p_{xx}qp + pq_{xx}p + 2pqp_{xx}) - 2(p_xq_xp + 3p_xqp_x + pq_xp_x) + 6pqpqp], \\ q_{t_4} = -\dfrac{1}{8}[q_{xxxx} - 2(2q_{xx}pq + qp_{xx}q + 2qpq_{xx}) - 2(q_xp_xq + 3q_xpq_x + qp_xq_x) + 6qpqpq]. \end{cases} \tag{2.16}$$

Appropriate reductions (e.g., $p(t,x) = v(it, ix),\ q(t,x) = v^*(it, ix)$) recover scalar integrable equations such as the classical NLS equation:

$$iv_t = -\frac{1}{2}(v_{xx} + 2|v|^2 v). \tag{2.17}$$

Finally, all flows in the hierarchy commute:

$$[\![X^{[k_1]}, X^{[k_2]}]\!] = X^{[k_1]\prime}(u)[X^{[k_2]}] - X^{[k_2]\prime}(u)[X^{[k_1]}] = 0,\ k_1, k_2 \geq 0, \tag{2.18}$$

ensuring the integrable structure of the hierarchy. This commutativity follows directly from the algebra of temporal spectral matrices or from the isospectral zero-curvature equations [42].

3. Group reductions for matrix spectral problems

Let Σ be a block diagonal matrix:

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \tag{3.1}$$

where $\Sigma_1 \in \mathbb{C}^{m\times m}$ and $\Sigma_2 \in \mathbb{C}^{n\times n}$. We consider group reductions which generate reduced integrable models for the fourth-order NLS system (2.16).

Journal of Physics: Conference Series **3264** (2026) 012012
doi:10.1088/1742-6596/3264/1/012012

By analyzing the structure of the Lax pair associated with system (2.16), we can identify four distinct pairs of group reductions—a total of eight group reductions—each of which leads to integrable reductions of the matrix spectral problem (see, e.g., [43, 44]).

First pair: The group reductions are given by

$$U(x,t,\lambda) = \Sigma U(x,t,\lambda)\Sigma^{-1},\ U^*(x,t,\lambda^*) = \Sigma U(x,t,\lambda)\Sigma^{-1}, \tag{3.2}$$

which lead to

$$\begin{cases} p(x,t) = \Sigma_1^{-1} p(x,t)\Sigma_2, \\ q(x,t) = \Sigma_2^{-1} q(x,t)\Sigma_1, \end{cases} \tag{3.3}$$

and

$$\begin{cases} p(x,t) = \Sigma_1^{-1} p^*(x,t)\Sigma_2, \\ q(x,t) = \Sigma_2^{-1} q^*(x,t)\Sigma_1, \end{cases} \tag{3.4}$$

respectively.

Second pair: The group reductions are given by

$$U(-x,t,-\lambda) = -\Sigma U(x,t,\lambda)\Sigma^{-1},\ U^*(-x,t,-\lambda^*) = -\Sigma U(x,t,\lambda)\Sigma^{-1}, \tag{3.5}$$

which lead to

$$\begin{cases} p(x,t) = -\Sigma_1^{-1} p(-x,t)\Sigma_2, \\ q(x,t) = -\Sigma_2^{-1} q(-x,t)\Sigma_1, \end{cases} \tag{3.6}$$

and

$$\begin{cases} p(x,t) = -\Sigma_1^{-1} p^*(-x,t)\Sigma_2, \\ q(x,t) = -\Sigma_2^{-1} q^*(-x,t)\Sigma_1, \end{cases} \tag{3.7}$$

respectively.

In the following two pairs, the reduction is expressed solely in terms of q, since p is determined accordingly.

Third pair: The group reductions are given by

$$U^T(x,-t,-\lambda) = -\Sigma U(x,t,\lambda)\Sigma^{-1},\ U^\dagger(x,-t,-\lambda^*) = -\Sigma U(x,t,\lambda)\Sigma^{-1}, \tag{3.8}$$

which lead to

$$q(x,t) = -\Sigma_2^{-1} p^T(x,-t)\Sigma_1,\ q(x,t) = -\Sigma_2^{-1} p^\dagger(x,-t)\Sigma_1, \tag{3.9}$$

respectively.

