

一族具有三Hamilton结构的 发展方程及其对称*

马文秀

(应用数学系)

摘 要 本文构造了三个可逆 Hamilton 算子, 它们两两组成一个 Hamilton 对, 由此生成了一族三 Hamilton 结构的具有对合守恒密度族的可积系, 并得到了这族发展方程的对称族。

关键词 Hamilton 算子, 遗传对称, 对称, Poisson 括弧, Hamilton 系统。

0 引 言

在 Soliton 理论研究中, 拓广可积方程族是其一个主要的课题, 文献中大多是从线性特征值问题, 即从

$$\begin{cases} \varphi_x = U\varphi \\ \varphi_t = V\varphi \end{cases}$$

的零曲率方程

$$U_t - V_x + [U, V] = 0$$

出发, 得到一族 Soliton 方程^[1, 2], 并有许多方法^[2-4]可以证明这种方程族具有广义 Hamilton 结构。双 Hamilton 理论^[5]的建立为构造可积系给出了另一个重要的方向, 即先直接构造出一个 Hamilton 对 J_1, J_2 , 其中一个可逆 (不妨设 J_1 可逆), 此时 $L = J_1^{-1}J_2$ 是一个遗传对称, 如果 J, L^* (L 的共轭算子) 再满足一定的条件, 且存在初始梯度向量 $f_0(u)$ 使 $Lf_0(u)$ 仍是梯度向量, 则可得到一族可积系 $u_t = J_2 L^n f_0(u)$ ($n \geq 0$)。

本文找出了三个可逆 Hamilton 算子, 它们两两构成 Hamilton 对, 从而得到一族具有三 Hamilton 结构的发展方程。设位势 r, q 关于空间变量 x 是速降函数, 且令

$$I = D^{-1} = \frac{1}{2} \left(\int_{-\infty}^x - \int_x^{+\infty} \right) dx', \quad D = \frac{d}{dx}$$

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1 Hamilton 算子与方程族

为了从 Hamilton 对构造遗传对称, 需要下列两条不证自明的引理:

引理 1 设 f, g 是两个 x 的函数, 则

$$(fD + g)^{-1} = PIQ$$

其中

$$P = \exp\left(-I \frac{g}{f}\right), \quad Q = \frac{1}{f} \exp\left(I \frac{g}{f}\right)$$

引理 2 设 $f_1, f_2, \dots, f_N$ 是 x 的函数, 则

$$\begin{bmatrix} -f_N & -f_{N-1} & \cdots & -f_1 & D \\ & -f_{N-1} & & & \\ & \cdots & & & \\ & -f_1 & & & \\ D & & & & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & & & & I \\ & & & & P_1 \\ & & & & \vdots \\ & & & & P_{N-1} \\ I & P_1 & \cdots & P_{N-1} & P_N \end{bmatrix}$$

其中 $P_1 = If_1I, P_i = If_iI + I(f_{i-1}P_1 + f_{i-2}P_2 + \cdots + f_1P_{i-1}) \quad 2 \leq i \leq N$

令

$$H = \begin{bmatrix} -\left(\frac{1}{2}q_x + qD\right) + D & D \\ D & 0 \end{bmatrix} \quad (1)$$

$$J = \begin{bmatrix} \frac{1}{2}r_x + rD & 0 \\ 0 & D \end{bmatrix} \quad (2)$$

$$K = \begin{bmatrix} 0 & \frac{1}{2}r_x + rD \\ \frac{1}{2}r_x + rD & \frac{1}{2}q_x + qD - D \end{bmatrix} \quad (3)$$

任取实数 a, b, c , 考虑 $aH + bJ + cK$. 令

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} r \\ q \end{pmatrix}$$

则

$$aH + bJ + cK = \begin{bmatrix} -a\left(\frac{1}{2}q_x + qD\right) + aD + b\left(\frac{1}{2}r_x + rD\right) & aD + c\left(\frac{1}{2}r_x + rD\right) \\ aD + c\left(\frac{1}{2}r_x + rD\right) & bD + c\left(\frac{1}{2}q_x + qD\right) - cD \end{bmatrix}$$

$$= \begin{bmatrix} -a \left(\frac{1}{2} q_x + qD \right) + b \left(\frac{1}{2} r_x + rD \right) & c \left(\frac{1}{2} r_x + rD \right) \\ c \left(\frac{1}{2} r_x + rD \right) & c \left(\frac{1}{2} q_x + qD \right) \end{bmatrix} + \begin{bmatrix} aD & aD \\ aD & (b-c)D \end{bmatrix}$$

$$= \left(\sum_{k=1}^2 c_{ij}^k \left(\frac{1}{2} u_k^1 + u_k D \right) \right)_{2 \times 2} + \left(\sum_{m \geq 0} b_{ijm} D^m \right)_{2 \times 2}$$

