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**Cite this article:** Ma W-X. 2017 Trigonal curves and algebro-geometric solutions to soliton hierarchies I. *Proc. R. Soc. A* **473**: 20170232.
http://dx.doi.org/10.1098/rspa.2017.0232

Received: 7 April 2017
Accepted: 5 June 2017

**Subject Areas:**
applied mathematics, differential equations

**Keywords:**
trigonal curve, Baker–Akhiezer function, Dubrovin-type equations

**Author for correspondence:**
Wen-Xiu Ma
e-mail: mawx@cas.usf.edu

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# Trigonal curves and algebro-geometric solutions to soliton hierarchies I

Wen-Xiu Ma[1,2,3,4]

[1]Department of Mathematics, Zhejiang Normal University, Jinhua 321004, People's Republic of China
[2]College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, Shandong, People's Republic of China
[3]Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA
[4]Department of Mathematical Sciences, North-West University, Mmabatho 2735, South Africa

W-XM, 0000-0001-5309-1493

This is the first part of a study, consisting of two parts, on Riemann theta function representations of algebro-geometric solutions to soliton hierarchies. In this part, using linear combinations of Lax matrices of soliton hierarchies, we introduce trigonal curves by their characteristic equations, explore general properties of meromorphic functions defined as ratios of the Baker–Akhiezer functions, and determine zeros and poles of the Baker–Akhiezer functions and their Dubrovin-type equations. We analyse the four-component AKNS soliton hierarchy in such a way that it leads to a general theory of trigonal curves applicable to construction of algebro-geometric solutions of an arbitrary soliton hierarchy.

## 1. Introduction

Algebro-geometric solutions to soliton equations are one important class of exact solutions, which describe periodic and quasi-periodic nonlinear phenomena in physical and engineering sciences [1–3]. With the development of solitons and finite-gap solutions to the Korteweg–de Vries equation, the mathematical theory of algebro-geometric solutions has been systematically developed since the early 1970s, particularly for the Korteweg–de Vries, modified Korteweg–de Vries, nonlinear Schrödinger, sine-Gordon, Kadomtsev–Petviashvili, Toda lattice and Camassa–Holm equations [1–12].

Dubrovin and Krichever proposed a systematic method of algebraic geometry to integration of nonlinear partial differential equations, which aims at constructing periodic and almost periodic solutions in terms of the Riemann theta functions for well-known integrable equations including the Korteweg–de Vries equation and the Kadomtsev–Petviashvili equation [13–15]. Cao and Geng made use of the nonlinearization technique of Lax pairs to generate algebro-geometric solutions of finite-dimensional integrable Hamiltonian systems and combined systems from lower dimensions to higher dimensions [16–18], and later, the nonlinearization technique was applied to constructing algebro-geometric solutions of a great number of soliton equations in both (1+1)- and (2+1)-dimensions [19–27]. Gesztesy *et al.* [3,28,29] established an alternative approach for constructing quasi-periodic solutions to soliton hierarchies associated with $2 \times 2$ matrix spectral problems, and by this approach, quasi-periodic solutions to many continuous and discrete soliton hierarchies have been constructed within a different kind of formulation using the Riemann theta functions [28,30,31].

The study of algebro-geometric solutions has opened up a new vista in the analysis of nonlinear partial differential equations. The adopted algebro-geometric techniques brought innovative ideas and led to inspiring results in soliton theory as well as algebraic geometry, for example, a solution of the Riemann–Schottky problem [3,32]. The successful idea in constructing algebro-geometric solutions is to employ the theory of algebraic curves associated with Lax pairs producing soliton hierarchies to represent the Baker–Akhiezer functions [33,34] in terms of the Riemann theta function [35,36]. The obtained algebro-geometric solutions satisfy a class of stationary counterparts of soliton equations, called Novikov-type equations [37]. It is also noted that symmetry constraints pave a way of separation of variables for soliton equations and the corresponding canonical variables solving the associated Jacobi inversion problems provide the so-called characteristic variables in the Riemann theta function presentation of algebro-geometric solutions [38,39]. There are primarily two types of research on algebro-geometric solutions. One is to explore asymptotics of the Baker–Akhiezer functions to construct algebro-geometric solutions to given nonlinear equations, and the other is to connect the Baker–Akhiezer functions possessing given asymptotics with potential nonlinear equations and their algebro-geometric solutions.

Very recently, Geng *et al.* successfully attempted a few $3 \times 3$ matrix spectral problems and constructed algebro-geometric solutions to the associated soliton hierarchies, including the modified Boussinesq hierarchy, the Kaup–Kupershmidt hierarchy and the hierarchy of three-wave resonant interaction equations [40–43]. In this paper, we would like to propose a general framework to analyse $3 \times 3$ matrix spectral problems and their corresponding trigonal curves, and to generate algebro-geometric solutions to soliton hierarchies by observing asymptotic behaviours of the Baker–Akhierzer functions. We analyse the four-component AKNS soliton hierarchy, particularly asymptotics of the Baker–Akhiezer functions, in such a way that it yields a general theory applicable to soliton hierarchies associated with arbitrary $3 \times 3$ matrix spectral problems.