Fourth pair: The group reductions are given by

$$U^T(-x,-t,\lambda) = \Sigma U(x,t,\lambda)\Sigma^{-1},\ U^\dagger(-x,-t,\lambda^*) = \Sigma U(x,t,\lambda)\Sigma^{-1}, \tag{3.10}$$

which lead to

$$q(x,t) = \Sigma_2^{-1} p^T(-x,-t)\Sigma_1,\ q(x,t) = \Sigma_2^{-1} p^\dagger(-x,-t)\Sigma_1, \tag{3.11}$$

respectively.

Here, the symbols A^*, A^T, and $A^\dagger$ denote the complex conjugate, transpose, and conjugate transpose of a matrix A, respectively. The block matrices Σ_i, $i = 1, 2$, are assumed to be arbitrary in reductions involving no transpose, symmetric in reductions involving transpose only, and Hermitian in reductions involving conjugate transpose.

The first pair of reductions yields only local reduced integrable systems, whereas the last three reductions give rise to nonlocal integrable systems.

4. Reduced integrable systems

Consider the stationary zero-curvature equation (2.5) associated with $\Sigma U(x,t,\lambda)\Sigma^{-1}$ under the group reductions. We can determine the following properties of their solutions.

For the first pair, we have

$$W(x,t,\lambda) = \Sigma W(x,t,\lambda)\Sigma^{-1},\ W^*(x,t,\lambda) = \Sigma W(x,t,\lambda)\Sigma^{-1}. \tag{4.1}$$

For the second pair, we have

$$W(-x,t,-\lambda) = \Sigma W(x,t,\lambda)\Sigma^{-1},\ W^*(-x,t,-\lambda^*) = \Sigma W(x,t,\lambda)\Sigma^{-1}. \tag{4.2}$$

For the third pair, we have

$$W^T(x,-t,-\lambda) = \Sigma W(x,t,\lambda)\Sigma^{-1},\ W^\dagger(x,-t,-\lambda^*) = \Sigma W(x,t,\lambda)\Sigma^{-1}. \tag{4.3}$$

For the four pair, we have

$$W^T(-x,-t,\lambda) = \Sigma W(x,t,\lambda)\Sigma^{-1},\ W^\dagger(-x,-t,\lambda^*) = \Sigma W(x,t,\lambda)\Sigma^{-1}. \tag{4.4}$$

These imply the following properties for the matrices for $V^{[2l]}$ $(l \geq 1)$:

$$\begin{cases} V^{[2l]}(x,t,\lambda) = \Sigma V^{[2l]}(x,t,\lambda)\Sigma^{-1},\ V^{[2l]*}(x,t,\lambda^*) = \Sigma V^{[2l]}(x,t,\lambda)\Sigma^{-1}; \\ V^{[2l]}(-x,t,-\lambda) = \Sigma V^{[2l]}(x,t,\lambda)\Sigma^{-1},\ V^{[2l]*}(-x,t,-\lambda^*) = \Sigma V^{[2l]}(x,t,\lambda)\Sigma^{-1}; \\ V^{[2l]T}(x,-t,-\lambda) = \Sigma V^{[2l]}(x,t,\lambda)\Sigma^{-1},\ V^{[2l]\dagger}(x,-t,-\lambda^*) = \Sigma V^{[2l]}(x,t,\lambda)\Sigma^{-1}; \\ V^{[2l]T}(-x,-t,\lambda) = \Sigma V^{[2l]}(x,t,\lambda)\Sigma^{-1},\ V^{[2l]\dagger}(-x,-t,\lambda^*) = \Sigma V^{[2l]}(x,t,\lambda)\Sigma^{-1}. \end{cases} \tag{4.5}$$

Then, under the four pairs of group reductions, the zero curvature equations satisfy

$$\begin{cases} (U_t - V_x^{[2l]} + [U, V^{[2l]}])(x,t,\lambda) = \Sigma(U_t - V_x^{[2l]} + [U, V^{[2l]}])(x,t,\lambda)\Sigma^{-1}, \\ (U_t - V_x^{[2l]} + [U, V^{[2l]}])^*(x,t,\lambda^*) = \Sigma(U_t - V_x^{[2l]} + [U, V^{[2l]}])(x,t,\lambda)\Sigma^{-1}; \end{cases} \tag{4.6}$$

