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A Matrix Integrable Enlargement of the DNLS Soliton Hierarchy Incorporating Two Diagonal Matrix Blocks

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Abstract

This paper presents a four-component integrable extension of the derivative nonlinear Schrödinger (DNLS) soliton hierarchy, namely, the Kaup–Newell hierarchy of soliton equations. Motivated by a general extension idea for the Kaup–Newell spectral matrix, we propose a specially constructed 4th-order matrix-valued eigenvalue problem involving four potentials and derive the corresponding integrable Hamiltonian hierarchy via the Lax pair framework. A recursion operator and a bi-Hamiltonian structure are established to demonstrate the Liouville integrability of the resulting hierarchy. As an illustrative example, we derive an integrable system of four DNLS equations, each containing two linear dispersion terms, which differs from standard integrable systems.

Keywords: matrix eigenvalue problem; zero-curvature equation; integrable hierarchy; derivate nonlinear Schrödinger equations

MSC: 37K10; 37K15; 35Q51

1. Introduction

Integrable models are typically organized into soliton hierarchies [1], endowed with hereditary recursion operators [2], and characterized by the existence of Lax pairs formulated as matrix eigenvalue problems [3]. These eigenvalue problems form the foundation of the inverse scattering transform, which offers an effective approach for solving Cauchy problems of nonlinear integrable equations. In addition, they naturally induce Hamiltonian structures that establish intrinsic connections between symmetries and conserved quantities. Owing to these profound mathematical features, integrable models have been extensively utilized in the physical and engineering sciences, particularly in water wave theory, nonlinear optics, quantum mechanics, and statistical physics.

A prototypical example is the well-known Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy [4], together with its various hierarchies of integrable couplings [5]. The theory of Lax pairs provides a fundamental framework for the investigation of exactly solvable models via the zero-curvature formulation [5–7]. Classical soliton hierarchies are typically associated with Lax pairs derived from semisimple Lie algebras, whereas integrable couplings are often generated by Lax pairs related to non-semisimple Lie algebras. This distinction motivates the exploration of new spectral problems capable of producing integrable systems. In this paper, by studying integrable extensions of Kaup–Newell-type Lax

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pairs, we develop a novel 4th-order matrix-valued eigenvalue problem and construct the corresponding integrable hierarchy of generalized Kaup–Newell models.

The zero-curvature formulation serves as a systematic and unified framework for the construction of integrable models (see [7,8] for detailed expositions). Let $u = (u_1, \cdots, u_r)^T$ denote the column vector of dependent variables, and let λ be the spectral parameter. We consider the loop algebra $\tilde{g} = g \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ associated with the matrix Lie algebra g and the loop parameter λ. An element E_0 belonging to this loop algebra is referred to as pseudo-regular if it satisfies

$$\mathrm{Im}\,\mathrm{ad}_{E_0} \oplus \mathrm{Ker}\,\mathrm{ad}_{E_0} = \tilde{g},\ [\mathrm{Ker}\,\mathrm{ad}_{E_0}, \mathrm{Ker}\,\mathrm{ad}_{E_0}] = 0, \tag{1}$$

where ad_{E_0} indicates the adjoint action of E_0 on the loop algebra. Fixing such a pseudo-regular element E_0, we choose r linearly independent elements $E_1, \cdots, E_r$ in $\tilde{g}$ and define the spatial matrix in the spectral problem

$$\mathcal{U} = \mathcal{U}(u, \lambda) = E_0(\lambda) + u_1 E_1(\lambda) + \cdots + u_r E_r(\lambda). \tag{2}$$

We then seek a Laurent-series expansion solution

$$\mathcal{W} = \sum_{k \geq 0} \lambda^{-k} \mathcal{W}^{[k]},\ \mathcal{W}^{[k]} \in g, \tag{3}$$

to the stationary zero-curvature equation

$$\mathcal{W}_x = [\mathcal{U}, \mathcal{W}] \tag{4}$$

in the underlying loop algebra $\tilde{g}$. The central task is to determine the coefficients $\mathcal{W}^{[k]}$ recursively from uitable initial data for $\mathcal{W}^{[0]}$.

In the next step, we construct an infinite sequence of temporal spectral matrices

$$\mathcal{V}^{[l]} = (\lambda^l \mathcal{W})_+ + \Delta_l = \sum_{k=0}^{l} \lambda^{l-k} \mathcal{W}^{[k]} + \Delta_l,\ l \geq 0, \tag{5}$$

where $\Delta_l \in \tilde{g}$, $l \geq 0$. These matrices form the temporal parts of the associated Lax pairs. The corresponding compatibility (zero-curvature) equations

$$\mathcal{U}_{t_l} - \mathcal{V}^{[l]}_x + [\mathcal{U}, \mathcal{V}^{[l]}] = 0,\ l \geq 0, \tag{6}$$

generate a hierarchy of integrable evolution equations

$$u_{t_l} = X^{[l]} = X^{[l]}(u),\ l \geq 0. \tag{7}$$

These equations arise as the solvability conditions of the spatial and temporal matrix eigenvalue problems

$$\varphi_x = \mathcal{U}\varphi,\ \varphi_{t_l} = \mathcal{V}^{[l]}\varphi,\ l \geq 0. \tag{8}$$

In practice, suitable choices of the correction terms Δ_l are often determined through a trial-and-error procedure.