Our study is divided into two parts. This is the first part, comprising five sections. In §2, with the aid of the zero-curvature formulation and the trace identity, we rederive the four-component AKNS soliton hierarchy and its bi-Hamiltonian structure. In §3, we introduce a class of trigonal curves by taking linear combinations of the Lax matrices and analyse the corresponding Baker–Akhiezer functions. In §4, we first present a general structure of Dubrovin-type dynamical equations [44] of zeros and poles of meromorphic functions as the characteristic variables, and then apply the resulting general theorems to the four-component AKNS case. In the last section, concluding remarks will be given.

## 2. Four-component AKNS soliton hierarchy

### (a) Soliton hierarchy

Let us recall the zero curvature formulation and the trace identity [45]. Let $U = U(u, \lambda)$ be a square spectral matrix belonging to a given matrix loop algebra, where $u$ is a potential and $\lambda$ is a spectral parameter. Assume that

$$W = W(u,\lambda) = \sum_{k=0}^{\infty} W_k \lambda^{-k} = \sum_{k=0}^{\infty} W_k(u)\lambda^{-k} \tag{2.1}$$

solves the corresponding stationary zero curvature equation

$$W_x = [U, W]. \tag{2.2}$$

Then, introduce a series of Lax matrices

$$V^{[r]} = V^{[r]}(u,\lambda) = (\lambda^r W)_+ + \Delta_r, \tag{2.3}$$

where the subscript $+$ denotes the operation of taking a polynomial part in $\lambda$ and $\Delta_r$, $r \geq 0$, are appropriate modification terms, such that a soliton hierarchy

$$u_{t_r} = K_r(u) = K_r(x, t, u, u_x, \ldots), \quad r \geq 0, \tag{2.4}$$

can be generated from a series of zero curvature equations

$$U_{t_r} - V_x^{[r]} + [U, V^{[r]}] = 0, \quad r \geq 0. \tag{2.5}$$

The two matrices $U$ and $V^{[r]}$ are called a Lax pair [46] of the $r$th soliton equation in the hierarchy (2.4). The zero curvature equations in (2.5) are the compatibility conditions of the spatial and temporal spectral problems:

$$\psi_x = U\psi = U(u,\lambda)\psi, \psi_{t_r} = V^{[r]}\psi = V^{[r]}(u,\lambda)\psi, \quad r \geq 0, \tag{2.6}$$

where $\psi$ is the vector eigenfunction.

One important task in soliton theory is to show the Liouville integrability of soliton equations in a hierarchy. This can be usually achieved by establishing a bi-Hamiltonian formulation [47]:

$$u_{t_r} = K_r = J\frac{\delta \tilde{H}_{r+1}}{\delta u} = M\frac{\delta \tilde{H}_r}{\delta u}, \quad r \geq 1, \tag{2.7}$$

where $J$ and $M$ constitute a Hamiltonian pair and $\delta/\delta u$ denotes the variational derivative [48]. The Hamiltonian structures can be furnished through the trace identity [45]:

$$\frac{\delta}{\delta u}\int \operatorname{tr}\left(W\frac{\partial U}{\partial \lambda}\right) \mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\left[\lambda^{\gamma}\operatorname{tr}\left(W\frac{\partial U}{\partial u}\right)\right], \quad \gamma = -\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln|\operatorname{tr}(W^2)| \tag{2.8}$$

or more generally, the variational identity [49]:

$$\frac{\delta}{\delta u}\int \left\langle W, \frac{\partial U}{\partial \lambda}\right\rangle \mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\left[\lambda^{\gamma}\left\langle W, \frac{\partial U}{\partial u}\right\rangle\right], \quad \gamma = -\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln|\langle W, W\rangle|, \tag{2.9}$$

where $\langle\cdot,\cdot\rangle$ is a non-degenerate, symmetric and ad-invariant bilinear form on the underlying matrix loop algebra [50]. The bi-Hamiltonian formulation guarantees the commutativity of infinitely many Lie symmetries $\{K_n\}_{n=0}^{\infty}$ and conserved quantities $\{\tilde{H}_n\}_{n=0}^{\infty}$:

$$[K_{n_1}, K_{n_2}] = K'_{n_1}[K_{n_2}] - K'_{n_2}[K_{n_1}] = 0 \tag{2.10}$$

and

$$\{\tilde{\mathcal{H}}_{n_1}, \tilde{\mathcal{H}}_{n_2}\}_N = \int \left(\frac{\delta \tilde{\mathcal{H}}_{n_1}}{\delta u}\right)^{\mathrm{T}} N \frac{\delta \tilde{\mathcal{H}}_{n_2}}{\delta u}\,\mathrm{d}x = 0, \tag{2.11}$$

where $n_1, n_2 \geq 0$, $N = J$ or $M$, and $K'$ denotes the Gateaux derivative of $K$:

$$K'(u)[S] = \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} K(u + \varepsilon S, u_x + \varepsilon S_x, \ldots). \tag{2.12}$$

It is known that, for an evolution equation $u_t = K(u)$, $\tilde{H} = \int H\,\mathrm{d}x$ is a conserved functional iff $\frac{\delta \tilde{H}}{\delta u}$ is an adjoint symmetry [51], and so, the Hamiltonian structures links conserved functionals to adjoint symmetries and further symmetries. When the underlying matrix loop algebra in the zero curvature formulation is simple, the associated zero curvature equations yield classical soliton hierarchies [52]; when semisimple, the associated zero curvature equations yield a collection of different soliton hierarchies; and when non-semisimple, we obtain a hierarchy of integrable couplings [53], which needs extra care in constructing exact solutions.