$$\begin{cases} (U_t - V_x^{[2l]} + [U, V^{[2l]}])(-x,t,-\lambda) = -\Sigma(U_t - V_x^{[2l]} + [U, V^{[2l]}])(x,t,\lambda)\Sigma^{-1}, \\ (U_t - V_x^{[2l]} + [U, V^{[2l]}])^*(-x,t,-\lambda) = -\Sigma(U_t - V_x^{[2l]} + [U, V^{[2l]}])(x,t,\lambda)\Sigma^{-1}; \end{cases} \tag{4.7}$$

$$\begin{cases} (U_t - V_x^{[2l]} + [U, V^{[2l]}])^T(x,-t,-\lambda) = \Sigma(U_t - V_x^{[2l]} + [U, V^{[2l]}])(x,t,\lambda)\Sigma^{-1}, \\ (U_t - V_x^{[2l]} + [U, V^{[2l]}])^\dagger(x,-t,-\lambda) = \Sigma(U_t - V_x^{[2l]} + [U, V^{[2l]}])(x,t,\lambda)\Sigma^{-1}, \end{cases} \tag{4.8}$$

$$\begin{cases} (U_t - V_x^{[2l]} + [U, V^{[2l]}])^T(-x,-t,\lambda) = -\Sigma(U_t - V_x^{[2l]} + [U, V^{[2l]}])(x,t,\lambda)\Sigma^{-1}, \\ (U_t - V_x^{[2l]} + [U, V^{[2l]}])^\dagger(-x,-t,\lambda) = -\Sigma(U_t - V_x^{[2l]} + [U, V^{[2l]}])(x,t,\lambda)\Sigma^{-1}. \end{cases} \tag{4.9}$$

Therefore, we can obtain the hierarchies of reduced integrable NLS-type systems under the four pairs of group reductions. The Lax operator theory [42] guarantees that these hierarchies consist of commuting flows, and thus, in every hierarchy, the corresponding vector fields are symmetries of each other.

We now compute several examples arising as reduced integrable systems of the fourth-order NLS-type system (2.16). In the cases of the first and second pairs of group reductions, we take $m = 1$ and $n = 2$, while in the cases of the third and fourth pairs of group reductions, m and n are arbitrary.

Journal of Physics: Conference Series **3264** (2026) 012012 doi:10.1088/1742-6596/3264/1/012012

First pair: Consider the case $m = 1$ and $n = 2$ with

$$\Sigma_1 = 1,\ \Sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{4.10}$$

Then the potential reductions yield

$$p = (p_1, p_1),\ q = (q_1, q_1)^T, \tag{4.11}$$

and

$$p = (p_1, p_1^*),\ q = (q_1, q_1^*)^T. \tag{4.12}$$

Consequently, the reduced fourth-order NLS-type system derived from (2.16) takes the form

$$\begin{cases} p_{1,t} = \dfrac{1}{8}\big(p_{1,xxxx} - 8p_1 p_{1,xx} q_1 - 4p_1^2 q_{1,xx} - 8p_1 p_{1,xx} q_1 - 4p_1 p_{1,x} q_{1,x} \\ \qquad -12p_{1,x}^2 q_1 - 4p_1 p_{1,x} q_{1,x} + 24p_1^3 q_1^2\big), \\ q_{1,t} = -\dfrac{1}{8}\big(q_{1,xxxx} - 8q_1 q_{1,xx} p_1 - 4q_1^2 p_{1,xx} - 8q_1 q_{1,xx} p_1 - 4q_1 q_{1,x} p_{1,x} \\ \qquad -12q_{1,x}^2 p_1 - 4q_1 q_{1,x} p_{1,x} + 24q_1^3 p_1^2\big), \end{cases} \tag{4.13}$$

