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An integrable counterpart of the D-AKNS soliton hierarchy from so(3, $\mathbb{R}$)

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A R T I C L E I N F O

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A B S T R A C T

An integrable counterpart of the D-AKNS soliton hierarchy is generated from a matrix spectral problem associated with so(3, $\mathbb{R}$). Hamiltonian structures of the resulting counterpart soliton hierarchy are furnished by using the trace identity, which yields its Liouville integrability.

1. Introduction

Soliton equations possessing Hamiltonian structures provide concrete examples of integrable equations. Matrix spectral problems (or Lax pairs) from given matrix loop algebras are starting points to generate soliton equations and their Hamiltonian structures (see, e.g., [1–4]). Among celebrated examples are the Korteweg-de Vries hierarchy [5], the Ablowitz–Kaup–Newell–Segur hierarchy [6], the Dirac hierarchy [7], the Kaup–Newell hierarchy [8] and the Wadati–Konno–Ichikawa hierarchy [9].

Recently, the three-dimensional real special orthogonal Lie algebra so(3, $\mathbb{R}$), which consists of 3×3 skew-symmetric matrices, has been used in constructing soliton hierarchies (see, e.g., [11,12]). This Lie algebra is simple and has the basis

$$e_1=\begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix},\qquad e_2=\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix},$$
$$e_3=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix},\tag{1.1}$$

with the circular commutator relations:

$$[e_1,e_2]=e_3,\qquad [e_2,e_3]=e_1,\qquad [e_3,e_1]=e_2.$$

Its derived algebra is itself, and so, it is 3-dimensional, too. The only other three-dimensional real Lie algebras with a three-dimensional derived algebra is the special linear algebra sl(2, $\mathbb{R}$), which has been widely used in generating soliton hierarchies in soliton theory (see, e.g., [5–10]).

The matrix loop algebra we shall adopt in what follows is

$$\widetilde{\mathrm{so}}(3,\mathbb{R})=\left\{\sum_{i\ge 0}M_i\lambda^{n-i}\;\middle|\;M_i\in \mathrm{so}(3,\mathbb{R}),\ i\ge 0,\ n\in\mathbb{Z}\right\},\tag{1.2}$$

that is, the space of all Laurent series in λ with coefficients in so(3, $\mathbb{R}$) and a finite regular part. Particular examples of this matrix loop algebra contain the following linear combinations:

$$\lambda^m d_1e_1+\lambda^n d_2e_2+\lambda^l d_3e_3$$

with arbitrary integers m,n,l and real constants d_1,d_2,d_3. The algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$ provides a structural basis for our study of soliton hierarchies, from which a few new soliton hierarchies have been generated of late (see, e.g., [11–16]).

We shall focus on the Liouville integrability of partial differential equations [17]. Let $x=(x^1,\cdots,x^p)$ be the vector of spatial variables and $u=(u^1,\cdots,u^q)^T$ the vector of dependent variables. We consider a Hamiltonian system of evolution equations

$$u_t=J\frac{\delta\mathcal{H}}{\delta u},\quad u=u(x,t),\tag{1.3}$$

where $J=J(x,u)$ is a Hamiltonian operator and $\frac{\delta}{\delta u}$ is the variational derivative with respect to u. A conserved functional of a Hamiltonian system (1.3) is a functional $\mathcal{T}=\int T\,dx$ which determines a conservation law of (1.3):

$$D_tT+\mathrm{Div}\,X=0,$$

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in which Div denotes spatial divergence. For a given differential function F, its corresponding one-form is defined by

$$dF := \sum_{i=1}^{p} \frac{\partial F}{\partial x^i} dx^i + \frac{\partial F}{\partial t} dt + \sum_{\alpha=1}^{q} \sum_{\#J \geq 0} \frac{\partial F}{\partial u_J^\alpha} du_J^\alpha, \tag{1.4}$$

where if $\#J = 0$, then $u_J^\alpha = u^\alpha$, and if $\#J = k \geq 1$, then $u_J^\alpha = \frac{\partial^k u^\alpha}{\partial x^{j_1} \cdots \partial x^{j_k}}$ for $J = (j_1, \cdots, j_k)$, $1 \leq j_i \leq p$, $1 \leq i \leq k$.

