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SCIENTIFIC RESEARCH PAPER

Dispersion-Driven Lump Waves in a (2+1)-Dimensional Generalized Bogoyavlensky-Konopelchenko-like Model

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Abstract
This study investigates dispersion-induced lump wave structures in a generalized Bogoyavlensky-Konopelchenko-type model in (2+1)-dimensions. A generalized bilinear representation of the governing equation is first established using generalized bilinear derivatives with $p = 3$. Positive quadratic wave solutions are then derived via symbolic computation, which in turn generate lump-type wave structures. Our analysis reveals that the stationary points of these quadratic waves lie on a straight line in the spatial plane and propagate with constant velocities. Along this characteristic trajectory, the lump amplitude vanishes. The emergence of these lump structures is attributed to the interplay between nonlinear and dispersive effects in the model.

Keywords Hirota bilinear form · Lump wave · Symbolic computation · Nonlinearity · Dispersion

Mathematics Subject Classification 37K35 · 35Q51 · 37K40

1 Introduction

Exact closed-form solutions play a fundamental role in mathematical physics and engineering sciences, as they not only reveal intrinsic structures of nonlinear models but also provide systematic frameworks for addressing complex phenomena. Yet, obtaining such solutions is often a nontrivial task. Much research has therefore been devoted to either deriving explicit expressions or identifying the precise conditions under which they can exist.

Within soliton theory and the analysis of integrable models, wave structures such as solitons, rogue waves, and lump waves are of central interest. These dispersive structures emerge from the intricate balance between nonlinearity and dispersion, and their construction, analytical or numerical, remains a central theme in the study of nonlinear wave dynamics.

Two cornerstone approaches for generating exact solutions are the inverse scattering transform (IST) [1, 2] and Hirota's bilinear method [3]. The IST, viewed as a nonlinear generalization of the Fourier transform, provides a powerful tool for solving initial value problems of integrable PDEs via their associated Lax pairs [4]. It has been widely applied to capture soliton dynamics and long-time asymptotics of dispersive waves, including those without solitons [5]. In contrast, Hirota's method offers a direct algebraic framework for constructing exact wave solutions, including solitons and lumps, articularly in higher-dimensional nonlinear dispersive systems [6]-[10].

Formally, let x, y denote spatial variables and t time. A general Hirota bilinear equation in $(2+1)$-dimensions can be written as

$$P(D_x, D_y, D_t) f \cdot f = 0, \tag{1.1}$$

where D_x, D_y and D_t are Hirota's bilinear operators [3], defined by

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$$
\begin{aligned}
&D_x^m D_y^n D_t^k f \cdot f \\
&\quad = (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^m (\frac{\partial}{\partial y} - \frac{\partial}{\partial y'})^n (\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})^k \\
&\quad f(x,y,t) f(x',y',t')|_{x'=x, y'=y, t'=t},
\end{aligned}
$$

with $m, n, k \geq 0$. Based on the Bell polynomial theory, nonlinear PDEs for a scalar field u can be obtained from bilinear forms through logarithmic derivative transformations (see, e.g., [11]), such as

$$u = \beta(\ln f)_{xx},\ \beta(\ln f)_{yy},\ \beta(\ln f)_{xy},\ \beta(\ln f)_x,\ \beta(\ln f)_y, \tag{1.2}$$

where $\beta \neq 0$ is a constant. The bilinear framework admits N-soliton solutions expressed in exponential superposition form (see, e.g., [6, 12]):

$$f = \sum_{\nu=0,1} \exp(\sum_{i=1}^{N} \nu_i \zeta_i + \sum_{i<j} \nu_i \nu_j d_{ij}), \tag{1.3}$$

where the summation runs over all $\nu_i \in 0, 1$. The phase shifts d_{ij} and linear phases ζ_i are determined from

$$\exp(d_{ij}) = -\frac{P(\omega_j - \omega_i, k_i - k_j, l_i - l_j)}{P(\omega_j + \omega_i, k_i + k_j, l_i + l_j)},\ \text{where } 1 \leq i < j \leq N, \tag{1.4}$$