Finally, we endow the hierarchy (7) with a bi-Hamiltonian structure. This is carried out through the derivation of a common recursion operator and the application of the trace identity

$$\frac{\delta}{\delta u} \int \mathrm{tr}\big(\mathcal{W} \frac{\partial \mathcal{U}}{\partial \lambda}\big)\, dx = \lambda^{-\kappa} \frac{\partial}{\partial \lambda} \lambda^{\kappa} \mathrm{tr}\big(\mathcal{W} \frac{\partial \mathcal{U}}{\partial u}\big), \tag{9}$$

where $\frac{\delta}{\delta u}$ stands for the functional derivative with respect to the potential vector u, and the constant κ, independent of λ, is given by

$$\kappa = -\frac{\lambda}{2}\frac{\partial}{\partial\lambda}\ln|\mathrm{tr}(\mathcal{W}^2)|. \tag{10}$$

As a consequence, each equation in the hierarchy admits a bi-Hamiltonian structure, thereby establishing its Liouville integrability (see, e.g., [7–9]).

A wide variety of Liouville-integrable hierarchies with different numbers of dependent variables have been extensively investigated in the literature (see, e.g., [4–18]). Representative single-component hierarchies include the NLS hierarchy, the KdV hierarchy, and the mKdV hierarchy [1]. The case of two components has been studied in particular depth, with representative examples including the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy [4], the Kaup–Newell hierarchy [19,20], the Heisenberg integrable hierarchy [21], and the Wadati–Konno–Ichikawa hierarchy [22]. All of these hierarchies of integrable flows are generated from 2×2 spectral matrices. By contrast, the construction of integrable hierarchies based on higher-dimensional spectral matrices is considerably more challenging and remains an active area of research (see, for example, [23]).

In this paper, motivated by a general extension idea, we propose a four-component integrable extension of the derivative nonlinear Schrödinger (DNLS) soliton hierarchy, namely the Kaup–Newell soliton hierarchy. Specifically, we introduce a novel 4×4 matrix in the spectral problem and generate a hierarchy of Liouville-integrable four-component dynamical systems within the zero-curvature framework. By exploiting the relations among the associated vector fields, a recursion operator is derived, and the trace identity is employed to establish a bi-Hamiltonian framework, thereby confirming the Liouville integrability of the resulting hierarchy. As an illustrative example, we obtain a class of generalized coupled integrable DNLS equations with four components. Concluding remarks are provided in the final section of the paper.

2. An Integrable Extension Featuring Two Diagonal Blocks

2.1. An Extension Framework

We consider multi-component extensions of the Kaup–Newell spectral matrix of the following block form:

$$\mathcal{U} = \begin{bmatrix} \lambda^2\Lambda_1 & \lambda p \\ \lambda q & \lambda^2\Lambda_2 \end{bmatrix}, \tag{11}$$

where Λ_1 and Λ_2 are two prescribed constant square matrices, and p and q are matrix-valued potentials, not necessarily square. To ensure that there exists a Laurent-series expansion solution

$$\mathcal{W} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{12}$$

partitioned compatibly with the spectral matrix $\mathcal{U}$, to the stationary zero-curvature Equation (4), it is necessary to impose the following conditions:

$$[A, \Lambda_1] = 0,\ [D, \Lambda_2] = 0,\ [A, pq] = 0,\ [D, qp] = 0. \tag{13}$$

The last two conditions do not arise in the AKNS-type framework and constitute a major source of difficulty in constructing integrable hierarchies of Kaup–Newell type. To overcome this obstacle, we consider specific block structures for the matrices involved, which permit the recursive construction of solutions to the stationary zero-curvature Equation (4) in Laurent series form.

From now on, we restrict our attention to the case where p and q are square matrices and Λ_1 and Λ_2 are diagonal matrix blocks.

2.2. A Four-Component Integrable Extension

Let $\alpha_1, \alpha_2 \in \mathbb{R}$ satisfy

$$\alpha = \alpha_1 - \alpha_2 \neq 0, \tag{14}$$

and let $u = u(x,t) = (u_1, u_2, u_3, u_4)^T$ be a vector of four potentials. Define two diagonal matrices

$$\Lambda_1 = \operatorname{diag}(\alpha_1, \alpha_1), \; \Lambda_2 = \operatorname{diag}(\alpha_2, \alpha_2), \tag{15}$$

and two 2×2 potential matrices

$$p = \begin{bmatrix} u_1 & u_2 \\ -u_2 & -u_1 \end{bmatrix}, \; q = \begin{bmatrix} u_3 & u_4 \\ -u_4 & -u_3 \end{bmatrix}. \tag{16}$$

Then, the generalized Kaup–Newell spectral problem reads

$$\varphi_x = \mathcal{U}\varphi = \mathcal{U}(u,\lambda)\varphi, \; \mathcal{U} = \begin{bmatrix} \alpha_1\lambda^2 & 0 & \lambda u_1 & \lambda u_2 \\ 0 & \alpha_1\lambda^2 & -\lambda u_2 & -\lambda u_1 \\ \lambda u_3 & \lambda u_4 & \alpha_2\lambda^2 & 0 \\ -\lambda u_4 & -\lambda u_3 & 0 & \alpha_2\lambda^2 \end{bmatrix}, \tag{17}$$

with spectral parameter λ. This spectral matrix $\mathcal{U}$ represents a generalization of the standard Kaup–Newell eigenvalue problem [19]. This specific eigenvalue problem provides an opportunity to successfully construct an associated integrable hierarchy of bi-Hamiltonian Kaup–Newell-type equations, all of which feature characteristic combined structures of dispersive terms.