Definition 1. Let I be a set of integers and $r \geq 1$ a natural number. A set of r-tuples of differential functions

$$\left\{ S_n = \left(S_n^1, \cdots, S_n^r\right)^T \,\middle|\, n \in I \right\}$$

is said to be independent, if all r-tuples of one-forms

$$\left\{ dS_n = \left(dS_n^1, \cdots, dS_n^r\right)^T \,\middle|\, n \in I \right\},$$

are linearly independent at every point in the infinite jet space. We say that a set of conserved functionals $\{\mathcal{H}_n \mid n \in I\}$ of a Hamiltonian system (1.3) is independent, if all characteristics $\{J \frac{\delta \mathcal{H}_n}{\delta u} \mid n \in I\}$ of the associated Hamiltonian vector fields are independent.

By the differential order of an r-tuple S of differential functions, we denote the order of the highest-order derivative of u with respect to x in S. Obviously, if a set of r-tuples of differential functions has distinct differential orders, then it is independent.

Definition 2. A Hamiltonian system (1.3) is called to be Liouville integrable, if there exists a sequence of conserved functionals $\{\mathcal{H}_n\}_{n=0}^\infty$, which are in involution with respect to the corresponding Poisson bracket:

$$\{\mathcal{H}_m, \mathcal{H}_n\}_J := \int \left(\frac{\delta \mathcal{H}_m}{\delta u}\right)^T J \frac{\delta \mathcal{H}_n}{\delta u} \, dx = 0, \quad m, n \geq 0, \tag{1.5}$$

and the characteristics of whose associated Hamiltonian vector fields

$$K_n := J \frac{\delta \mathcal{H}_n}{\delta u}, \quad n \geq 0, \tag{1.6}$$

are independent.

In this paper, we would like to use the matrix loop algebra $\widetilde{\mathrm{so}}(3, \mathbb{R})$ to introduce a counterpart matrix spectral problem of the D-AKNS spectral problem, and compute an integrable Hamiltonian counterpart hierarchy of the D-AKNS soliton hierarchy [18,19] by using the zero curvature formulation. The corresponding Hamiltonian structures will be furnished by applying the trace identity, and all soliton equations in the resulting soliton hierarchy will be shown to be Liouville integrable. The resulting hierarchy provides another example of soliton hierarchies associated with the loop algebra $\widetilde{\mathrm{so}}(3, \mathbb{R})$. A few concluding remarks will round off the paper.

2. An integrable counterpart of the D-AKNS hierarchy

We apply the generating scheme using the zero curvature formulation [10,17], to present an integrable counterpart associated with the matrix loop algebra $\mathrm{so}(3, \mathbb{R})$ for the D-AKNS soliton hierarchy [18,19]. Let us introduce a new 3×3 matrix spectral problem:

$$\phi_x = U\phi = U(u, \lambda)\phi, \quad u = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}, \tag{2.1}$$

where the spectral matrix U is chosen as

$$\begin{aligned} U = U(u, \lambda) &= (\lambda + r)e_1 + pe_2 + qe_3 \\ &= \begin{bmatrix} 0 & -q & -\lambda - r \\ q & 0 & -p \\ \lambda + r & p & 0 \end{bmatrix} \in \widetilde{\mathrm{so}}(3, \mathbb{R}). \end{aligned} \tag{2.2}$$

This is the selfsame linear combination of the basis matrices as the D-AKNS one associated with $\mathrm{sl}(2, \mathbb{R})$ [18,19].

Once a matrix spectral problem is chosen, it is inherently feasible to compute a soliton hierarchy associated with the spectral problem. First, we solve the stationary zero curvature equation

$$W_x = [U, W], \quad W \in \widetilde{\mathrm{so}}(3, \mathbb{R}). \tag{2.3}$$

If W is assumed to be

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} 0 & -c & -a \\ c & 0 & -b \\ a & b & 0 \end{bmatrix}, \tag{2.4}$$

then the stationary zero curvature equation (2.3) becomes

$$\begin{cases} a_x = pc - qb, \\ b_x = -\lambda c - rc + qa, \\ c_x = -pa + \lambda b + rb. \end{cases} \tag{2.5}$$