and

$$\zeta_i = k_i x + l_i y - \omega_i t + \zeta_{i,0},\ \text{ where } 1 \leq i \leq N, \tag{1.5}$$

subject to the dispersion relations

$$P(-\omega_i, k_i, l_i) = 0,\ \text{ where } 1 \leq i \leq N. \tag{1.6}$$

Verifying that f of the form (1.3) satisfies (1.1) under conditions (1.6) is a central task, for which systematic algorithms have been established in both $(1+1)$- and $(2+1)$-dimensional cases [12, 13].

Beyond solitons, lump waves and rogue waves constitute another important class of explicit structures (see [14] for a general study). Lump waves are rationally localized in space, decaying algebraically in all directions at fixed t [14, 15]. For instance, the KPI equation is well known to admit diverse lump solutions [7], some of which arise as long-wave limits of multi-soliton configurations [16]. Such structures are not restricted to integrable models: lump-type solutions also appear in nonintegrable KP-, BKP-, and KP–Boussinesq-type systems [17], and even in linear higher-dimensional wave models via superposition principles [18, 19].

A widely used approach to constructing lumps is the sum-of-squares ansatz, where a positive quadratic form is substituted into a bilinear equation [7, 14]. Applying logarithmic derivative transformations then produces lump solutions for various nonlinear PDEs. In this work, we adopt this method to investigate a $(2+1)$-dimensional generalized Bogoyavlensky-Konopelchenko-like (gBK-like) equation that incorporates eight nonlinearities and five distinct dispersive terms. These nonlinear and dispersive contributions together provide the balancing mechanisms that sustain lump structures. Symbolic computations are employed to derive lump solutions and analyze the stationary points of the quadratic form, offering insight into the associated wave dynamics.

2 A Generalized BK-like Model

It is well established that Hirota's bilinear derivatives can be extended to encompass all-order differential terms. A broad class of such operators, known as generalized bilinear differential operators, was introduced in [20] and is defined by

$$
\begin{aligned}
&D_{p,x}^m D_{p,y}^n D_{p,t}^k f \cdot f \\
&\quad = (\frac{\partial}{\partial x} + \alpha_p \frac{\partial}{\partial x'})^m (\frac{\partial}{\partial y} + \alpha_p \frac{\partial}{\partial y'})^n (\frac{\partial}{\partial t} + \alpha_p \frac{\partial}{\partial t'})^k \\
&\quad f(x,y,t) f(x',y',t')|_{x'=x, y'=y, t'=t},
\end{aligned} \tag{2.1}
$$

where

$$\alpha_p^k = (-1)^{r(k)} \text{ where } k \equiv r(k) \bmod p,\ 0 \leq r(k) < p. \tag{2.2}$$

For example, when $p = 3$, the sequence of coefficients is

$$\alpha_3 = -1,\ \alpha_3^2 = \alpha_3^3 = 1,\ \alpha_3^4 = -1,\ \alpha_3^5 = \alpha_3^6 = 1,\ \cdots, \tag{2.3}$$

and when $p = 5$, it takes the form

$$
\begin{aligned}
&\alpha_5 = -1,\ \alpha_5^2 = 1,\ \alpha_5^3 = -1,\ \alpha_5^4 = \alpha_5^5 = 1, \\
&\alpha_5^6 = -1,\ \alpha_5^7 = 1,\ \alpha_5^8 = -1,\ \alpha_5^9 = \alpha_5^{10} = 1,\ \cdots.
\end{aligned} \tag{2.4}
$$