We look for a matrix-valued solution $\mathcal{W}$ to the stationary zero-curvature Equation (4) in the form

$$\mathcal{W} = \begin{bmatrix} a & f & -e & -b \\ f & a & b & e \\ g & c & -a & f \\ -c & -g & f & -a \end{bmatrix} = \sum_{k\geq 0} \lambda^{-k}\mathcal{W}^{[k]}. \tag{18}$$

Substituting this ansatz into the stationary zero-curvature Equation (4) yields the following equivalent system:

$$\begin{cases} a_x = \lambda g u_1 - \lambda c u_2 + \lambda e u_3 - \lambda b u_4, \\ b_x = \alpha\lambda^2 b + 2\lambda a u_2 - 2\lambda f u_1, \\ c_x = -\alpha\lambda^2 c + 2\lambda a u_4 + 2\lambda f u_3, \end{cases} \tag{19}$$

and

$$\begin{cases} e_x = \alpha\lambda^2 e - 2\lambda f u_2 + 2\lambda a u_1, \\ g_x = -\alpha\lambda^2 g + 2\lambda f u_4 + 2\lambda a u_3, \\ f_x = \lambda c u_1 - \lambda g u_2 - \lambda b u_3 + \lambda e u_4. \end{cases} \tag{20}$$

From (19) and (20), one can directly verify the following identities:

$$\begin{cases} -\alpha\lambda f_x = u_3 b_x + u_1 c_x - u_4 e_x - u_2 g_x, \\ -\alpha\lambda a_x = u_4 b_x - u_2 c_x - u_3 e_x + u_1 g_x, \end{cases} \tag{21}$$

which play a key role in deriving the recursion relations. We further assume that the entries of $\mathcal{W}$ allow Laurent expansions with respect to λ of the following particular form

$$\begin{cases} a = \sum_{n\geq 0} \lambda^{-2k} a^{[k]},\ b = \sum_{k\geq 0} \lambda^{-2k-1} b^{[k]},\ c = \sum_{k\geq 0} \lambda^{-2k-1} c^{[k]}, \\ e = \sum_{k\geq 0} \lambda^{-2k-1} e^{[k]},\ f = \sum_{k\geq 0} \lambda^{-2k} f^{[k]},\ g = \sum_{k\geq 0} \lambda^{-2k-1} g^{[k]}. \end{cases} \tag{22}$$

Comparing powers of λ in Equations (19) and (20) leads to the initial relations

$$\begin{cases} a_x^{[0]} = u_1 g^{[0]} - u_2 c^{[0]} + u_3 e^{[0]} - u_4 b^{[0]}, \\ f_x^{[0]} = u_1 c^{[0]} - u_2 g^{[0]} - u_3 b^{[0]} + u_4 e^{[0]}, \end{cases} \tag{23}$$

and the recursion relations for $k \geq 0$:

$$\begin{cases} a_x^{[k+1]} = -\frac{1}{\alpha}(u_4 b_x^{[k]} - u_2 c_x^{[k]} - u_3 e_x^{[k]} + u_1 g_x^{[k]}), \\ f_x^{[k+1]} = -\frac{1}{\alpha}(u_3 b_x^{[k]} + u_1 c_x^{[k]} - u_4 e_x^{[k]} - u_2 g_x^{[k]}), \end{cases} \tag{24}$$

$$\begin{cases} b^{[k+1]} = \frac{1}{\alpha}(b_x^{[k]} - 2u_2 a^{[k+1]} + 2u_1 f^{[k+1]}), \\ c^{[k+1]} = \frac{1}{\alpha}(-c_x^{[k]} + 2u_4 a^{[k+1]} + 2u_3 f^{[k+1]}), \end{cases} \tag{25}$$

and

$$\begin{cases} e^{[k+1]} = \frac{1}{\alpha}(e_x^{[k]} + 2u_2 f^{[k+1]} - 2u_1 a^{[k+1]}), \\ g^{[k+1]} = \frac{1}{\alpha}(-g_x^{[k]} + 2u_4 f^{[k+1]} + 2u_3 a^{[k+1]}). \end{cases} \tag{26}$$

These relations determine the Laurent-series solution $\mathcal{W}$ and thus generate the associated integrable hierarchy.

The uniqueness of the Laurent-series solution can be guaranteed via prescribing the combined initial data for the first coefficients as

$$\begin{cases} b^{[0]} = \beta u_1 - \gamma u_2,\ c^{[0]} = \beta u_3 + \gamma u_4, \\ e^{[0]} = \beta u_2 - \gamma u_1,\ g^{[0]} = \beta u_4 + \gamma u_3, \end{cases} \tag{27}$$

where β and γ are two arbitrary constants. It follows from (19) and (20) that

$$\begin{cases} \alpha b^{[0]} + 2a^{[0]} u_2 - 2f^{[0]} u_1 = 0,\ -\alpha c^{[0]} + 2a^{[0]} u_4 + 2f^{[0]} u_3 = 0, \\ \alpha e^{[0]} - 2f^{[0]} u_2 + 2a^{[0]} u_1 = 0, -\alpha g^{[0]} + 2f^{[0]} u_4 + 2a^{[0]} u_3 = 0, \end{cases}$$

from which one obtains

$$a^{[0]} = \frac{\alpha}{2}\gamma,\ f^{[0]} = \frac{\alpha}{2}\beta. \tag{28}$$

Additionally, to fix higher-order coefficients consistently, we impose the normalization conditions

$$a^{[k]}|_{u=0} = 0,\ f^{[k]}|_{u=0} = 0,\ k \geq 1. \tag{29}$$

These normalization conditions, together with the prescribed initial choices, guarantee that the recursion relations uniquely determine all subsequent Laurent coefficients $\mathcal{W}^{[k]}$. Different values of β and γ produce different integrable hierarchies, whereas the specific values of $a^{[0]}$ and $f^{[0]}$ do not affect the resulting hierarchies.