Further, let a, b and c possess the Laurent expansions:

$$a = \sum_{i \geq 0} a_i \lambda^{-i}, \qquad b = \sum_{i \geq 0} b_i \lambda^{-i}, \qquad c = \sum_{i \geq 0} c_i \lambda^{-i}, \tag{2.6}$$

and take the initial data

$$a_0 = -1, \qquad b_0 = 0, \qquad c_0 = 0, \tag{2.7}$$

which are required by the equations on the highest powers of λ in (2.5):

$$a_{0,x} = pc_0 - qb_0, \qquad b_0 = 0, \qquad c_0 = 0.$$

Then, the system (2.5) leads equivalently to

$$\begin{cases} a_{i+1,x} = -pb_{i,x} - qc_{i,x} - ra_{i,x}, \\ b_{i+1} = c_{i,x} + pa_i - rb_i, \\ c_{i+1} = -b_{i,x} + qa_i - rc_i, \end{cases} \quad i \geq 0, \tag{2.8}$$

the first of which is because from (2.5), we have

$$\begin{aligned} \lambda a_x &= \lambda(pc - qb) \\ &= p(-b_x - rc + qa) - q(c_x + pa - rb) \\ &= -pb_x - qc_x - r(pc - qb) \\ &= -pb_x - qc_x - ra_x. \end{aligned}$$

While using the above recursion relations (2.8), we impose the condition that the constants of integration take the value of zero:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1, \tag{2.9}$$

to determine the sequence of $\{a_i, b_i, c_i \mid i \geq 1\}$ uniquely. This way, the first three sets can be worked out as follows:

$$\begin{aligned} &a_1 = 0, \qquad b_1 = -p, \qquad c_1 = -q; \\ &a_2 = \frac{1}{2}\left(p^2 + q^2\right), \qquad b_2 = -q_x + pr, \qquad c_2 = p_x + qr; \\ &a_3 = pq_x - p_x q - \left(p^2 + q^2\right)r, \\ &b_3 = p_{xx} + 2q_x r + qr_x + \frac{1}{2}p\left(p^2 + q^2\right) - pr^2, \\ &c_3 = q_{xx} - 2p_x r - pr_x - qr^2 + \frac{1}{2}\left(p^2 + q^2\right)q. \end{aligned}$$

We saw above the localness of the first three sets of $\{a_i, b_i, c_i \mid i \geq 1\}$. This is not an accident, and in fact, the functions a_i, b_i, c_i, $i \geq 1$, are all local. We verify this fact as follows. First from $W_x = [U, W]$, we can have

$$\frac{d}{dx}\operatorname{tr}\big(W^2\big) = 2\operatorname{tr}(WW_x) = 2\operatorname{tr}\big(W[U, W]\big) = 0,$$

and so, due to $\operatorname{tr}(W^2) = -2(a^2+b^2+c^2)$, we have

$$a^2+b^2+c^2 = \big(a^2+b^2+c^2\big)\big|_{u=0} = 1, \tag{2.10}$$

the last step of which follows from the initial data in (2.7). Then, by using the Laurent expansions in (2.6) and noting the initial data in (2.7) again, a balance of coefficients of λ^i in (2.10) for each $i \geq 1$ tells that

$$a_i = -\frac{1}{2}\sum_{k+l=i,\, k,l\geq 1}(a_k a_l + b_k b_l + c_k c_l), \quad i \geq 1. \tag{2.11}$$

Based on this recursion relation (2.11) and the last two recursion relations in (2.8), an application of the mathematical induction finally shows that all functions a_i, b_i, c_i, $i \geq 1$, are differential polynomials in u, i.e., they are all local; and that for each $i \geq 2$, the differential orders of the functions a_i, b_i and c_i are $i-2, i-1$ and $i-1$, respectively.

Now as usual, we compute

$$\big(\big(\lambda^m W\big)_+\big)_x - \big[U, \big(\lambda^m W\big)_+\big] = \begin{bmatrix} 0 & -b_{m+1} & 0 \\ b_{m+1} & 0 & c_{m+1} \\ 0 & -c_{m+1} & 0 \end{bmatrix}, \quad m \geq 0, \tag{2.12}$$

where P_+ denotes the polynomial part of P in λ. This is not the same type matrix as the Gateaux derivative matrix U', and so, we need to introduce modification terms for Lax operators. Noting

$$[U, e_1] = qe_2 - pe_3,$$

we take a sequence of Lax operators with modification terms:

$$V^{[m]} = \big(\lambda^m W\big)_+ + \Delta_m,$$
$$\Delta_m = \beta a_{m+1} e_1 \in \widetilde{\mathrm{so}}(3, \mathbb{R}), \quad m \geq 0, \tag{2.13}$$