2.1 A Generalized Bilinear Form

For the case $p = 3$, we propose the following generalized Bogoyavlensky-Konopelchenko-like (gBK-like) bilinear equation:

$$
\begin{aligned}
&F_{\text{gBK-like}}(f) \\
&:= \big(\sigma_1 D_{3,x}^4 + \sigma_2 D_{3,x}^3 D_{3,y} + \delta_1 D_{3,t} D_{3,x} + \delta_2 D_{3,t} D_{3,y} \\
&+ \delta_3 D_{3,x}^2 + \delta_4 D_{3,x} D_{3,y} + \delta_5 D_{3,y}^2\big) f \cdot f \\
&= 2[3\sigma_1 f_{xx}^2 + 3\sigma_2 f_{xx} f_{xy} + \delta_1 (f_{tx} f - f_t f_x) \\
&+ \delta_2 (f_{ty} f - f_t f_y) + \delta_3 (f_{xx} f - f_x^2) \\
&\quad + \delta_4 (f_{xy} f - f_x f_y) + \delta_5 (f_{yy} f - f_y^2)] = 0,
\end{aligned} \tag{2.5}
$$

where $D_{3,x}, D_{3,y}$ and $D_{3,t}$ are the generalized bilinear derivatives, and σ_1, σ_2, δ_i for $1 \le i \le 5$ are arbitrary constants. This construction extends the classical Bogoyavlensky-Konopelchenko equation; the $p = 2$ case was previously analyzed in [21].

2.2 Nonlinear Form of the Model

By redefining the dependent variable as

$$u = 2(\ln f)_x,\ \ v = 2(\ln f)_y, \tag{2.6}$$

we obtain the generalized gBK-like model in nonlinear form:

$$\begin{aligned}
P_{\text{gBK-like}}(u, v) := {} & \sigma_1\big(3u_x u_{xx} + 3uu_x^2 + \frac{3}{2}u^3 u_x + \frac{3}{2}u^2 u_{xx}\big) \\
& + \sigma_2\big(\frac{9}{8}u^2 u_x v + \frac{3}{8}u^3 u_y \\
& + \frac{3}{4}uu_{xx}v + \frac{3}{4}u_x^2 v + \frac{3}{4}u^2 u_{xy} + \frac{9}{4}uu_x u_y \\
& + \frac{3}{2}u_{xx}u_y + \frac{3}{2}u_x u_{xy}\big) \\
& + \delta_1 u_{tx} + \delta_2 u_{ty} + \delta_3 u_{xx} + \delta_4 u_{xy} + \delta_5 u_{yy} = 0,
\end{aligned} \tag{2.7}$$

subject to the compatibility condition $v_x = u_y$. This equation incorporates two sets of nonlinear terms together with five linear dispersive terms. The nonlinear terms differ significantly from those in the classical generalized Bogoyavlensky-Konopelchenko equation associated with $p = 2$ (see, e.g., [21]). Their complexity arises from two fourth-order generalized bilinear derivatives associated with the prime number $p = 3$.

Special reductions occur when only one dispersive coefficient is active: If $\delta_1 = 1$ and others vanish, the generalized equation (2.7) reduces to a BK-like equation with u_{tx}:

$$\begin{aligned}
& \sigma_1\big(3u_x u_{xx} + 3uu_x^2 + \frac{3}{2}u^3 u_x + \frac{3}{2}u^2 u_{xx}\big) \\
& + \sigma_2\big(\frac{9}{8}u^2 u_x v + \frac{3}{8}u^3 u_y + \frac{3}{4}uu_{xx}v \\
& + \frac{3}{4}u_x^2 v + \frac{3}{4}u^2 u_{xy} + \frac{9}{4}uu_x u_y \\
& + \frac{3}{2}u_{xx}u_y + \frac{3}{2}u_x u_{xy}\big) + u_{tx} = 0.
\end{aligned} \tag{2.8}$$

If $\delta_2 = 1$ and others vanish, it reduces reduces instead to a BK-like equation with u_{ty}:

$$\begin{aligned}
& \sigma_1\big(3u_x u_{xx} + 3uu_x^2 + \frac{3}{2}u^3 u_x + \frac{3}{2}u^2 u_{xx}\big) \\
& + \sigma_2\big(\frac{9}{8}u^2 u_x v + \frac{3}{8}u^3 u_y + \frac{3}{4}uu_{xx}v \\
& + \frac{3}{4}u_x^2 v + \frac{3}{4}u^2 u_{xy} + \frac{9}{4}uu_x u_y \\
& + \frac{3}{2}u_{xx}u_y + \frac{3}{2}u_x u_{xy}\big) + u_{ty} = 0.
\end{aligned} \tag{2.9}$$

Here, $v_x = u_y$ is defined as before.