## (b) Four-component AKNS hierarchy

Let us consider a $3\times 3$ matrix spectral problem

$$\psi_x = U\psi = U(u,\lambda)\psi,\ U=(U_{ij})_{3\times 3} = \begin{bmatrix} -2\lambda & p_1 & p_2 \\ q_1 & \lambda & 0 \\ q_2 & 0 & \lambda \end{bmatrix}, \quad \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}, \tag{2.13}$$

where $\lambda$ is a spectral parameter and $u$ is a four-component potential

$$u=(p,q^{\mathrm{T}})^{\mathrm{T}}, \quad p=(p_1,p_2), \quad q=(q_1,q_2)^{\mathrm{T}}. \tag{2.14}$$

Since $U_0=\mathrm{diag}(-2,1,1)$ has a multiple eigenvalue, the spectral problem (2.13) is degenerate. Under the special reduction of $p_2=q_2=0$, (2.13) is equivalent to the AKNS spectral problem [54], and thus it is called a four-component AKNS spectral problem.

To derive the associated soliton hierarchy, we first solve the stationary zero curvature equation (2.2) corresponding to (2.13). We suppose that a solution $W$ is given by

$$W=\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2.15}$$

where $a$ is a scalar, $b^{\mathrm{T}}$ and $c$ are two-dimensional columns and $d$ is a $2\times 2$ matrix. Then, the stationary zero curvature equation (2.2) becomes

$$a_x = pc-bq, \quad b_x=-3\lambda b+pd-ap, \quad c_x=3\lambda c+qa-dq \quad \text{and} \quad d_x=qb-cp. \tag{2.16}$$

We seek a formal series solution as

$$W=\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\sum_{k=0}^{\infty} W_k\lambda^{-k}, \quad W_k=W_k(u)=\begin{bmatrix} a^{[k]} & b^{[k]} \\ c^{[k]} & d^{[k]} \end{bmatrix}, \quad k\geq 0, \tag{2.17}$$

with $b^{[k]}, c^{[k]}$ and $d^{[k]}$ being assumed to be

$$b^{[k]}=(b_1^{[k]},b_2^{[k]}), \quad c^{[k]}=(c_1^{[k]},c_2^{[k]})^{\mathrm{T}} \quad \text{and} \quad d^{[k]}=(d_{ij}^{[k]})_{2\times 2}, \quad k\geq 0. \tag{2.18}$$

Thus, the system (2.16) equivalently leads to the following recursion relations:

$$b^{[0]}=0, \quad c^{[0]}=0, \quad a_x^{[0]}=0, \quad d_x^{[0]}=0, \tag{2.19a}$$

$$b^{[k+1]}=\tfrac{1}{3}(-b_x^{[k]}+pd^{[k]}-a^{[k]}p), \quad k\geq 0, \tag{2.19b}$$

$$c^{[k+1]}=\tfrac{1}{3}(c_x^{[k]}-qa^{[k]}+d^{[k]}q), \quad k\geq 0 \tag{2.19c}$$

and

$$a_x^{[k]}=pc^{[k]}-b^{[k]}q,\ d_x^{[k]}=qb^{[k]}-c^{[k]}p, \quad k\geq 1. \tag{2.19d}$$

We choose the initial values as follows:

$$a^{[0]}=-2 \quad \text{and} \quad d^{[0]}=I_2, \tag{2.20}$$

where $I_2=\mathrm{diag}(1,1)$, and take constants of integration in (2.19*d*) to be zero:

$$W_k|_{u=0}=0, \quad k\geq 1. \tag{2.21}$$

Therefore, with $a^{[0]}$ and $d^{[0]}$ given by (2.20), all matrices $W_k$, $k \geq 1$, will be uniquely determined. For example, it follows from (2.19) that

$$\left.\begin{aligned}
&b_i^{[1]} = p_i, \quad c_i^{[1]} = q_i, \quad a^{[1]} = 0, \quad d_{ij}^{[1]} = 0;\\
&b_i^{[2]} = -\tfrac{1}{3} p_{i,x}, \quad c_i^{[2]} = \tfrac{1}{3} q_{i,x}, \quad a^{[2]} = \tfrac{1}{3}(p_1q_1 + p_2q_2), \quad d_{ij}^{[2]} = -\tfrac{1}{3} p_j q_i;\\
&b_i^{[3]} = \tfrac{1}{9}[p_{i,xx} - 2(p_1q_1 + p_2q_2)p_i], \quad c_i^{[3]} = \tfrac{1}{9}[q_{i,xx} - 2(p_1q_1 + p_2q_2)q_i],\\
&a^{[3]} = \tfrac{1}{9}(p_1q_{1,x} - p_{1,x}q_1 + p_2q_{2,x} - p_{2,x}q_2), \quad d_{ij}^{[3]} = \tfrac{1}{9}(p_{j,x}q_i - p_jq_{i,x});\\
&b_i^{[4]} = -\tfrac{1}{27}[p_{i,xxx} - 3(p_1q_1 + p_2q_2)p_{i,x} - 3(p_{1,x}q_1 + p_{2,x}q_2)p_i],\\
&c_i^{[4]} = \tfrac{1}{27}[q_{i,xxx} - 3(p_1q_1 + p_2q_2)q_{i,x} - 3(p_1q_{1,x} + p_2q_{2,x})q_i],\\
&a^{[4]} = -\tfrac{1}{27}[3(p_1q_1 + p_2q_2)^2 - p_1q_{1,xx} + p_{1,x}q_{1,x} - p_{1,xx}q_1 - p_2q_{2,xx} + p_{2,x}q_{2,x} - p_{2,xx}q_2]\\
\text{and}\quad &d_{ij}^{[4]} = \tfrac{1}{27}[3p_j(p_1q_1 + p_2q_2)q_i - p_{j,xx}q_i + p_{j,x}q_{i,x} - p_jq_{i,xx}],
\end{aligned}\right\} \tag{2.22}$$