其中

$$(c_{ij}^1)_{2 \times 2} = \begin{pmatrix} b & c \\ c & 0 \end{pmatrix}, (c_{ij}^2) = \begin{pmatrix} -a & 0 \\ 0 & c \end{pmatrix}$$

$$(b_{ij1})_{2 \times 2} = \begin{pmatrix} a & a \\ a & b-c \end{pmatrix}, (b_{ijm})_{2 \times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad m \neq 1 \quad m \geq 0$$

由文献[7]中的结论2与推论2不难证明, $aH+bJ+cK$ 是 Hamilton 算子, 故得:

定理 1 设 H, J, K 分别由式(1~3)确定, 则 $aH+bJ+cK$ ($\forall a, b, c \in R$) 是 Hamilton 算子, 从而 H, J, K 两两构成 Hamilton 对.

由引理 1, 2 知, H, J, K 都是可逆算子, 并由引理 2 可得:

$$H^{-1} = \begin{bmatrix} 0 & I \\ I & \frac{1}{2} I q_x I + I q - I \end{bmatrix}$$

$$L = H^{-1}J = \begin{bmatrix} 0 & 1 \\ r - \frac{1}{2} I_x & q - \frac{1}{2} I q_x - 1 \end{bmatrix} \quad (4)$$

则

$$N = L^+ = \begin{bmatrix} 0 & r + \frac{1}{2} r_x I \\ 1 & q + \frac{1}{2} q_x I - 1 \end{bmatrix} \quad (5)$$

是遗传对称^[7, 8] (或 Nijenhuis 算子^[9]), 且

$$HL = J, JL = K \quad (6)$$

由引理 1 易得:

$$J^{-1} = \begin{bmatrix} \frac{1}{\sqrt{r}} I \frac{1}{\sqrt{r}} & 0 \\ 0 & I \end{bmatrix}$$

于是

$$L^{-1} = J^{-1}H = \begin{bmatrix} -\frac{1}{2\sqrt{r}} I \frac{q_x}{\sqrt{r}} - \frac{1}{\sqrt{r}} I \frac{q}{\sqrt{r}} D + \frac{1}{\sqrt{r}} I \frac{1}{\sqrt{r}} D & \frac{1}{\sqrt{r}} I \frac{1}{\sqrt{r}} D \\ 1 & 0 \end{bmatrix} \quad (7)$$

取

$$f_0(u) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\delta H_0}{\delta u}, \quad H_0 = q$$

则
$$L_0(u) = \left[\begin{array}{c} 1 \\ \frac{1}{2}q-1 \end{array} \right] = -\frac{\delta H_1}{\delta u}, \quad H_1 = r + \frac{1}{4}q^2 - q$$

于是由 Magri 第一定理^[9]知: $L^n f_0(u)$ 都是梯度向量, 故存在 $H_n = H_n(u)$, 使

$$-\frac{\delta H_n}{\delta u} = L^n f_0(u) \quad n \geq 0 \quad (8)$$

于是方程族

$$u_t = J L^n f_0(u) \quad n \geq 1 \quad (9)$$

是一族广义 Hamilton 系统族。

2 算子 J 与 N 的耦合条件

1) 第一耦合条件

因 $JL = K$ 是 Hamilton 算子, 自然是斜对称, 故

$$JL = -(JL)^+ = -L^+ J^+ = L^+ J$$

从而

$$NJ = JN^+$$

2) 第二耦合条件

设
$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

易知

$$J\alpha = \begin{bmatrix} \frac{1}{2}r_x\alpha_1 + r\alpha_{1x} \\ \alpha_{2x} \end{bmatrix}, \quad J\beta = \begin{bmatrix} \frac{1}{2}r_x\beta_1 + r\beta_{1x} \\ \beta_{2x} \end{bmatrix}$$