and

$$\begin{cases} p_{1,t} = \dfrac{1}{8}\big\{p_{1,xxxx} - 2\big[2\big(p_{1,xx} q_1 + p_{1,xx}^* q_1^*\big) + \big(p_1 q_{1,xx} + p_1^* q_{1,xx}^*\big)\big]p_1 \\ \qquad -4\big(p_1 q_1 + p_1^* q_1^*\big)p_{1,xx} - 2\big(p_{1,x} q_{1,x} + p_{1,x}^* q_{1,x}^*\big)p_1 - 6\big(p_{1,x} q_1 \\ \qquad +p_{1,x}^* q_1^*\big)p_{1,x} - 2\big(p_1 q_{1,x} + p_1^* q_{1,x}^*\big)p_{1,x} + 6\big(p_1 q_1 + p_1^* q_1^*\big)^2 p_1\big\}, \\ q_{1,t} = -\dfrac{1}{8}\big\{q_{1,xxxx} - 2\big[2\big(q_{1,xx} p_1 + q_{1,xx}^* p_1^*\big) + \big(q_1 p_{1,xx} + q_1^* p_{1,xx}^*\big)\big]q_1 \\ \qquad -4\big(q_1 p_1 + q_1^* p_1^*\big)q_{1,xx} - 2\big(q_{1,x} p_{1,x} + q_{1,x}^* p_{1,x}^*\big)q_1 - 6\big(q_{1,x} p_1 \\ \qquad +q_{1,x}^* p_1^*\big)q_{1,x} - 2\big(q_1 p_{1,x} + q_1^* p_{1,x}^*\big)q_{1,x} + 6\big(q_1 p_1 + q_1^* p_1^*\big)^2 q_1\big\}. \end{cases} \tag{4.14}$$

respectively. Here, in these two examples, the potentials p_1 and q_1 are complex-valued functions. The first system corresponds to the reduction with no complex conjugates, while the second system corresponds to the complex-conjugate reduction, where each occurrence of a potential in the second component is replaced by its complex.

Second pair: Consider the case $m = 1$ and $n = 2$ with the same matrices in the group reductions as in (4.10). Then, the potentials take the forms

$$p = (p_1, -p_1(-x)),\ q = (q_1, -q_1(-x))^T, \tag{4.15}$$

for the real-reduction case, and and

$$p = (p_1, -p_1^*(-x)),\ q = (q_1, -q_1^*(-x))^T, \tag{4.16}$$

for the complex-conjugate reduction case. Under these reductions, the fourth-order NLS-type system (2.16) reduces to

$$\begin{cases} p_{1,t} = \frac{1}{8}\big\{p_{1,xxxx} - 2\big[2\big(p_{1,xx}q_1 + p_{1,xx}(-x)q_1(-x)\big) + \big(p_1 q_{1,xx} + p_1(-x)q_{1,xx}(-x)\big)\big]p_1 \\ \quad -4\big(p_1q_1 + p_1(-x)q_1(-x)\big)p_{1,xx} - 2\big(p_{1,x}q_{1,x} + p_{1,x}(-x)q_{1,x}(-x)\big)p_1 - 6\big(p_{1,x}q_1 \\ \quad -p_{1,x}(-x)q_1(-x)\big)p_{1,x} - 2\big(p_1q_{1,x} - p_1(-x)q_{1,x}(-x)\big)p_{1,x} + 6\big(p_1q_1 + p_1(-x)q_1(-x)\big)^2 p_1\big\}, \\ q_{1,t} = -\frac{1}{8}\big\{q_{1,xxxx} - 2\big[2\big(q_{1,xx}p_1 + q_{1,xx}(-x)p_1(-x)\big) + \big(q_1 p_{1,xx} + q_1(-x)p_{1,xx}(-x)\big)\big]q_1 \\ \quad -4\big(q_1p_1 + q_1(-x)p_1(-x)\big)q_{1,xx} - 2\big(q_{1,x}p_{1,x} + q_{1,x}(-x)p_{1,x}(-x)\big)q_1 - 6\big(q_{1,x}p_1 \\ \quad -q_{1,x}(-x)p_1(-x)\big)q_{1,x} - 2\big(q_1p_{1,x} - q_1(-x)p_{1,x}(-x)\big)q_{1,x} + 6\big(q_1p_1 + q_1(-x)p_1(-x)\big)^2 q_1\big\}. \end{cases} \tag{4.17}$$