By means of the recursion relations derived above, one can compute the first nontrivial set of coefficients explicitly. Specifically, we obtain

$$\begin{cases} a^{[1]} = -\frac{1}{\alpha}[(\beta u_1 - \gamma u_2)u_4 + (\gamma u_1 - \beta u_2)u_3], \\ f^{[1]} = -\frac{1}{\alpha}[(\gamma u_1 - \beta u_2)u_4 + (\beta u_1 - \gamma u_2)u_3], \end{cases}$$

$$\begin{cases} b^{[1]} = \frac{1}{\alpha}\{\beta u_{1,x} - \gamma u_{2,x} - \frac{2}{\alpha}[(\beta u_3 + \gamma u_4)u_1 - (\gamma u_3 + \beta u_4)u_2]u_1 \\ \qquad + \frac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 + \gamma u_4)u_2]u_2\}, \\ c^{[1]} = \frac{1}{\alpha}\{-\beta u_{3,x} - \gamma u_{4,x} - \frac{2}{\alpha}[(\beta u_3 + \gamma u_4)u_1 - (\gamma u_3 + \beta u_4)u_2]u_3 \\ \qquad - \frac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 + \gamma u_4)u_2]u_4\}, \end{cases}$$

and

$$\begin{cases} e^{[1]} = \frac{1}{\alpha}\{-\gamma u_{1,x} + \beta u_{2,x} + \frac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 + \gamma u_4)u_2]u_1 \\ \qquad - \frac{2}{\alpha}[(\beta u_3 + \gamma u_4)u_1 - (\gamma u_3 + \beta u_4)u_2]u_2\}, \\ g^{[1]} = \frac{1}{\alpha}\{-\gamma u_{3,x} - \beta u_{4,x} - \frac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 + \gamma u_4)u_2]u_3 \\ \qquad - \frac{2}{\alpha}[(\beta u_3 + \gamma u_4)u_1 - (\gamma u_3 + \beta u_4)u_2]u_4\}. \end{cases}$$

These recursion relations enable us to introduce the temporal matrix eigenvalue problems:

$$\varphi_{t_l} = \mathcal{V}^{[l]}\varphi = \mathcal{V}^{[l]}(u,\lambda)\varphi,\ \mathcal{V}^{[l]} = \lambda(\lambda^{2l+1}\mathcal{W})_+,\ l \geq 0, \tag{30}$$

where the subscript $+$ indicates the polynomial part of the expansion in λ. The compatibility conditions between the spatial and temporal matrix eigenvalue problems (17) and (30) recover precisely the zero-curvature equations given in (6). As a consequence, these compatibility (zero-curvature) equations generate the four-potential hierarchy of integrable equations:

$$u_{t_l} = X^{[l]} = X^{[l]}(u) = (-e_x^{[l]}, -b_x^{[l]}, g_x^{[l]}, c_x^{[l]})^T,\ l \geq 0. \tag{31}$$

Equivalently, component-wise,

$$u_{1,t_l} = -e_x^{[l]},\ u_{2,t_l} = -b_x^{[l]},\ u_{3,t_l} = g_x^{[l]},\ u_{4,t_l} = c_x^{[l]},\ l \geq 0. \tag{32}$$

2.3. An Illustrative Example

The first nontrivial nonlinear member of this hierarchy corresponds to $l = 1$ and yields a system of combined integrable DNLS equations:

$$\begin{cases} u_{1,t_1} = \frac{1}{\alpha}(\gamma u_{1,xx} - \beta u_{2,xx}) - \frac{2}{\alpha^2}\{[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 + \gamma u_4)u_2]u_1\}_x \\ \qquad + \frac{2}{\alpha^2}\{[(\beta u_3 + \gamma u_4)u_1 - (\gamma u_3 + \beta u_4)u_2]u_2\}_x, \\ u_{2,t_1} = -\frac{1}{\alpha}(\beta u_{1,xx} - \gamma u_{2,xx}) + \frac{2}{\alpha^2}\{[(\beta u_3 + \gamma u_4)u_1 - (\gamma u_3 + \beta u_4)u_2]u_1\}_x \\ \qquad - \frac{2}{\alpha^2}\{[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 + \gamma u_4)u_2]u_2\}_x, \\ u_{3,t_1} = -\frac{1}{\alpha}(\gamma u_{3,xx} + \beta u_{4,xx}) - \frac{2}{\alpha^2}\{[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 + \gamma u_4)u_2]u_3\}_x \\ \qquad - \frac{2}{\alpha^2}\{[(\beta u_3 + \gamma u_4)u_1 - (\gamma u_3 + \beta u_4)u_2]u_4\}_x, \\ u_{4,t_1} = -\frac{1}{\alpha}(\beta u_{3,xx} + \gamma u_{4,xx}) - \frac{2}{\alpha^2}\{[(\beta u_3 + \gamma u_4)u_1 - (\gamma u_3 + \beta u_4)u_2]u_3\}_x \\ \qquad - \frac{2}{\alpha^2}\{[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 + \gamma u_4)u_2]u_4\}_x. \end{cases} \tag{33}$$

This system defines a combined coupled integrable model with four potentials, extending the class of coupled integrable DNLS equations (see, e.g., [24–29]). It is worth noting that each equation contains a linear combination of two second-order dispersion terms, and this distinctive feature justifies the terminology "combined models".

Two special reductions, corresponding to $\beta = 0$ or $\gamma = 0$, lead to uncoupled dispersive terms while preserving integrability.