where $1 \leq i, j \leq 2$. Based on (2.19*d*), we can obtain, from (2.19*b*) and (2.19*c*), a recursion relation for $b^{[k]}$ and $c^{[k]}$:

$$\begin{bmatrix} c^{[k+1]} \\ b^{[k+1]\mathrm{T}} \end{bmatrix} = \Psi \begin{bmatrix} c^{[k]} \\ b^{[k]\mathrm{T}} \end{bmatrix}, \quad k \geq 1, \tag{2.23}$$

where $\Psi$ is a $4 \times 4$ matrix operator

$$\Psi = \frac{1}{3}\begin{bmatrix} \left(\partial - \sum_{i=1}^{2} q_i \partial^{-1} p_i\right) I_2 - q\partial^{-1}p & q\partial^{-1}q^{\mathrm{T}} + (q\partial^{-1}q^{\mathrm{T}})^{\mathrm{T}} \\ -p^{\mathrm{T}}\partial^{-1}p - (p^{\mathrm{T}}\partial^{-1}p)^{\mathrm{T}} & \left(-\partial + \sum_{i=1}^{2} p_i\partial^{-1}q_i\right) I_2 + p^{\mathrm{T}}\partial^{-1}q^{\mathrm{T}}. \end{bmatrix} \tag{2.24}$$

As usual, for all integers $r \geq 0$, we introduce the following Lax matrices:

$$V^{[r]} = V^{[r]}(u, \lambda) = (V_{ij}^{[r]})_{3\times 3} = (\lambda^r W)_+ = \sum_{k=0}^{r} W_k \lambda^{r-k}, \quad r \geq 0, \tag{2.25}$$

where the modification terms are taking as zero. Note that we have

$$V^{[r+1]} = \sum_{k=0}^{r+1} W_k \lambda^{r-k+1} = \lambda \sum_{k=0}^{r+1} W_k \lambda^{r-k} = \lambda V^{[r]} + W_{r+1}, \quad r \geq 0. \tag{2.26}$$

The compatibility conditions of (2.6), i.e. the zero curvature equation (2.5), generate the four-component AKNS soliton hierarchy

$$u_{t_r} = \begin{bmatrix} p^{\mathrm{T}} \\ q \end{bmatrix}_{t_r} = K_r = \begin{bmatrix} -3b^{[r+1]\mathrm{T}} \\ 3c^{[r+1]} \end{bmatrix}, \quad r \geq 0. \tag{2.27}$$

The first two nonlinear systems in this soliton hierarchy (2.27) read

$$p_{i,t_2} = -\tfrac{1}{3}[p_{i,xx} - 2(p_1q_1 + p_2q_2)p_i] \quad \text{and} \quad q_{i,t_2} = \tfrac{1}{3}[q_{i,xx} - 2(p_1q_1 + p_2q_2)q_i], \quad 1 \leq i \leq 2 \tag{2.28}$$

and

$$p_{i,t_3} = \tfrac{1}{9}[p_{i,xxx} - 3(p_1q_1 + p_2q_2)p_{i,x} - 3(p_{1,x}q_1 + p_{2,x}q_2)p_i], \quad 1 \leq i \leq 2, \tag{2.29a}$$

$$q_{i,t_3} = \tfrac{1}{9}[q_{i,xxx} - 3(p_1q_1 + p_2q_2)q_{i,x} - 3(p_1q_{1,x} + p_2q_{2,x})q_i], \quad 1 \leq i \leq 2, \tag{2.29b}$$

which are the four-component versions of the AKNS systems of nonlinear Schrödinger equations and modified Korteweg–de Vries equations, respectively. The four-component AKNS equations (2.28) can be reduced to the Manokov system [55], for which a decomposition into finite-dimensional integrable Hamiltonian systems was given in [56], whereas the four-component

AKNS equations (2.29$b$) contain various mKdV equations, for which there exist different kinds of integrable decompositions [57,58].