Similarly, in the complex-conjugate case, the system becomes

$$\begin{cases} p_{1,t} = \frac{1}{8}\big\{p_{1,xxxx} - 2\big[2\big(p_{1,xx}q_1 + p^*_{1,xx}(-x)q^*_1(-x)\big) + \big(p_1 q_{1,xx} + p^*_1(-x)q^*_{1,xx}(-x)\big)\big]p_1 \\ \quad -4\big(p_1q_1 + p^*_1(-x)q^*_1(-x)\big)p_{1,xx} - 2\big(p_{1,x}q_{1,x} + p^*_{1,x}(-x)q^*_{1,x}(-x)\big)p_1 - 6\big(p_{1,x}q_1 \\ \quad -p^*_{1,x}(-x)q^*_1(-x)\big)p_{1,x} - 2\big(p_1q_{1,x} - p^*_1(-x)q^*_{1,x}(-x)\big)p_{1,x} + 6\big(p_1q_1 + p^*_1(-x)q^*_1(-x)\big)^2 p_1\big\}, \\ q_{1,t} = -\frac{1}{8}\big\{q_{1,xxxx} - 2\big[2\big(q_{1,xx}p_1 + q^*_{1,xx}(-x)p^*_1(-x)\big) + \big(q_1 p_{1,xx} + q^*_1(-x)p^*_{1,xx}(-x)\big)\big]q_1 \\ \quad -4\big(q_1p_1 + q^*_1(-x)p^*_1(-x)\big)q_{1,xx} - 2\big(q_{1,x}p_{1,x} + q^*_{1,x}(-x)p^*_{1,x}(-x)\big)q_1 - 6\big(q_{1,x}p_1 \\ \quad -q^*_{1,x}(-x)p^*_1(-x)\big)q_{1,x} - 2\big(q_1p_{1,x} - q^*_1(-x)p^*_{1,x}(-x)\big)q_{1,x} + 6\big(q_1p_1 + q^*_1(-x)p^*_1(-x)\big)^2 q_1\big\}. \end{cases} \tag{4.18}$$

Here, in both cases, the potentials p_1 and q_1 are complex-valued functions. The negative argument $-x$ represents a nonlocal reduction in space, while in the complex-conjugate reduction, the values at $-x$ are replaced by their complex conjugates.
Third pair: Using the potential reductions in (3.9), the reduced fourth-order NLS-type system of (2.16) becomes

$$\begin{aligned} p_t = \frac{1}{8}\big[&p_{xxxx} + 2\big(2p_{xx}\Sigma_2^{-1}p^T(x,-t)\Sigma_1 p + p\Sigma_2^{-1}p^T_{xx}(x,-t)\Sigma_1 p \\ &+2p\Sigma_2^{-1}p^T(x,-t)\Sigma_1 p_{xx}\big) + 2\big(p_x\Sigma_2^{-1}p^T_x(x,-t)\Sigma_1 p + 3p_x\Sigma_2^{-1}p^T(x,-t)\Sigma_1 p_x \\ &+p\Sigma_2^{-1}p^T_x(x,-t)\Sigma_1 p_x\big) + 6p\Sigma_2^{-1}p^T(x,-t)\Sigma_1 p\Sigma_2^{-1}p^T(x,-t)\Sigma_1 p\big], \end{aligned} \tag{4.19}$$

for the symmetric case (Σ_1, Σ_2 symmetric), and

$$\begin{aligned} p_t = \frac{1}{8}\big[&p_{xxxx} + 2\big(2p_{xx}\Sigma_2^{-1}p^\dagger(x,-t)\Sigma_1 p + p\Sigma_2^{-1}p^\dagger_{xx}(x,-t)\Sigma_1 p \\ &+2p\Sigma_2^{-1}p^\dagger(x,-t)\Sigma_1 p_{xx}\big) + 2\big(p_x\Sigma_2^{-1}p^\dagger_x(x,-t)\Sigma_1 p + 3p_x\Sigma_2^{-1}p^\dagger(x,-t)\Sigma_1 p_x \\ &+p\Sigma_2^{-1}p^\dagger_x(x,-t)\Sigma_1 p_x\big) + 6p\Sigma_2^{-1}p^\dagger(x,-t)\Sigma_1 p\Sigma_2^{-1}p^\dagger(x,-t)\Sigma_1 p\big], \end{aligned} \tag{4.20}$$

for the Hermitian case (Σ_1, Σ_2 Hermitian). Here, p is a complex matrix-valued potential, and the difference between the two examples lies in whether the matrices Σ_i are symmetric or Hermitian. The negative argument $-t$ reflects a nonlocal reduction in time, with the transpose used in the symmetric case and the Hermitian conjugate in the Hermitian case.