2.3 Relation Between Bilinear and Nonlinear Forms

The bilinear form (2.5) and the nonlinear Equation (2.7) are related through

$$P_{\text{gBK-like}}(u, v) = \Big[\frac{F_{\text{gBK-like}}(f)}{f^2}\Big]_x. \tag{2.10}$$

Thus, if f satisfies the bilinear equation, the fields u, v defined in (2.6) solve the corresponding nonlinear gBK-like model.

This generalized framework raises important questions regarding integrability, such as the existence of lump solutions, a hallmark of integrable systems. In the next section, we investigate lump wave solutions generated by dispersive contributions.

3 Lump Waves Induced by Dispersion

We now turn to the explicit construction of lump wave solutions for the gBK-like Equation (2.7), by analyzing its generalized bilinear form (2.5) with the aid of symbolic computation. In particular, we emphasize the role of the five dispersion terms, which are indispensable for the formation of lump structures, and we investigate the stationary points of the associated quadratic function.

3.1 Sum-of-Squares Ansatz

A standard approach for deriving lump solutions in higher-dimensional nonlinear evolution equations is the sum-of-squares ansatz, originally proposed and applied to the KPI equation in [7]. The central idea is to represent the dependent variable in terms of a logarithmic derivative of a positive quadratic function. Specifically, we adopt the quadratic form

$$\begin{aligned}
f = \zeta_1^2 + \zeta_2^2 + a_9,\ & \zeta_1 = a_1 x + a_2 y + a_3 t + a_4, \\
& \zeta_2 = a_5 x + a_6 y + a_7 t + a_8,
\end{aligned} \tag{3.1}$$

which guarantees rational localization in both spatial directions. Substituting (3.1) into the bilinear Equation (2.5) results in a large algebraic system for the parameters a_i, reducing the problem to solving this nonlinear system. Fortunately, this system is solvable for the parameters a_3, a_7, and a_9. Symbolic computation then yields explicit expressions for a_3, a_7, and a_9, while leaving the remaining coefficients free:

$$\begin{aligned} a_3 = &-\frac{1}{(a_1\delta_1+a_2\delta_2)^2+(a_5\delta_1+a_6\delta_2)^2}\big[a_1(a_1^2+a_5^2)\delta_1\delta_3 \\ &+a_2(a_1^2+a_5^2)\delta_1\delta_4 \\ &+(a_1a_2^2-a_1a_6^2+2a_2a_5a_6)\delta_1\delta_5 \\ &+(a_1^2a_2+2a_1a_5a_6-a_2a_5^2)\delta_2\delta_3 \\ &+a_1(a_2^2+a_6^2)\delta_2\delta_4+a_2(a_2^2+a_6^2)\delta_2\delta_5\big], \end{aligned} \tag{3.2}$$

$$\begin{aligned} a_7 = &-\frac{1}{(a_1\delta_1+a_2\delta_2)^2+(a_5\delta_1+a_6\delta_2)^2}\big[a_5(a_1^2+a_5^2)\delta_1\delta_3 \\ &+a_6(a_1^2+a_5^2)\delta_1\delta_4 \\ &+(2a_1a_2a_6-a_2^2a_5+a_5a_6^2)\delta_1\delta_5 \\ &+(2a_1a_2a_5-a_1^2a_6+a_5^2a_6)\delta_2\delta_3 \\ &+a_5(a_2^2+a_6^2)\delta_2\delta_4+a_6(a_2^2+a_6^2)\delta_2\delta_5\big], \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} a_9 = &-\frac{3\sigma_1(a_1^2+a_5^2)^2[(a_1\delta_1+a_2\delta_2)^2+(a_5\delta_1+a_6\delta_2)^2]}{(a_1a_6-a_2a_5)^2(\delta_1^2\delta_5-\delta_1\delta_2\delta_4+\delta_2^2\delta_3)} \\ &-\frac{3\sigma_2(a_1a_2+a_5a_6)(a_1^2+a_5^2)[(a_1\delta_1+a_2\delta_2)^2+(a_5\delta_1+a_6\delta_2)^2]}{(a_1a_6-a_2a_5)^2(\delta_1^2\delta_5-\delta_1\delta_2\delta_4+\delta_2^2\delta_3)}. \end{aligned} \tag{3.4}$$