下面来验证第二耦合条件:

$$\begin{aligned} \langle \alpha, N'[\varphi]J\beta \rangle - \langle \alpha, N'[J\beta]\varphi \rangle + \langle \beta, N'[J\alpha]\varphi \rangle \\ + \langle \beta, NJ'[\varphi]\alpha \rangle - \langle \beta, J'[N\varphi]\alpha \rangle = 0 \end{aligned}$$

依次计算左边各项:

$$\begin{aligned} N'[\varphi]J\beta &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} N(u + \varepsilon\varphi) J\beta \\ &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \begin{bmatrix} 0 & r + \varepsilon\varphi_1 + \frac{1}{2}(r + \varepsilon\varphi_1)_x I \\ 1 & q + \varepsilon\varphi_2 + \frac{1}{2}(q + \varepsilon\varphi_2)_x I - 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}r_x\beta_1 + r\beta_{1x} \\ \beta_{2x} \end{bmatrix} \\ &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \begin{bmatrix} (r + \varepsilon\varphi_1)\beta_{2x} + \frac{1}{2}(r + \varepsilon\varphi_1)_x\beta_2 \\ \frac{1}{2}\beta_1 r_x + r\beta_{1x} + (q + \varepsilon\varphi_2)\beta_{2x} + \frac{1}{2}(q + \varepsilon\varphi_2)_x\beta_2 - \beta_{2x} \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \varphi_1 \beta_{2x} + \frac{1}{2} \varphi_{1x} \beta_2 \\ \varphi_2 \beta_{2x} + \frac{1}{2} \varphi_{2x} \beta_2 \end{bmatrix}$$

故

$$\langle \alpha, N'[\varphi]J\beta \rangle = \int_{-\infty}^{\infty} dx \left(\alpha_1 \beta_{2x} \varphi_1 + \frac{1}{2} \alpha_1 \beta_2 \varphi_{1x} + \alpha_2 \beta_{2x} \varphi_2 + \frac{1}{2} \alpha_2 \beta_2 \varphi_{2x} \right) \quad (10)$$

同理

$$N'[J\beta]\varphi = \begin{bmatrix} \left(\frac{1}{2} \beta_1 r_x + r \beta_{1x} \right) \varphi_2 + \frac{1}{2} \left(\frac{1}{2} \beta_1 r_x + r \beta_{1x} \right)_x I \varphi_2 \\ \beta_{2x} \varphi_2 + \frac{1}{2} \beta_{2xx} I \varphi_2 \end{bmatrix}$$

$$\langle \alpha, N'[J\beta]\varphi \rangle = \int_{-\infty}^{\infty} dx \left[\alpha_1 \left(\frac{1}{2} \beta_1 r_x + r \beta_{1x} \right) \varphi_2 + \frac{1}{2} \alpha_1 \left(\frac{1}{2} \beta_1 r_x + r \beta_{1x} \right)_x I \varphi_2 \right. \\ \left. + \alpha_2 \beta_{2x} \varphi_2 + \frac{1}{2} \alpha_2 \beta_{2xx} I \varphi_2 \right] \quad (11)$$

上式交换 α 和 β 得:

$$\langle \beta, N'[J\alpha]\varphi \rangle = \int_{-\infty}^{\infty} dx \left[\left(\frac{1}{2} \alpha_1 r_x + r \alpha_{1x} \right) \beta_1 \varphi_2 + \frac{1}{2} \left(\frac{1}{2} \alpha_1 r_x + r \alpha_{1x} \right)_x \beta_1 I \varphi_2 \right. \\ \left. + \alpha_{2x} \beta_2 \varphi_2 + \frac{1}{2} \alpha_{2xx} \beta_2 I \varphi_2 \right] \quad (12)$$

$$J'[\varphi]\alpha = \begin{bmatrix} \frac{1}{2} \alpha_1 \varphi_{1x} + \alpha_{1x} \varphi_1 \\ 0 \end{bmatrix}, \quad NJ'[\varphi]\alpha = \begin{bmatrix} 0 \\ \frac{1}{2} \alpha_1 \varphi_{1x} + \alpha_{1x} \varphi_1 \end{bmatrix}$$