Journal of Physics: Conference Series **3264** (2026) 012012 doi:10.1088/1742-6596/3264/1/012012

Fourth pair: Using the potential reductions in (3.11), the reduced fourth-order NLS-type system of (2.16) becomes

$$\begin{aligned} p_t = \frac{1}{8}\big[&p_{xxxx} - 2\big(2p_{xx}\Sigma_2^{-1}p^T(-x,-t)\Sigma_1 p + p\Sigma_2^{-1}p_{xx}^T(-x,-t)\Sigma_1 p \\ &+2p\Sigma_2^{-1}p^T(-x,-t)\Sigma_1 p_{xx}\big) - 2\big(p_x\Sigma_2^{-1}p_x^T(-x,-t)\Sigma_1 p + 3p_x\Sigma_2^{-1}p^T(-x,-t)\Sigma_1 p_x \\ &+p\Sigma_2^{-1}p_x^T(-x,-t)\Sigma_1 p_x\big) + 6p\Sigma_2^{-1}p^T(-x,-t)\Sigma_1 p\Sigma_2^{-1}p^T(-x,-t)\Sigma_1 p\big], \end{aligned} \tag{4.21}$$

for the symmetric case (Σ_1, Σ_2 symmetric), and

$$\begin{aligned} p_t = \frac{1}{8}\big[&p_{xxxx} - 2\big(2p_{xx}\Sigma_2^{-1}p^\dagger(-x,-t)\Sigma_1 p + p\Sigma_2^{-1}p_{xx}^\dagger(-x,-t)\Sigma_1 p \\ &+2p\Sigma_2^{-1}p^\dagger(-x,-t)\Sigma_1 p_{xx}\big) - 2\big(p_x\Sigma_2^{-1}p_x^\dagger(-x,-t)\Sigma_1 p + 3p_x\Sigma_2^{-1}p^\dagger(-x,-t)\Sigma_1 p_x \\ &+p\Sigma_2^{-1}p_x^\dagger(-x,-t)\Sigma_1 p_x\big) + 6p\Sigma_2^{-1}p^\dagger(-x,-t)\Sigma_1 p\Sigma_2^{-1}p^\dagger(-x,-t)\Sigma_1 p\big], \end{aligned} \tag{4.22}$$

for the Hermitian case (Σ_1, Σ_2 Hermitian). Here, p s a complex matrix-valued potential. The distinction between the two cases lies in the nature of the matrices Σ_i: they are symmetric in the first case and Hermitian in the second. The negative arguments $(-x, -t)$ indicate a nonlocal reduction in space and time., while in the complex-conjugate reduction, the entries at $(-x, -t)$ are replaced by their complex conjugates.

5. Concluding remarks

In this study, by considering block-diagonal matrices Σ and analyzing the Lax pair of the fourth-order matrix NLS system (2.16), we identified four distinct pairs of group reductions—a total of eight group reductions. These reductions lead to integrable matrix models with either local or nonlocal interactions, and with potentials that are real, complex, or related by complex conjugation. The structure of Σ_1 and Σ_2 (symmetric or Hermitian) plays a key role in determining the type of reduction and the corresponding integrable system (see, e.g., [45]-[48] for other examples). Overall, these results provide a systematic framework for constructing and classifying reduced fourth-order NLS-type equations with rich mathematical and physical properties.

The use of Laurent-series solutions to solve the stationary zero-curvature equation is a crucial step in constructing these integrable models, as it reveals their underlying structure and integrability. Applying the trace identity to the matrix eigenvalue problem further elucidates the bi-Hamiltonian structures inherent in these systems. Binary Darboux transformations are then employed to generate soliton solutions from the vacuum state, providing powerful analytical tools for exploring nonlinear dynamics [4, 5]. Specific reductions of the general soliton solutions can yield breathers, kinks, anti-kinks, lumps, rogue waves, and various mixed interaction patterns [49, 50].

The concrete examples presented here illustrate explicit coupled integrable systems, demonstrating the practical applicability of the theoretical framework and highlighting the richness of their integrability structures. Similar approaches can be extended to odd-order systems in the matrix AKNS hierarchy. Moreover, under dual group reductions [43], these systems can be further reduced to specialized integrable classes, including those with more intricate reflections in space and time [44]. Although the increasing complexity of these systems poses analytical challenges, their classification and the study of soliton structures provide deep insights into the fundamental principles governing nonlinear dynamics and integrability in mathematical physics [44].

Acknowledgment The work was supported in part by the Ministry of Science and Technology of China (G2021016032L and G2023016011L) and the National Natural Science Foundation of China (12271488 and 11975145).

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