We point out that the four-component AKNS soliton hierarchy (2.27) has a Hamiltonian structure [51], which can be generated through the trace identity [45], or more generally, the variational identity [49]. Actually, we have

$$\operatorname{tr}\left(W\frac{\partial U}{\partial \lambda}\right) = -2a + \operatorname{tr}(d) = \sum_{k=0}^{\infty}(-2a^{[k]} + d_{11}^{[k]} + d_{22}^{[k]})\lambda^{-k}$$

and

$$\operatorname{tr}\left(W\frac{\partial U}{\partial u}\right) = \begin{bmatrix} c \\ b^{\mathrm{T}} \end{bmatrix} = \sum_{k\geq 0} G_{k-1}\lambda^{-k}.$$

Inserting these expressions into the trace identity and considering the case of $k=2$, we get $\gamma = 0$ and thus we have

$$\frac{\delta \tilde{H}_k}{\delta u} = G_{k-1}, \quad \tilde{H}_k = \frac{1}{k}\int (2a^{[k+1]} - d_{11}^{[k+1]} - d_{22}^{[k+1]})\,\mathrm{d}x, \quad G_{k-1} = \begin{bmatrix} c^{[k]} \\ b^{[k]T} \end{bmatrix}, \quad k \geq 1. \tag{2.30}$$

A bi-Hamiltonian structure of the four-component AKNS equations (2.27) then follows:

$$u_{t_r} = K_r = JG_r = J\frac{\delta \tilde{H}_{r+1}}{\delta u} = M\frac{\delta \tilde{H}_r}{\delta u}, \quad r \geq 1, \tag{2.31}$$

where the Hamiltonian pair $(J, M = J\Psi)$ is given by

$$J = \begin{bmatrix} 0 & -3I_2 \\ 3I_2 & 0 \end{bmatrix} \tag{2.32a}$$

and

$$M = \begin{bmatrix} p^{\mathrm{T}}\partial^{-1}p + (p^{\mathrm{T}}\partial^{-1}p)^{\mathrm{T}} & \left(\partial - \sum_{i=1}^{2} p_i\partial^{-1}q_i\right) I_2 - p^{\mathrm{T}}\partial^{-1}q^{\mathrm{T}} \\ \left(\partial - \sum_{i=1}^{2} p_i\partial^{-1}q_i\right) I_2 - q\partial^{-1}p & q\partial^{-1}q^{\mathrm{T}} + (q\partial^{-1}q^{\mathrm{T}})^{\mathrm{T}} \end{bmatrix}. \tag{2.32b}$$

Adjoint symmetry constraints or equivalently symmetry constraints separate the above four-component AKNS equations into two commuting finite-dimensional Liouville integrable Hamiltonian systems [51].

## 3. Trigonal curves and Baker–Akhiezer functions

For each integer $n \geq 1$, let us take a linear combination of the Lax matrices

$$W^{[n]} = W^{[n]}(u,\lambda) = (W_{ij}^{[n]})_{3\times 3} = \sum_{k=0}^{n} \alpha_k V^{[n-k]}, \tag{3.1}$$

where the Lax matrices $V^{[k]}$, $0 \leq k \leq n$, are given by (2.25) and $\alpha_k$, $0 \leq k \leq n$, are arbitrary constants but $\alpha_0 \neq 0$. Its corresponding characteristic polynomial reads

$$\mathcal{F}_m(\lambda, y) = \det(yI_3 - W^{[n]}) = y^3 + yS_m(\lambda) - T_m(\lambda), \tag{3.2}$$

where $I_3 = \operatorname{diag}(1,1,1)$, $S_m$ and $T_m$ are two polynomials of $\lambda$ with degrees $\deg(S_m) = 2n$ and $\deg(T_m) = 3n$, defined by

$$S_m = \sum_{1\leq i<j\leq 3} \begin{vmatrix} W_{ii}^{[n]} & W_{ij}^{[n]} \\ W_{ji}^{[n]} & W_{jj}^{[n]} \end{vmatrix}, \quad T_m = \det W^{[n]} = \begin{vmatrix} W_{11}^{[n]} & W_{12}^{[n]} & W_{13}^{[n]} \\ W_{21}^{[n]} & W_{22}^{[n]} & W_{23}^{[n]} \\ W_{31}^{[n]} & W_{32}^{[n]} & W_{33}^{[n]} \end{vmatrix} \tag{3.3}$$

and $m = \max(\deg(S_m), \deg(T_m)) = 3n$.

Using the combined Lax matrix $W^{[n]}$, we introduce a trigonal curve $\mathcal{K}_g$ of degree $m$ as follows:

$$\mathcal{K}_g = \{P = (\lambda, y) \in \mathbb{C}^2 \mid \det(yI_3 - W^{[n]}) = y^3 + yS_m(\lambda) - T_m(\lambda) = 0\}. \tag{3.4}$$

Note that the corresponding discriminant $\Delta = -27T_m^2 - 4S_m^3$, a polynomial of $\lambda$ of degree $4n-2$, is not zero at infinity, and thus, the curve has three non-branch points at infinity [59], which we denote by $P_{\infty_i}$, $1 \le i \le 3$. The curve $\mathcal{K}_g$ is compactified by adding those three points at infinity and its compactification is still denoted by $\mathcal{K}_g$ for the sake of convenience. The curve $\mathcal{K}_g$ is said to be non-singular, if we have $(\partial \mathcal{F}_m/\partial\lambda, \partial\mathcal{F}_m/\partial y) \ne 0$, while $\mathcal{F}_m(\lambda, y) = 0$. When $\mathcal{K}_g$ is non-singular, it becomes a three-sheeted Riemann surface of arithmetical genus determined by the Riemann–Hurwitz formula:

$$g = \frac{f}{2} - k + 1 = 2n - 3, \tag{3.5}$$

where $f = 4n - 2$ is the total multiplicity of its branch points and $k = 3$ is the number of sheets. The compact Riemann surface $\mathcal{K}_g$ consists of points satisfying $\mathcal{F}_m(\lambda, y) = 0$ and the three points at infinity: $\{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}\}$.