For example, choosing $\alpha = \beta = 1$ and $\gamma = 0$ in (33) gives the coupled integrable DNLS system

$$\begin{cases} u_{1,t_1} = -u_{2,xx} - 2[(u_1u_4 - u_2u_3)u_1 - (u_1u_3 - u_2u_4)u_2]_x, \\ u_{2,t_1} = -u_{1,xx} - 2[-(u_1u_3 - u_2u_4)u_1 + (u_1u_4 - u_2u_3)u_2]_x, \\ u_{3,t_1} = -u_{4,xx} - 2[(u_1u_4 - u_2u_3)u_3 + (u_1u_3 - u_2u_4)u_4]_x, \\ u_{4,t_1} = -u_{3,xx} - 2[(u_1u_3 - u_2u_4)u_3 + (u_1u_4 - u_2u_3)u_4]_x, \end{cases} \tag{34}$$

while $\alpha = \gamma = 1$ and $\beta = 0$ produces another coupled integrable DNLS model,

$$\begin{cases} u_{1,t_1} = u_{1,xx} - 2[(u_1u_3 - u_2u_4)u_1 - (u_1u_4 - u_2u_3)u_2]_x, \\ u_{2,t_1} = u_{2,xx} - 2[-(u_1u_4 - u_2u_3)u_1 + (u_1u_3 - u_2u_4)u_2]_x, \\ u_{3,t_1} = -u_{3,xx} - 2[(u_1u_3 - u_2u_4)u_3 + (u_1u_4 - u_2u_3)u_4]_x, \\ u_{4,t_1} = -u_{4,xx} - 2[(u_1u_4 - u_2u_3)u_3 + (u_1u_3 - u_2u_4)u_4]_x. \end{cases} \tag{35}$$

These two reduced systems are equivalent under the replacement

$$(u_{1,t_1}, u_{2,t_1}, u_{3,t_1}, u_{4,t_1}) \to (-u_{2,t_1}, -u_{1,t_1}, u_{4,t_1}, u_{3,t_1}),$$

corresponding to a permutation of the components together with sign changes in the time derivatives.

3. Bi-Hamiltonian Formulation

The spatial matrix eigenvalue problem (17) forms the foundation for a bi-Hamiltonian framework of the soliton hierarchy (32). In particular, the Liouville integrability of the hierarchy can be rigorously established via the trace identity (9).

Using the Laurent expansion corresponding to the stationary zero-curvature solution $\mathcal{W}$ given in (18), one can explicitly compute the corresponding Hamiltonian densities and associated conserved quantities of the hierarchy. The basic quantities in the trace identity are

$$\begin{cases} \mathrm{tr}\big(Y\dfrac{\partial \mathcal{M}}{\partial \lambda}\big) = 2(2\alpha\lambda a + bu_4 - cu_2 - eu_3 + gu_1), \\ \mathrm{tr}\big(Y\dfrac{\partial \mathcal{M}}{\partial u}\big) = 2(\lambda g, -\lambda c, -\lambda e, \lambda b)^T. \end{cases} \tag{36}$$

An application of the trace identity leads to

$$\begin{aligned} &\frac{\delta}{\delta u}\int \lambda^{-2k-1}(2\alpha a^{[k+1]} + u_4b^{[k]} - u_2c^{[k]} - u_3e^{[k]} + u_1g^{[k]})\,dx \\ &= \lambda^{-\kappa}\frac{\partial}{\partial\lambda}\lambda^{\kappa-2k}(g^{[k]}, -c^{[k]}, -e^{[k]}, b^{[k]})^T,\ k \geq 0. \end{aligned} \tag{37}$$

We first consider the case ($k = 0$). Since

$$2\alpha a^{[1]} + u_4b^{[0]} - u_2c^{[0]} - u_3e^{[0]} + u_1g^{[0]} = 0,$$

it follows that

$$\kappa(g^{[0]}, -c^{[0]}, -e^{[0]}, b^{[0]}) = 0.$$

Since $(g^{[0]}, -c^{[0]}, -e^{[0]}, b^{[0]}) \neq 0$, we conclude that $\kappa = 0$. Therefore, the following equalities hold for the Hamiltonian functionals:

$$\frac{\delta}{\delta u}\mathcal{H}^{[k]} = (-c^{[k]}, -c^{[k]}, -e^{[k]}, b^{[k]})^T,\ k \geq 0, \tag{38}$$

where the Hamiltonian functionals are defined as

$$\begin{cases} \mathcal{H}^{[0]} = \int [\beta(u_1u_4 - u_2u_3) + \gamma(u_1u_3 - u_2u_4)]\, dx, \\ \mathcal{H}^{[k]} = -\int \frac{1}{2k}(2\alpha a^{[k+1]} + u_4b^{[k]} - u_2c^{[k]} - u_3e^{[k]} + u_1g^{[k]})\, dx,\ k \geq 1. \end{cases} \tag{39}$$

As a result, the hierarchy (32) admits the Hamiltonian formulation

$$u_{t_l} = X^{[l]} = J_1\frac{\delta\mathcal{H}^{[l]}}{\delta u},\ l \geq 0, \tag{40}$$

where the Hamiltonian operator J_1 is given by

$$J_1 = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \partial & 0 \\ & & 0 & -\partial \\ \hline \partial & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & -\partial & & \end{array}\right]. \tag{41}$$

Therefore, for each flow in the hierarchy, there exists a one-to-one correspondence

$$X = J_1\frac{\delta\mathcal{H}}{\delta u}$$

between a conserved functional $\mathcal{H}$ to a Lie symmetry X, exhibiting the integrable Hamiltonian structure of the hierarchy.

Furthermore, the vector fields $X^{[k]}$ commute pairwise, satisfying the characteristic property

$$[\![X^{[k_1]}, X^{[k_2]}]\!] = X^{[k_1]\prime}(u)[X^{[k_2]}] - X^{[k_2]\prime}(u)[X^{[k_1]}] = 0,\ k_1, k_2 \geq 0, \tag{42}$$

which reflects the integrability of the hierarchy. This commutativity follows directly from the algebraic relations of the temporal spectral matrices:

$$[\![\mathcal{N}^{[k_1]}, \mathcal{N}^{[k_2]}]\!] = \mathcal{N}^{[k_1]\prime}(u)[X^{[k_2]}] - \mathcal{N}^{[k_2]\prime}(u)[X^{[k_1]}] + [\mathcal{N}^{[k_1]}, \mathcal{N}^{[k_2]}] = 0,\ k_1, k_2 \geq 0. \tag{43}$$

These identities can be verified directly through the zero-curvature flows being isospectral; see, for instance, ref. [30,31] for further details.