$$\langle \beta, NJ'[\varphi]\alpha \rangle = \int_{-\infty}^{\infty} dx \left(\frac{1}{2} \alpha_1 \beta_2 \varphi_{1x} + \alpha_{1x} \beta_2 \varphi_1 \right) \quad (13)$$

$$J'[N\varphi]\alpha = \begin{bmatrix} \frac{1}{2} \alpha_1 \left(r \varphi_2 + \frac{1}{2} r_x I \varphi_2 \right)_x + \alpha_{1x} \left(r \varphi_2 + \frac{1}{2} r_x I \varphi_2 \right) \\ 0 \end{bmatrix}$$

$$\langle \beta, J'[N\varphi]\alpha \rangle = \int_{-\infty}^{\infty} dx \left[\frac{1}{2} \alpha_1 \beta_1 \left(r \varphi_2 + \frac{1}{2} r_x I \varphi_2 \right)_x + \alpha_{1x} \beta_1 \left(r \varphi_2 + \frac{1}{2} r_x I \varphi_2 \right) \right] \quad (14)$$

于是由式(10~14)得:

$$\begin{aligned} & \langle \alpha, N'[\varphi]J\beta \rangle - \langle \alpha, N'[J\beta]\varphi \rangle + \langle \beta, N'[J\alpha]\varphi \rangle \\ & \quad + \langle \beta, NJ'[\varphi]\alpha \rangle - \langle \beta, J'[N\varphi]\alpha \rangle \\ &= \int_{-\infty}^{\infty} dx \left[\alpha_1 \beta_{2x} \varphi_1 + \frac{1}{2} \alpha_2 \beta_2 \varphi_{1x} + \alpha_2 \beta_{2x} \varphi_2 + \frac{1}{2} \alpha_2 \beta_2 \varphi_{2x} \right. \\ & \quad \left. - \alpha_1 \left(\frac{1}{2} \beta_1 r_x + r \beta_{1x} \right) \varphi_2 - \frac{1}{2} \alpha_1 \left(\frac{1}{2} \beta_1 r_x + r \beta_{1x} \right)_x I \varphi_2 - \alpha_2 \beta_{2x} \varphi_2 \right] \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}a_2\beta_{2xx}I\varphi_2 + \left(\frac{1}{2}a_1r_x + a_{1x}r\right)\beta_1\varphi_2 + \frac{1}{2}\left(\frac{1}{2}a_1r_x + ra_{1x}\right)_x\beta_1I\varphi_2 \\
& + a_{2x}\beta_2\varphi_2 + \frac{1}{2}a_{2xx}\beta_2I\varphi_2 + \frac{1}{2}a_1\beta_2\varphi_{1x} + a_{1x}\beta_2\varphi_1 \\
& - \frac{1}{2}a_1\beta_1\left(r\varphi_2 + \frac{1}{2}r_xI\varphi_2\right)_x - a_{1x}\beta_1\left(r\varphi_2 + \frac{1}{2}r_xI\varphi_2\right) \\
& = \int_{-\infty}^{\infty} dx \left\{ (a_1\beta_{2x}\varphi_1 + a_1\beta_2\varphi_{1x} + a_{1x}\beta_2\varphi_1) \right. \\
& + \left(\frac{1}{2}a_2\beta_2\varphi_{2x} - \frac{1}{2}a_{2x}\beta_{2x}I\varphi_2 + a_{2x}\beta_2\varphi_2 + \frac{1}{2}a_{2xx}\beta_2I\varphi_2 \right. \\
& + \left[-a_1\beta_{1x}r\varphi_2 - \frac{1}{2}a_1\left(\frac{1}{2}\beta_1r_x + r\beta_{1x}\right)_xI\varphi_2 \right. \\
& + \left.\frac{1}{2}\left(\frac{1}{2}a_1r_x + ra_{1x}\right)_x\beta_1I\varphi_2 - \frac{1}{2}a_1\beta_1\left(r\varphi_2 + \frac{1}{2}r_xI\varphi_2\right)_x \right. \\
& \left. \left. - \frac{1}{2}a_{1x}\beta_1r_xI\varphi_2 \right] \right\} \quad (15)
\end{aligned}$$