where β is an arbitrarily given constant. Then, we arrive at

$$V_x^{[m]} - \big[U, V^{[m]}\big] = \begin{bmatrix} 0 & -b_{m+1} - \beta p a_{m+1} & -\beta a_{m+1,x} \\ b_{m+1} + \beta p a_{m+1} & 0 & c_{m+1} + \beta q a_{m+1} \\ \beta a_{m+1,x} & -c_{m+1} - \beta q a_{m+1} & 0 \end{bmatrix},$$
$$m \geq 0, \tag{2.14}$$

and further, the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + \big[U, V^{[m]}\big] = 0, \quad m \geq 0, \tag{2.15}$$

equivalently yield a hierarchy of soliton equations:

$$u_{t_m} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{t_m} = K_m = \begin{bmatrix} -c_{m+1} - \beta q a_{m+1} \\ b_{m+1} + \beta p a_{m+1} \\ \beta a_{m+1,x} \end{bmatrix}, \quad m \geq 0. \tag{2.16}$$

All systems of evolution equations in this counterpart soliton hierarchy are local, since the functions a_i, b_i, c_i, $i \geq 1$, are differential polynomials in u. The first two nonlinear systems in the hierarchy are as follows:

$$\begin{cases} p_{t_1} = -p_x - qr - \frac{\beta}{2}(p^2+q^2)q, \\ q_{t_1} = -q_x + pr + \frac{\beta}{2}p(p^2+q^2), \\ r_{t_1} = \beta(pp_x + qq_x); \end{cases} \tag{2.17}$$

and

$$\begin{cases} p_{t_2} = -q_{xx} + 2p_x r + pr_x + qr^2 - \frac{1}{2}(p^2+q^2)q \\ \qquad + \beta(p_x q - pq_x)q + \beta(p^2+q^2)qr, \\ q_{t_2} = p_{xx} + 2q_x r + qr_x + \frac{1}{2}p(p^2+q^2) - pr^2 \\ \qquad - \beta p(p_x q - pq_x) - \beta p(p^2+q^2)r, \\ r_{t_2} = \beta[-p_{xx}q + pq_{xx} - 2(p_x p + q_x q)r - (p^2+q^2)r_x]. \end{cases} \tag{2.18}$$

3. Hamiltonian structures and Liouville integrability

We shall show that all systems in the counterpart soliton hierarchy (2.16) are Liouville integrable. Towards this end, let us first establish Hamiltonian structures for the counterpart hierarchy (2.16). We shall use the trace identity [10,17]:

$$\frac{\delta}{\delta u}\int \operatorname{tr}\bigg(\frac{\partial U}{\partial \lambda}W\bigg)\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\operatorname{tr}\bigg(\frac{\partial U}{\partial u}W\bigg),$$
$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln\big|\operatorname{tr}\big(W^2\big)\big|, \tag{3.1}$$

or generally, the variational identity (see [20,21]).

It is direct to see that

$$\frac{\partial U}{\partial \lambda} = e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \frac{\partial U}{\partial p} = e_2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\frac{\partial U}{\partial q} = e_3 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \frac{\partial U}{\partial r} = e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and so, we have

$$\operatorname{tr}\bigg(W\frac{\partial U}{\partial \lambda}\bigg) = -2a, \qquad \operatorname{tr}\bigg(W\frac{\partial U}{\partial p}\bigg) = -2b,$$
$$\operatorname{tr}\bigg(W\frac{\partial U}{\partial q}\bigg) = -2c, \qquad \operatorname{tr}\bigg(W\frac{\partial U}{\partial r}\bigg) = -2a.$$

Now we readily see that the trace identity (3.1) gives rise to

$$\frac{\delta}{\delta u}\int(-2a)\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} -2b \\ -2c \\ -2a \end{bmatrix}. \tag{3.2}$$

A balance of coefficients of λ^{-m-1} for each $m \geq 0$ in this equality leads to

$$\frac{\delta}{\delta u}\int a_{m+1}\,dx = (\gamma - m)\begin{bmatrix} b_m \\ c_m \\ a_m \end{bmatrix}, \quad m \geq 0.$$

The identity with $m = 1$ tells $\gamma = 0$, and thus, we obtain

$$\frac{\delta}{\delta u}\mathcal{H}_m = \begin{bmatrix} b_m \\ c_m \\ a_m \end{bmatrix}, \quad m \geq 1, \tag{3.3}$$

with the Hamiltonian functionals being defined by

$$\mathcal{H}_m = \int\bigg(-\frac{a_{m+1}}{m}\bigg)\,dx, \quad m \geq 1. \tag{3.4}$$