For a fixed $\lambda \in \mathbb{C}$, we denote the three branches of $y(\lambda)$ satisfying $\mathcal{F}_m(\lambda, y) = 0$ by $y_i = y_i(\lambda)$, $1 \le i \le 3$, and thus, we have

$$(y - y_1(\lambda))(y - y_2(\lambda))(y - y_3(\lambda)) = y^3 + yS_m - T_m = 0, \tag{3.6}$$

from which we can easily get

$$\left.\begin{aligned}
&y_1 + y_2 + y_3 = 0, \quad y_1y_2 + y_1y_3 + y_2y_3 = S_m, \quad y_1y_2y_3 = T_m,\\
&y_1^2 + y_2^2 + y_3^2 = -2S_m, \quad y_1^3 + y_2^3 + y_3^3 = 3T_m,\\
&(y_1 + y_2)y_3^2 + (y_2 + y_3)y_1^2 + (y_3 + y_1)y_2^2 = -3T_m\\
&y_1^2y_2^2 + y_1^2y_3^2 + y_2^2y_3^2 = S_m^2, \quad (3y_1^2 + S_m)(3y_2^2 + S_m)(3y_3^2 + S_m) = -\Delta
\end{aligned}\right\} \tag{3.7}$$

and further we have

$$\sum_{i=1}^{3} \frac{1}{3y_i^2 + S_m} = 0 \quad \text{and} \quad \sum_{i=1}^{3} \frac{y_i}{3y_i^2 + S_m} = 0.$$

The points $(\lambda, y_1(\lambda))$, $(\lambda, y_2(\lambda))$ and $(\lambda, y_3(\lambda))$ are on the three different sheets of the Riemann surface $\mathcal{K}_g$. The holomorphic map $*$, changing sheets, is defined by

$$* : \mathcal{K}_g \to \mathcal{K}_g, \quad P = (\lambda, y_i(\lambda)) \to P^* = (\lambda, y_{i+1\,(\mathrm{mod}\,3)}(\lambda)), \quad 1 \le i \le 3 \tag{3.8}$$

and $P^{**} = (P^*)^*$, etc. Moreover, positive divisors on $\mathcal{K}_g$ of degree $k$ are denoted by

$$\begin{aligned}
&\mathcal{D}_{P_1,\dots,P_k} : \mathcal{K}_g \to \mathbb{N}_0 = \mathbb{N} \cup \{0\},\\
&P \mapsto \mathcal{D}_{P_1,\dots,P_k}(P) = \begin{cases} l, & \text{if } P \text{ occurs } l \text{ times in } \{P_1, \dots, P_k\},\\ 0, & \text{if } P \notin \{P_1, \dots, P_k\}. \end{cases}
\end{aligned} \tag{3.9}$$

Therefore, a divisor of a meromorphic function $f$ on $\mathcal{K}_g$ reads

$$(f(P)) = \mathcal{D}_{P_1,\dots,P_k}(P) - \mathcal{D}_{Q_1,\dots,Q_l}(P), \tag{3.10}$$

if $f$ has zeros $P_i$, $1 \le i \le k$, and poles $Q_i$, $1 \le i \le l$. The space of divisors on $\mathcal{K}_g$ is denoted by $\mathrm{Div}(\mathcal{K}_g)$.

We now introduce a vector of associated Baker–Akhiezer functions $\psi(P, x, x_0, t_r, t_{0,r})$ as follows:

$$\psi_x(P, x, x_0, t_r, t_{0,r}) = U(u(x, t_r), \lambda(P))\psi(P, x, x_0, t_r, t_{0,r}), \tag{3.11}$$

$$\psi_{t_r}(P, x, x_0, t_r, t_{0,r}) = V^{[r]}(u(x, t_r), \lambda(P))\psi(P, x, x_0, t_r, t_{0,r}), \tag{3.12}$$

$$W^{[n]}(u(x, t_r), \lambda(P))\psi(P, x, x_0, t_r, t_{0,r}) = y(P)\psi(P, x, x_0, t_r, t_{0,r}) \tag{3.13}$$

and

$$\psi_i(P, x_0, x_0, t_{0,r}, t_{0,r}) = 1, \quad 1 \le i \le 3, \tag{3.14}$$

where $x, t_r, x_0, t_{0,r}, \lambda(P), y(P) \in \mathbb{C}$ and $P = (\lambda, y) \in \mathcal{K}_g \backslash \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}\}$. The compatibility conditions of the equations (3.11)–(3.13) engender that

$$W_x^{[n]} = [U, W^{[n]}] \tag{3.15}$$

and

$$W_{t_r}^{[n]} = [V^{[r]}, W^{[n]}], \tag{3.16}$$

besides the $r$th zero curvature equation in (2.5). Note that the matrix $yI_3 - W^{[n]}$ also satisfies the Lax equations in (3.15) and (3.16), and so, the characteristic polynomial $\mathcal{F}_m(\lambda, y) = \det(yI_3 - W^{[n]})$ of the combined Lax matrix $W^{[n]}$ is a constant, independent of the variables $x$ and $t_r$, when $u$ solves the $r$th four-component AKNS equation (2.27).