上式右第一项

$$\int_{-\infty}^{\infty} (a_1\beta_2\varphi_1)_x dx = 0$$

第二项

$$\begin{aligned}
& \int_{-\infty}^{\infty} dx \left(\frac{1}{2}a_2\beta_2\varphi_{2x} + \frac{1}{2}a_{2x}\beta_{2x}I\varphi_2 + \frac{1}{2}a_{2xx}\beta_2\varphi_2 \right. \\
& \left. + a_{2x}\beta_2\varphi_2 - \frac{1}{2}a_{2x}\beta_{2x}I\varphi_2 - \frac{1}{2}a_{2xx}\beta_2\varphi_2 \right) \\
& = \int_{-\infty}^{\infty} dx \left(\frac{1}{2}a_2\beta_2\varphi_{2x} + \frac{1}{2}a_{2x}\beta_{2x}\varphi_2 + \frac{1}{2}a_{2xx}\beta_2\varphi_2 \right) \\
& = \int_{-\infty}^{\infty} \left(\frac{1}{2}a_2\beta_2\varphi_2 \right)_x dx = 0
\end{aligned}$$

第三项

$$\begin{aligned}
& \int_{-\infty}^{\infty} dx \left[-a_1\beta_{1x}r\varphi_2 + \frac{1}{2}\left(\frac{1}{2}\beta_1r_x + r\beta_{1x}\right)(a_{1x}I\varphi_2 + a_1\varphi_2) \right. \\
& \left. - \frac{1}{2}\left(\frac{1}{2}a_1r_x + ra_{1x}\right)(\beta_{1x}I\varphi_2 + \beta_1\varphi_2) \right. \\
& \left. + \frac{1}{2}(a_{1x}\beta_1 + a_1\beta_{1x})\left(r\varphi_2 + \frac{1}{2}r_xI\varphi_2\right) - \frac{1}{2}a_{1x}\beta_1r_xI\varphi_2 \right] \\
& = \int_{-\infty}^{\infty} dx \left\{ \left[-a_1\beta_{1x}r + \frac{1}{2}a_1\left(\frac{1}{2}\beta_1r_x + r\beta_{1x}\right) \right. \right. \\
& \left. \left. - \frac{1}{2}\left(\frac{1}{2}a_1r_x + ra_{1x}\right)\beta_1 + \frac{1}{2}(a_{1x}\beta_1 + a_1\beta_{1x})r \right] \varphi_2 \right.
\end{aligned}$$

$$\begin{aligned}
& + \left[\frac{1}{2} \alpha_{1x} \left(\frac{1}{2} \beta_{1x} r_x + r \beta_{1x} \right) - \frac{1}{2} \left(\frac{1}{2} \alpha_{1x} r_x + r \alpha_{1x} \right) \beta_{1x} \right. \\
& \left. + \frac{1}{4} (\alpha_{1x} \beta_1 + \alpha_1 \beta_{1x}) r_x - \frac{1}{2} \alpha_{1x} \beta_1 r_x \right] I \varphi_2 \} \\
& = \int_{-\infty}^{\infty} (0 \cdot \varphi_2 + 0 \cdot I \varphi_2) dx = 0
\end{aligned}$$

由此及式(15)证得了第二耦合条件.

综合1)和2)可推出:

定理 2 由式(2)和式(5)确定的 Hamilton 算子 J 和遗传对称 N 满足第一和第二耦合条件, 即

$$NJ = JN^+ \quad (16)$$

$$\begin{aligned}
\langle \alpha, N'[J\varphi]J\beta \rangle - \langle \alpha, N'[J\beta]J\varphi \rangle + \langle \beta, N'[J\alpha]\varphi \rangle + \langle \beta, NJ'[\varphi]\alpha \rangle \\
- \langle \beta, J'[N\varphi]\alpha \rangle = 0
\end{aligned} \quad (17)$$

式中 α, β, φ 是任意的三个函数向量.