On the other hand, beginning with the recursion relation

$$X^{[l+1]} = \Phi X^{[l]},$$

we can explicitly construct a hereditary recursion operator $\Phi = (\Phi_{jk})_{4\times 4}$ [32]. The individual entries of Φ are given by

$$\begin{cases} \Phi_{11} = \frac{1}{\alpha}\partial_x - \frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_3 - \partial u_2\partial^{-1}u_4),\ \Phi_{12} = -\frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_4 + \partial u_2\partial^{-1}u_3), \\ \Phi_{13} = \frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_1 - \partial u_2\partial^{-1}u_2),\ \Phi_{14} = \frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_2 + \partial u_2\partial^{-1}u_1); \end{cases} \tag{44}$$

$$\begin{cases} \Phi_{21} = \frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_4 + \partial u_2\partial^{-1}u_3),\ \Phi_{22} = \frac{1}{\alpha}\partial_x - \frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_3 - \partial u_2\partial^{-1}u_4), \\ \Phi_{23} = -\frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_2 + \partial u_2\partial^{-1}u_1),\ \Phi_{24} = \frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_1 - \partial u_2\partial^{-1}u_2); \end{cases} \tag{45}$$

$$\begin{cases} \Phi_{31} = \frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_3 - \partial u_4\partial^{-1}u_4),\ \Phi_{32} = \frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_4 + \partial u_4\partial^{-1}u_3), \\ \Phi_{33} = -\frac{1}{\alpha}\partial_x - \frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_1 - \partial u_4\partial^{-1}u_2),\ \Phi_{34} = -\frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_2 + \partial u_4\partial^{-1}u_1); \end{cases} \tag{46}$$

$$\begin{cases} \Phi_{41} = -\frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_4 + \partial u_4\partial^{-1}u_3),\ \Phi_{42} = \frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_3 - \partial u_4\partial^{-1}u_4), \\ \Phi_{43} = \frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_2 + \partial u_4\partial^{-1}u_1),\ \Phi_{44} = -\frac{1}{\alpha}\partial_x - \frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_1 - \partial u_4\partial^{-1}u_2). \end{cases} \tag{47}$$

A straightforward calculation shows that the Hamiltonian operators J_1 and $J_2 = \Phi J_1$ consitute a Hamiltonian pair; that is, any linear combination of J_1 and J_2 preserves the Hamiltonian structure. Consequently, the hierarchy (32) admits a bi-Hamiltonian formulation in the sense of Magri [33]:

$$u_{t_l} = X^{[l]} = J_1\frac{\delta\mathcal{H}^{[l]}}{\delta u} = J_2\frac{\delta\mathcal{H}^{[l-1]}}{\delta u},\ \ l \geq 1. \tag{48}$$

As a result, the Hamiltonian functionals $\mathcal{H}^{[l]}$ are pairwise in involution under both Poisson brackets defined by J_1 and J_2 [7]:

$$\{\mathcal{H}^{[k_1]}, \mathcal{H}^{[k_2]}\}_{J_1} = \int (\frac{\delta\mathcal{H}^{[k_1]}}{\delta u})^T J_1\frac{\delta\mathcal{H}^{[k_2]}}{\delta u}\, dx = 0,\ k_1, k_2 \geq 0, \tag{49}$$

and

$$\{\mathcal{H}^{[k_1]}, \mathcal{H}^{[k_2]}\}_{J_2} = \int (\frac{\delta\mathcal{H}^{[k_1]}}{\delta u})^T J_2\frac{\delta\mathcal{H}^{[k_2]}}{\delta u}\, dx = 0,\ k_1, k_2 \geq 0. \tag{50}$$

This bi-Hamiltonian structure also guarantees the recursion operator Φ is hereditary, and hence serves as a unified recursion operator governing the entire hierarchy (32).

In summary, every model within the hierarchy (32) is integrable in the Liouville sense, admitting infinitely many mutually commuting Lie symmetries $\{X^{[k]}\}_{k=0}^{\infty}$ and conserved Hamiltonian functionals $\{\mathcal{H}^{[k]}\}_{k=0}^{\infty}$. Notably, the coupled system (33) serves as a concrete example of a nonlinear, combined, Liouville-integrable Hamiltonian four-component Kaup–Newell-type model, thereby extending the known class of such integrable systems (see, e.g., [34,35]).

4. Concluding Remarks

By studying integrable extensions of the matrix eigenvalue problem corresponding to the Kaup–Newell hierarchy, we introduce a specially constructed 4th-order matrix eigenvalue formulation and derive, via the zero-curvature approach, a hierarchy of Liouville-integrable four-component dynamical ntegrable flows. A central step is the construction of a particular Laurent-series solution of the stationary compatibility (zero-curvature) equation with appropriately combined initial data. Via the trace identity, we demonstrate that the resulting hierarchy is equipped with a bi-Hamiltonian formulation associated with a hereditary recursion operator, providing a natural four-component extension of the Kaup–Newell soliton hierarchy.

Understanding the mathematical structures of soliton solutions for these integrable models is of particular interest. Among the powerful tools for this purpose are Darboux transformations [36–38], the Riemann–Hilbert method [39], the Zakharov–Shabat dressing approach [40], and determinant-based techniques [41]. Beyond solitons, other coherent structures, in particular, lumps, kinks, breathers, and rogue waves, and their interaction solutions are also of significant interest (see, e.g., [42–49]). Many of these solutions can be obtained by applying suitable reductions to multi-soliton solutions.