Associated with the Baker–Akhiezer functions, we define a set of meromorphic functions

$$\phi_{ij} = \phi_{ij}(P, x, x_0, t_r, t_{0,r}) = \frac{\psi_i(P, x, x_0, t_r, t_{0,r})}{\psi_j(P, x, x_0, t_r, t_{0,r})}, \quad 1 \le i, j \le 3. \tag{3.17}$$

Based on (3.13), we can have

$$\phi_{ij} = \frac{yW_{ik}^{[n]} + C_{ij}^{[m]}}{yW_{jk}^{[n]} + A_{ij}^{[m]}} = \frac{F_{ij}^{[m]}}{y^2 W_{ik}^{[n]} - yC_{ij}^{[m]} + D_{ij}^{[m]}} = \frac{y^2 W_{jk}^{[n]} - yA_{ij}^{[m]} + B_{ij}^{[m]}}{E_{ij}^{[m]}}, \tag{3.18}$$

with

$$A_{ij}^{[m]} = W_{ji}^{[n]} W_{ik}^{[n]} - W_{jk}^{[n]} W_{ii}^{[n]}, \tag{3.19}$$

$$B_{ij}^{[m]} = W_{jk}^{[n]}(W_{jj}^{[n]} W_{kk}^{[n]} - W_{jk}^{[n]} W_{kj}^{[n]}) + W_{ji}^{[n]}(W_{jj}^{[n]} W_{ik}^{[n]} - W_{jk}^{[n]} W_{ij}^{[n]}), \tag{3.20}$$

$$C_{ij}^{[m]} = A_{ji}^{[m]}, \quad D_{ij}^{[m]} = B_{ji}^{[m]}, \tag{3.21}$$

$$E_{ij}^{[m]} = (W_{jk}^{[n]})^2 W_{ki}^{[n]} + W_{ji}^{[n]} W_{jk}^{[n]}(W_{ii}^{[n]} - W_{kk}^{[n]}) - (W_{ji}^{[n]})^2 W_{ik}^{[n]} \tag{3.22}$$

and

$$F_{ij}^{[m]} = E_{ji}^{[m]}, \tag{3.23}$$

where $\{i, j, k\} = \{1, 2, 3\}$. By the notation $\{i, j, k\} = \{1, 2, 3\}$, we mean here and hereafter to take $1 \le i, j, k \le 3$ arbitrarily, but as three different natural numbers. Obviously from (3.19) and (3.22), we can obtain

$$E_{ij}^{[m]} = -E_{kj}^{[m]} \tag{3.24}$$

and

$$E_{ij}^{[m]} = W_{jk}^{[n]} A_{kj}^{[m]} - W_{ji}^{[n]} A_{ij}^{[m]}, \tag{3.25}$$

where $\{i, j, k\} = \{1, 2, 3\}$.

From the expressions of the meromorphic functions $\phi_{ij}$, $1 \le i, j \le 3$, in (3.18), using $y^3 = -yS_m + T_m$, we can also directly derive the following relations:

$$W_{ik}^{[n]} E_{ij}^{[m]} = -(W_{jk}^{[n]})^2 S_m + W_{jk}^{[n]} B_{ij}^{[m]} - (A_{ij}^{[m]})^2, \tag{3.26}$$

$$C_{ij}^{[m]} E_{ij}^{[m]} = (W_{jk}^{[n]})^2 T_m + A_{ij}^{[m]} B_{ij}^{[m]}, \tag{3.27}$$

$$-(W_{ik}^{[n]})^2 S_m - (C_{ij}^{[m]})^2 + W_{ik}^{[n]} D_{ij}^{[m]} = W_{jk}^{[n]} F_{ij}^{[m]} \tag{3.28}$$

$$(W_{ik}^{[n]})^2 T_m + C_{ij}^{[m]} D_{ij}^{[m]} = A_{ij}^{[m]} F_{ij}^{[m]}, \tag{3.29}$$

where $\{i, j, k\} = \{1, 2, 3\}$, and

$$-W_{ik}^{[n]} W_{jk}^{[n]} S_m + W_{ik}^{[n]} B_{ij}^{[m]} + W_{jk}^{[n]} D_{ij}^{[m]} + A_{ij}^{[m]} C_{ij}^{[m]} = 0, \tag{3.30}$$

$$W_{ik}^{[n]} W_{jk}^{[n]} T_m + W_{ik}^{[n]} A_{ij}^{[m]} S_m + W_{jk}^{[n]} C_{ij}^{[m]} S_m - B_{ij}^{[m]} C_{ij}^{[m]} - A_{ij}^{[m]} D_{ij}^{[m]} = 0 \tag{3.31}$$

$$W_{ik}^{[n]} A_{ij}^{[m]} T_m + W_{jk}^{[n]} C_{ij}^{[m]} T_m - B_{ij}^{[m]} D_{ij}^{[m]} + E_{ij}^{[m]} F_{ij}^{[m]} = 0, \tag{3.32}$$

where $\{i,j,k\}=\{1,2,3\}$. Actually, owing to (3.21) and (3.23), (3.28) and (3.29) are also consequences of (3.26) and (3.27), respectively.