3 式(9)方程族的无穷多守恒律及对称

1) 式(9)方程族的守恒律

由于 J 和 N 满足第一耦合条件式(16), 故

$$\begin{aligned}
\{H_i, H_j\}_J &= \left\langle \frac{\delta H_i}{\delta u}, J \frac{\delta H_j}{\delta u} \right\rangle = \langle L^i f_0(u), JL^j f_0(u) \rangle \\
&= \langle L^i f_0(u), L^+ J L^{j-1} f_0(u) \rangle = \langle L^{i+1} f_0(u), J L^{j-1} f_0(u) \rangle \\
&= \left\langle \frac{\delta H_{i+1}}{\delta u}, J \frac{\delta H_{j-1}}{\delta u} \right\rangle = \{H_{i+1}, H_{j-1}\}_J.
\end{aligned}$$

由此可得

$$\{H_i, H_j\}_J = \{H_j, H_i\}_J$$

又 Poisson 括弧 $\{\cdot, \cdot\}_J$ 是反对称, 得 $\{H_i, H_j\}_J = -\{H_j, H_i\}_J$, 因此 $\{H_i, H_j\} = 0$.

于是有:

定理 3 由式(8)确定的 $\{H_i\}_{i=0}^{\infty}$ 是式(9)方程族的一族公共的关于 Poisson 括弧 $\{\cdot, \cdot\}_J$ 守恒的守恒密度.

2) 式(9)方程族的对称

易知 $u_t = J L f_0(u)$, 即为:

$$\begin{cases} r_t = \frac{1}{2} r_x \\ q_t = \frac{1}{2} q_x \end{cases}$$

由 J 和 N 满足两个耦合条件, 故由 Magri 第二定理^[9]知: N 是式(9)方程族中任一方程的强对称, 又 N 可逆, 故 N^{-1} 也是遗传对称和式(9)方程族的公共强对称^[10], 于是方程

$u_t = J L^n f_0(u)$. 有对称族 $\{N^m J L^n f_0(u)\}_{m \in \mathbb{Z}}$, 即 $\left\{ N^m \left(\frac{1}{2} u_x \right) \right\}_{m \in \mathbb{Z}}$, 容易算得:

$$N^{-1}u_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (18)$$

故

$$\left\{ N^m \left(\frac{1}{2} u_x \right) \right\}_{m \in \mathbb{Z}} = \left\{ N^m \left(\frac{1}{2} u_x \right) \right\}_{m=0}^{\infty}$$

又

$$\begin{aligned} \left[N^i \left(\frac{1}{2} u_x \right), N^j \left(\frac{1}{2} u_x \right) \right] &= \left[J - \frac{\delta H_{i+1}}{\delta u}, J - \frac{\delta H_{j+1}}{\delta u} \right] \\ &= J \frac{\delta}{\delta u} \{ H_{i+1}, H_{j+1} \}_x = 0 \quad 0 \leq i, j < +\infty \end{aligned}$$

于是得到:

定理 4 式(9)方程族具有一族公共对称 $\left\{ N^m \left(\frac{1}{2} u_x \right) \right\}_{m=0}^{\infty}$, 且

$$\left[N^i \left(\frac{1}{2} u_x \right), N^j \left(\frac{1}{2} u_x \right) \right] = 0, \quad 0 \leq i, j < +\infty$$

4 三 Hamilton 结构与拟线性性

因为 $u_i = H L^{n+1} f_0(u) = J L^n f_0(u) = K L^{n-1} f_0(u) \quad n \geq 1$, 又 H 和 J 及 K 两两构成 Hamilton 对, 故式(9)方程族具有三 Hamilton 结构. 于是有:

定理 5 式(9)方程族具有三 Hamilton 结构, 即

$$u_i = H \frac{\delta H_{n+1}}{\delta u} = J \frac{\delta H_n}{\delta u} = K \frac{\delta H_{n-1}}{\delta u} \quad n \geq 1$$

从而其守恒密度族 $\{H_i\}_{i=0}^{\infty}$ 关于 H 和 K 确定的 Poisson 括弧对合, 即

$$\{H_i, H_j\}_H = \{H_i, H_j\} = 0 \quad 0 \leq i, j < +\infty$$

容易算得:

$$\begin{aligned} L f_0(u) &= \begin{bmatrix} 1 \\ \frac{1}{2} q - 1 \end{bmatrix} \\ L^2 f_0(u) &= \begin{bmatrix} \frac{1}{2} q - 1 \\ \frac{1}{2} r + \frac{3}{8} q^2 - q + 1 \end{bmatrix} \\ L^3 f_0(u) &= \begin{bmatrix} \frac{1}{2} r + \frac{3}{8} q^2 - q + 1 \\ \frac{3}{4} r q - r + \frac{5}{16} q^3 - \frac{9}{8} q^2 + \frac{3}{2} q - 1 \end{bmatrix} \end{aligned}$$