Recently, nonlocal integrable reductions have attracted considerable attention. These arise from nonlocal symmetry reductions or similarity transformations of Lax pairs, and their soliton solutions exhibit novel mathematical structures and physical behaviors [50,51]. Nonlocality introduces a rich variety of linear and nonlinear phenomena, warranting detailed study [52].

Integrable models are of broad interest because they reveal deep connections between mathematics and physical phenomena. They are intimately related to areas such as Lie theory, algebraic geometry, and the theory of special functions. Studying these models not only enhances our understanding of universal dynamical behaviors in physical systems but also provides a foundation for analyzing complex wave phenomena across mathematics and physics.

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References

1. Ablowitz, M.J.; Segur, H. *Solitons and the Inverse Scattering Transform*; SIAM: Philadelphia, PA, USA, 1981. [CrossRef]
2. Fuchssteiner, B. Application of hereditary symmetries to nonlinear evolution equations. *Nonlinear Anal.* **1979**, *3*, 849–862. [CrossRef]
3. Lax, P.D. Integrals of nonlinear equations of evolution and solitary waves. *Commun. Pure Appl. Math.* **1968**, *21*, 467–490. [CrossRef]
4. Ablowitz, M.J.; Kaup, D.J.; Newell, A.C.; Segur, H. The inverse scattering transform-Fourier analysis for nonlinear problems. *Stud. Appl. Math.* **1974**, *53*, 249–315. [CrossRef]
5. Ma, W.X. Integrable couplings and matrix loop algebras. *AIP Conf. Proc.* **2013**, *1562*, 105–122. [CrossRef]
6. Drinfel'd, V.; Sokolov, V.V. Lie algebras and equations of Korteweg–de Vries type. *Sov. J. Math.* **1985**, *30*, 1975–2036. [CrossRef]
7. Tu, G.Z. On Liouville integrability of zero-curvature equations and the Yang hierarchy. *J. Phys. A Math. Gen.* **1989**, *22*, 2375–2392. [CrossRef]
8. Ma, W.X. A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction. *Chin. Ann. Math. Ser. A* **1992**, *13*, 115–123.
9. Liu, C.S. How many first integrals imply integrability in infinite-dimensional Hamilton system. *Rep. Math. Phys.* **2011**, *67*, 109–123. [CrossRef]
10. Ma, W.X. Novel integrable Hamiltonian hierarchies with six potentials. *Acta Math. Sci.* **2024**, *44B*, 2498–2508. [CrossRef]
11. Antonowicz, M.; Fordy, A.P. A family of completely integrable multi-hamiltonian systems. *Phys. Lett. A* **1987**, *122*, 95–99. [CrossRef]
12. Geng, X.G.; Du, D.L. Two hierarchies of new nonlinear evolution equations associated with 3×3 matrix spectral problems. *Chaos Solitons Fractals* **2006**, *29*, 1165–1172. [CrossRef]
13. Ma, W.X. An extended AKNS eigenvalue problem and its affiliated integrable Hamiltonian hierarchies. *Chaos Solitons Fractals* **2024**, *188*, 115580. [CrossRef]
14. Ma, W.X. Reduced AKNS spectral problems and associated complex matrix integrable models. *Acta Appl. Math.* **2023**, *187*, 17. [CrossRef]
15. Manukure, S. Finite-dimensional Liouville integrable Hamiltonian systems generated from Lax pairs of a bi-Hamiltonian soliton hierarchy by symmetry constraints. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *57*, 125–135. [CrossRef]
16. Gerdjikov, V.S. Nonlinear evolution equations related to Kac-Moody algebras $A_r^{(1)}$: Spectral aspects. *Turk. J. Math.* **2022**, *46*, 1828–1844. [CrossRef]
17. Ma, W.X. AKNS type reduced integrable hierarchies with Hamiltonian formulations. *Rom. J. Phys.* **2023**, *68*, 116. [CrossRef]