The resulting expressions, (3.2), (3.3), and (3.4), encode the dispersion relations and the structural conditions underlying the lump. In particular, a_3 and a_7 characterize temporal frequencies linked to higher-order dispersive effects, while a_9 is determined by a delicate balance among the wave numbers and dispersion coefficients. Similar dispersion relations appear in lump solutions associated with the second flow of the KP hierarchy [22] and in generalized KP-type models (see, e.g., [23]-[27]).

Well-posedness of the solution requires two essential non-degeneracy conditions, including a dispersion condition

$$\delta_1^2\delta_5-\delta_1\delta_2\delta_4+\delta_2^2\delta_3\neq 0, \tag{3.5}$$

which guarantees

$$\delta_1^2+\delta_2^2\neq 0, \tag{3.6}$$

and the determinant condition

$$a_1a_6-a_2a_5\neq 0, \tag{3.7}$$

which ensures

$$a_1^2+a_5^2\neq 0,\ \ a_2^2+a_6^2\neq 0, \tag{3.8}$$

thereby guaranteeing that the solution u and v, defined via the logarithmic derivative transformation (2.6), decays to zero as $x^2+y^2\to\infty$, confirming spatial localization.

The positivity of f, and thus analyticity of the resulting lump waves u and v, is secured by the following necessary and sufficient mixed condition involving the wave numbers and dispersion parameters:

$$\frac{\sigma_1(a_1^2+a_5^2)+\sigma_2(a_1a_2+a_5a_6)}{\delta_1^2\delta_5-\delta_1\delta_2\delta_4+\delta_2^2\delta_3}<0. \tag{3.9}$$

This condition can be ensured by either of the following two sufficient conditions:

$$\frac{\sigma_1}{\delta_1^2\delta_5-\delta_1\delta_2\delta_4+\delta_2^2\delta_3}\leq 0,\ \ \frac{\sigma_2(a_1a_2+a_5a_6)}{\delta_1^2\delta_5-\delta_1\delta_2\delta_4+\delta_2^2\delta_3}<0. \tag{3.10}$$

$$\frac{\sigma_1}{\delta_1^2\delta_5-\delta_1\delta_2\delta_4+\delta_2^2\delta_3}< 0,\ \ \frac{\sigma_2(a_1a_2+a_5a_6)}{\delta_1^2\delta_5-\delta_1\delta_2\delta_4+\delta_2^2\delta_3}\leq 0. \tag{3.11}$$

The condition (3.9) ensures that $a_9>0$, as seen from (3.4), and consequently that f, defined in (3.1), is positive. This, in turn, guarantees that the corresponding solutions u and v are analytic throughout the (x, y, t)-domain.

In summary, the construction of lump wave solutions via the logarithmic derivative transformation requires the two essential conditions (3.9) and (3.7). The former ensures the well-posedness of u and v across the spatial-temporal domain, while the latter guarantees spatial localization in all directions. The specific conditions (3.10) or (3.11) are sufficient to ensure analytic regularity. Under the two essential requirements, the resulting u and v indeed represent lump wave solutions.

Obersving that the conditions in (3.9), (3.10) and (3.11) depend strongly on the coefficients of the five dispersion terms, the resulting lump wave solutions are governed by the dispersive effects in the model.