Obviously, noting $a_{m+1,x} = pc_{m+1} - qb_{m+1}$, $m \geq 0$, we can find that

$$K_m = J\begin{bmatrix} b_{m+1} \\ c_{m+1} \\ a_{m+1} \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & -1 & -\beta q \\ 1 & 0 & \beta p \\ \beta q & -\beta p & 2\beta\partial \end{bmatrix}, \quad m \geq 0. \tag{3.5}$$

A simple and straightforward computation shows that J is a Hamiltonian operator. In the case of $\mathrm{sl}(2,\mathbb{R})$, the Hamiltonian operator J is different, which can be generated by exchanging two dependent variables p and q [18,19]. It follows now that the counterpart soliton hierarchy (2.16) has the Hamiltonian structures:

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_{m+1}}{\delta u}, \quad m \ge 0, \tag{3.6}$$

where the Hamiltonian functionals $\mathcal{H}_m$'s are given by (3.4) and the Hamiltonian operator is defined as in (3.5).

The resulting functionals correspond to common conservation laws for each soliton system in the counterpart soliton hierarchy (2.16). We point out that such differential polynomial conservation laws can also be generated either by computer algebra codes (see, e.g., [22]) or from some Riccati equation associated with the underlying matrix spectral problem (see, e.g., [23–25]).

Upon observation of the Hamiltonian structures displayed in (3.6) and the differential orders of the sequence $\{a_i, b_i, c_i \mid i \ge 1\}$ shown in the last section, we can now state that the counterpart soliton hierarchy (2.16) is Liouville integrable. More concretely, every member in the counterpart hierarchy (2.16) possesses infinitely many independent commuting conserved functionals and symmetries:

$$\{\mathcal{H}_k, \mathcal{H}_l\}_J := \int \left(\frac{\delta \mathcal{H}_k}{\delta u}\right)^T J \frac{\delta \mathcal{H}_l}{\delta u}\, dx = 0, \quad k, l \ge 0, \tag{3.7}$$

and

$$\begin{aligned}[K_k, K_l] &:= K_k'(u)[K_l] - K_l'(u)[K_k] \\ &= J\frac{\delta}{\delta u}\{\mathcal{H}_k, \mathcal{H}_l\}_J = 0, \quad k, l \ge 0,\end{aligned} \tag{3.8}$$

where K' denotes the Gateaux derivative. These commuting relations are also consequences of the Virasoro algebra of Lax operators. See [26–28] for a detailed and systematical study on algebraic structures related to Lax operators and zero curvature equations.

4. Concluding remarks

Based on the matrix loop algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$, we introduced a counterpart matrix spectral problem of the D-AKNS spectral problem, and generated an integrable Hamiltonian counterpart of the D-AKNS soliton hierarchy. All members in the resulting counterpart soliton hierarchy are Hamiltonian and Liouville integrable.

Among typical discussed spectral matrices associated with $\widetilde{\mathrm{so}}(3,\mathbb{R})$ are the following three:

$$U(u,\lambda) = \lambda e_1 + pe_2 + qe_3,$$
$$U(u,\lambda) = \lambda^2 e_1 + \lambda pe_2 + \lambda qe_3,$$
$$U(u,\lambda) = \lambda e_1 + \lambda pe_2 + \lambda qe_3,$$

which correspond to the Ablowitz–Kaup–Newell–Segur spectral matrix, the Kaup–Newell spectral matrix and the Wadati–Konno–Ichikawa spectral matrix associated with $\mathrm{sl}(2,\mathbb{R})$, respectively. In those three examples, the vector u consists of only two dependent variables p and q. Our example is a new soliton hierarchy with three dependent variables, and we hope more examples of such soliton hierarchies with three dependent variables or even more dependent variables, such as four or five dependent variables, can be presented. Given a starting matrix loop algebra, it only needs a considerable amount of time and computational dexterity to compute hierarchies of integrable Hamiltonian systems.

There are many other higher-order matrix spectral problems which engender soliton hierarchies (see, e.g., [29–36]). The study of integrable couplings [37] associated with enlarged matrix loop algebras [38] will definitely provide more specific examples of soliton hierarchies generated from higher-order matrix spectral problems.

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