$$L^4 f_0(u) = \left[\begin{array}{l} \frac{3}{4}rq - r + \frac{5}{16}q^3 - \frac{9}{8}q^2 + \frac{3}{2}q - 1 \\ \frac{3}{8}r^2 + \frac{15}{16}rq^2 - \frac{9}{4}rq + \frac{3}{2}r + \frac{35}{128}q^4 - \frac{5}{4}q^3 - \frac{9}{4}q^2 - 2q + 1 \end{array} \right]$$

由此按变分反问题计算公式^[11]得:

$$H_1 = r + \frac{1}{4}q^2 - q$$

$$H_2 = \frac{1}{2}rq - r + \frac{1}{8}q^3 - \frac{1}{2}q^2 + q$$

$$H_3 = \frac{1}{4}r^2 + \frac{3}{8}rq^2 - rq + r + \frac{5}{64}q^4 - \frac{3}{8}q^3 + \frac{3}{4}q^2 - q$$

$$H_4 = \frac{3}{8}r^2q - \frac{1}{2}r^2 - r + \frac{5}{16}rq^3 - \frac{9}{8}rq^2 + \frac{3}{2}rq + \frac{7}{128}q^5 \\ - \frac{5}{16}q^4 + \frac{3}{4}q^3 - q^2 + q$$

注意守恒密度 H_1, H_2, H_3, H_4 都是 r 和 q 的多项式, 又

$$J = \begin{bmatrix} \frac{1}{2}r_x + rD & 0 \\ 0 & D \end{bmatrix}$$

从易知: 式(9)方程族中第一个方程组是一个特殊的线性方程组, 但第二、三、四方程组则都是一阶拟线性方程组。我们猜想式(9)方程族的守恒密度 $H_i (i \geq 0)$ 都是位势 r 和 q 的二元多项式, 而式(9)方程族除 $n=1$ 外, 其余方程组都是一阶的拟线性方程组。

容易算出前三个方程组为:

$$\begin{cases} r_t = \frac{1}{2}r_x \\ q_t = \frac{1}{2}q_x \end{cases} \quad (n=1) \quad (19)$$

$$\begin{cases} r_t = \frac{1}{4}r_xq + \frac{1}{2}rq_x - \frac{1}{2}r_x \\ q_t = \frac{1}{2}r_x + \frac{3}{4}qq_x - q_x \end{cases} \quad (n=2) \quad (20)$$

$$\begin{cases} r_t = \frac{3}{4}rr_x + \frac{3}{16}r_xq^2 - \frac{1}{2}r_xq + \frac{1}{2}r_x + \frac{3}{4}rqq_x - rq_x \\ q_t = \frac{3}{4}r_xq + \frac{3}{4}rq_x - r_x + \frac{15}{16}q^2q_x - \frac{9}{4}qq_x + \frac{2}{3}q_x \end{cases} \quad (n=3) \quad (21)$$

关于式(20)和式(21)方程组有没有孤子解需要作进一步的探讨。

另外,因 L 可逆,故可生成具有 Hamilton 结构的逆族,即

$$u_n = J L^{-n} f_0(u) \quad n \geq 1 \quad (22)$$

但此方程族很特殊,即有:

$$L^{-1} f_0(u) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (23)$$

故逆族都为零方程组。

定理 5 说明:式(9)方程族不仅只是双 Hamilton 系统族,而且还是三 Hamilton 系统族,从而可以断定式(9)方程族还具有其他一般孤子方程所具有的共性。

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A Hierarchy of Evolution Equations with Three-Hamiltonian Structures and Its Symmetries

Ma Wenxiu

Abstract This paper introduces three Hamiltonian operators, which form a Hamiltonian pair with one another, and herefrom a hierarchy integrable systems with three Hamiltonian structures and conserved quantities in evolution is found. A series of symmetries of this hierarchy evolution equation has also been obtained.

Key words Hamiltonian operator, hereditary symmetry, symmetry, Poisson bracket, Hamiltonian system.