3.2 Path of Stationary Points

The dynamical behaviour of the lump waves can be further elucidated by computing the stationary points of f. Setting $f_x=f_y=0$ leads to the linear system

$$a_1\zeta_1+a_5\zeta_2=0,\ \ a_2\zeta_1+a_6\zeta_2=0, \tag{3.12}$$

which, under the non-degeneracy condition (3.7), reduces to

$$\zeta_1=0,\ \ \zeta_2=0, \tag{3.13}$$

with ζ_1, ζ_2 given by (3.1). Solving (3.13) yields explicit linear relations for $x(t)$ and $y(t)$, stationary points of the quadratic function:

$$x(t) = \frac{[(a_1^2+a_5^2)\delta_3 - (a_2^2+a_6^2)\delta_5]\delta_1 + [2(a_1a_2+a_5a_6)\delta_3 + (a_2^2+a_6^2)\delta_4]\delta_2}{(a_1\delta_1+a_2\delta_2)^2 + (a_5\delta_1+a_6\delta_2)^2} t + \frac{a_2a_8 - a_4a_6}{a_1a_6 - a_2a_5}, \tag{3.14}$$

$$y(t) = \frac{[(a_1^2+a_5^2)\delta_4 + 2(a_1a_2+a_5a_6)\delta_5]\delta_1 - [(a_1^2+a_5^2)\delta_3 - (a_2^2+a_6^2)\delta_5]\delta_2}{(a_1\delta_1+a_2\delta_2)^2 + (a_5\delta_1+a_6\delta_2)^2} t - \frac{a_1a_8 - a_4a_5}{a_1a_6 - a_2a_5}. \tag{3.15}$$

These parametric formulas describe a straight-line trajectory in the (x, y)-plane, along which the lump attains zero amplitude. In other words, the lump vanishes identically on this characteristic path, while retaining its rational localization elsewhere.

4 Concluding Remarks

A (2+1)-dimensional generalized Bogoyavlensky–Konopelchenko-type model is formulated and analyzed, and its lump wave structures are investigated through symbolic computation with computer algebra systems. The constructed lump waves vanish along a characteristic trajectory determined by the stationary points of the associated quadratic function.

Lump waves arise in a broad range of physical and mathematical contexts, reflecting both their versatility and the complexity of modeling nonlinear dispersive phenomena. Previous studies have explored lump solutions in linear models [18, 19], as well as in nonlinear and nonintegrable models across (2+1) dimensions [28]–[37], (3+1) dimensions [23, 38, 39], and (4+1) dimensions [40]. Their construction frequently relies on Hirota bilinear forms and their generalizations, which serve as powerful tools for identifying localized coherent structures [14].

Moreover, lump waves display rich interactions with other coherent structures in (2+1)-dimensional integrable systems, including homoclinic and heteroclinic waves [41–43]. At the same time, N-soliton solutions and integrability have been systematically investigated in both local and nonlocal frameworks, for example via Riemann–Hilbert techniques and bi-Hamiltonian formulations [44]–[49]. Long-time asymptotics of Cauchy problems have been studied for various integrable models, including the three-component coupled nonlinear Schrödinger system [50], the spin-1 Gross–Pitaevskii equation [51], and the modified Korteweg-de Vries hierarchy [52], the latter two of which can be reduced from matrix integrable models via group reductions (see (4.10) in [53] and (3.9) with $m = n = 1$ and $\Sigma_1 = -1,\ \Sigma_2 = 1,\ \Delta_1 = 1,$ and $\Delta_2 = 1$ in [54], respectively), as well as the modified Camassa-Holm equation [55], for which the soliton resolution conjecture and the asymptotic stability of N-soliton solutions has been established and studied [55]. The existence, structure, and dynamics of lump waves in (2+1)-dimensional extensions of integrable systems–whether scalar or multi-component, standard or generalized bilinear–remain active and compelling topics for further study (see, e.g., [56]-[60]).

In summary, the investigation of lump waves deepens our understanding of nonlinear dispersive dynamics and offers potential applications in physical and engineering systems where localized, coherent, and energy-concentrated structures play a fundamental role.

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