18. Ma, W.X. A matrix integrable Hamiltonian hierarchy and its two integrable reductions. *Int. J. Geom. Methods Mod. Phys.* **2024**, *21*, 2450293. [CrossRef]
19. Kaup, D.J.; Newell, A.C. An exact solution for a derivative nonlinear Schrödinger equation. *J. Math. Phys.* **1978**, *19*, 798–801. [CrossRef]
20. Smirnov, A.O.; Filimonova, E.G.; Matveev, V.B. The spectral curve method for the Kaup-Newell hierarchy. *IOP Conf. Ser. Mater. Sci. Eng.* **2020**, *919*, 052051. [CrossRef]
21. Takhtajan, L.A. Integration of the continuous Heisenberg spin chain through the inverse scattering method. *Phys. Lett. A* **1977**, *64*, 235–237. [CrossRef]
22. Wadati, M.; Konno, K.; Ichikawa, Y.H. New integrable nonlinear evolution equations. *J. Phys. Soc. Jpn.* **1979**, *47*, 1698–1700. [CrossRef]
23. Hu, X.B. A powerful approach to generate new integrable systems. *J. Phys. A Math. Gen.* **1994**, *27*, 2497–2514. [CrossRef]
24. Ma, W.X. A combined derivative nonlinear Schrödinger soliton hierarchy. *Rep. Math. Phys.* **2024**, *93*, 313–325. [CrossRef]
25. Ma, W.X. Four-component integrable hierarchies and their Hamiltonian structures. *Commun. Nonlinear Sci. Numer. Simul.* **2023**, *126*, 107460. [CrossRef]
26. Ma, W.X. A combined integrable hierarchy with four potentials and its recursion operator and bi-Hamiltonian structure. *Indian J. Phys.* **2025**, *99*, 1063–1069. [CrossRef]
27. Zhaqilao. A generalized AKNS hierarchy, bi-Hamiltonian structure, and Darboux transformation. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 2319–2332. [CrossRef]
28. Ma, W.X. Novel Liouville integrable Hamiltonian models with six components and three signs. *Chin. J. Phys.* **2023**, *86*, 292–299. [CrossRef]
29. Ma, W.X. A real four-component integrable extension of the standard Kaup–Newell hierarchy with two diagonal blocks. *Axioms* **2026**, *15*, 411. [CrossRef]
30. Ma, W.X. The algebraic structures of isospectral Lax operatorn and applicationn to integrable equations. *J. Phys. A Math. Gen.* **1992**, *25*, 5329–5343. [CrossRef]
31. Ma, W.X. The algebraic structure of zero-curvature representations and application to coupled KdV systems. *J. Phys. A Math. Gen.* **1993**, *26*, 2573–2582. [CrossRef]
32. Fuchssteiner, B.; Fokas, A.S. Symplectic structures, their Bäcklund transformations and hereditary symmetries. *Phys. D Nonlinear Phenom.* **1981**, *4*, 47–66. [CrossRef]
33. Magri, F. A simple model of the integrable Hamiltonian equation. *J. Math. Phys.* **1978**, *19*, 1156–1162. [CrossRef]
34. Ma, W.X. A combined Kaup-Newell type integrable hierarchy with four potentials and its bi-Hamiltonian formulation. *Rev. Math. Phys.* **2025**, *37*, 2450049. [CrossRef]
35. Ma, W.X. Integrable matrix modified Korteweg-de Vries equations derived from reducing AKNS Lax pairs. *Rom. J. Phys.* **2025**, *70*, 110. [CrossRef]
36. Matveev, V.B.; Salle, M.A. *Darboux Transformations and Solitons*; Springer: Berlin, Germany, 1991. [CrossRef]
37. Geng, X.G.; Li, R.M.; Xue, B. A vector general nonlinear Schrödinger equation with $(m+n)$ components. *J. Nonlinear Sci.* **2020**, *30*, 991–1013. [CrossRef]
38. Ye, R.S.; Zhang, Y. A vectorial Darboux transformation for the Fokas-Lenells system. *Chaos Solitons Fractals* **2023**, *169*, 113233. [CrossRef]
39. Novikov, S.P.; Manakov, S.V.; Pitaevskii, L.P.; Zakharov, V.E. *Theory of Solitons: The Inverse Scattering Method*; Consultantn Bureau: New York, NY, USA, 1984.
40. Doktorov, E.V.; Leble, S.B. *A Dressing Method in Mathematical Physics*; Springer: Dordrecht, The Netherlands, 2007. [CrossRef]
41. Aktosun, T.; Busse, T.; Demontis, F.; van der Mee, C. Symmetries for exact solutions to the nonlinear Schrödinger equation. *J. Phys. A Math. Theor.* **2010**, *43*, 025202. [CrossRef]
42. Cheng, L.; Zhang, Y.; Lin, M.J. Lax pair and lump solutions for the (2+1)-dimensional DJKM equation associated with bilinear Bäcklund transformations. *Anal. Math. Phys.* **2019**, *9*, 1741–1752. [CrossRef]
43. Manukure, S.; Chowdhury, A.; Zhou, Y. Complexiton solutions to the asymmetric Nizhnik-Novikov-Veselov equation. *Int. J. Mod. Phys. B* **2019**, *33*, 1950098. [CrossRef]
44. Zhou, Y.; Manukure, S.; McAnally, M. Lump and rogue wave solutions to a (2+1)-dimensional Boussinesq type equation. *J. Geom. Phys.* **2021**, *167*, 104275. [CrossRef]
45. Ma, W.X. Lump waves and their dynamics in a generalized Kadomtsev-Petviashvili-like model. *Mod. Phys. Lett. A* **2026**, *41*, 2550215. [CrossRef]
46. Manukure, S.; Zhou, Y. A study of lump and line rogue wave solutions to a (2+1)-dimensional nonlinear equation. *J. Geom. Phys.* **2021**, *167*, 104274. [CrossRef]
47. Ma, W.X. Dispersion-governed lump waves in a generalized Calogero–Bogoyavlenskii–Schiff-like model with spatially symmetric nonlinearity. *Axioms* **2025**, *14*, 869. [CrossRef]

48. Yang, S.X.; Wang, Y.F.; Zhang, X. Conservation laws, Darboux transformation and localized waves for the N-coupled nonautonomous Gross-Pitaevskii equations in the Bose-Einstein condensates. *Chaos Solitons Fractals* **2023**, *169*, 113272. [CrossRef]
49. Li, Y.; Yao, R.X.; Lou, S.Y. Novel nonlinear wave transitions and interactions for (2+1)-dimensional generalized fifth-order KdV equation. *Commun. Theor. Phys.* **2024**, *76*, 125003. [CrossRef]
50. Ma, W.X.; Huang, Y.H.; Wang, F.D.; Zhang, Y.; Ding, L.Y. Darboux transformation of vector nonlocal reverse-space nonlinear Schrödinger equations. *Int. J. Geom. Methods Mod. Phys.* **2024**, *21*, 2450182. [CrossRef]
51. Ma, W.X. Nonlocality, integrability, and solitons. *Theor. Math. Phys.* **2025**, *224*, 1220–1233. [CrossRef]
52. Ma, W.X. Solving a non-local linear differential equation model of the Newtonian-type. *Pramana—J. Phys.* **2024**, *98*, 68. [CrossRef]