In what follows, we first derive two derivative formulae with respect to $x$ and $t_r$ for the meromorphic functions $\phi_{ij}$, $1\le i,j\le 3$.

**Lemma 3.1.** *Suppose that* (3.11) *and* (3.12) *hold. Then, the meromorphic functions* $\phi_{ij}$, $1\le i,j\le 3$, *defined by* (3.17), *satisfy the following Riccati-type equations*:

$$\phi_{ij,x}=(U_{ii}-U_{jj})\phi_{ij}+U_{ij}+U_{ik}\phi_{kj}-U_{ji}\phi_{ij}^2-U_{jk}\phi_{ij}\phi_{kj} \tag{3.33}$$

*and*

$$\phi_{ij,t_r}=(V_{ii}^{[r]}-V_{jj}^{[r]})\phi_{ij}+V_{ij}^{[r]}+V_{ik}^{[r]}\phi_{kj}-V_{ji}^{[r]}\phi_{ij}^2-V_{jk}^{[r]}\phi_{ij}\phi_{kj}, \tag{3.34}$$

*where* $\{i,j,k\}=\{1,2,3\}$.

*Proof.* We prove the $x$-derivative part. The proof of the $t_r$-derivative part is similar. Observing (3.11), we have

$$\frac{\phi_{ij,x}}{\phi_{ij}}=(\ln\frac{\psi_i}{\psi_j})_x=\frac{\psi_{i,x}}{\psi_i}-\frac{\psi_{j,x}}{\psi_j}=\frac{\sum_{k=1}^3 U_{ik}\psi_k}{\psi_i}-\frac{\sum_{k=1}^3 U_{jk}\psi_k}{\psi_j}=\sum_{k=1}^3(U_{ik}\phi_{ki}-U_{jk}\phi_{kj}).$$

The $x$-derivative part (3.33) follows. ■

Secondly, we directly verify the following relations between $B_{ij}^{[m]}$, $D_{ij}^{[m]}$, $E_{ij}^{[m]}$ and $F_{ij}^{[m]}$.

**Lemma 3.2.** *Let* $B_{ij}^{[m]}$, $D_{ij}^{[m]}$, $E_{ij}^{[m]}$ *and* $F_{ij}^{[m]}$, $1\le i,j\le 3$, *be defined by* (3.20) –(3.23), *respectively. Then,*

$$W_{jj}^{[n]}E_{ij}^{[m]}+W_{ji}^{[n]}B_{ij}^{[m]}-W_{jk}^{[n]}B_{kj}^{[m]}=0 \tag{3.35}$$

*and*

$$W_{ii}^{[n]}F_{ij}^{[m]}+W_{ij}^{[n]}D_{ij}^{[m]}-W_{ik}^{[n]}D_{ik}^{[m]}=0, \tag{3.36}$$

*where* $\{i,j,k\}=\{1,2,3\}$.

*Proof.* From the definitions of $B_{ij}^{[m]}$ and $E_{ij}^{[m]}$ in (3.20) and (3.22), a direct computation verifies the relation in (3.35).

Further, using (3.21) and (3.23), from (3.35), we immediately get the relation in (3.36). ■

Now, we consider how to compute derivatives of $E_{ij}^{[m]}$ and $F_{ij}^{[m]}$. Owing to $\mathrm{tr}(W^{[n]})=0$, we can directly prove the following statements.

**Theorem 3.3.** *Let* $S_m$, $B_{ij}^{[m]}$, $D_{ij}^{[m]}$, $E_{ij}^{[m]}$ *and* $F_{ij}^{[m]}$, $1\le i,j\le 3$, *be defined by* (3.3), (3.20)–(3.23), *respectively. If* $W_z^{[n]}=[V,W^{[n]}]$, *where* $V=(V_{ij})_{3\times3}$, *then we have*

$$E_{ij,z}^{[m]}=(2V_{jj}-V_{ii}-V_{kk})E_{ij}^{[m]}-V_{ji}(2W_{jk}^{[n]}S_m-3B_{ij}^{[m]})+V_{jk}(2W_{ji}^{[n]}S_m-3B_{kj}^{[m]}) \tag{3.37}$$

*and*

$$F_{ij,z}^{[m]}=(2V_{ii}-V_{jj}-V_{kk})F_{ij}^{[m]}-V_{ij}(2W_{ik}^{[n]}S_m-3D_{ij}^{[m]})+V_{ik}(2W_{ij}^{[n]}S_m-3D_{ik}^{[m]}), \tag{3.38}$$

*where* $\{i,j,k\}=\{1,2,3\}$.

Applying this theorem, we can easily obtain the following relations between two derivatives of $E_{ij}^{[m]}$ and $F_{ij}^{[m]}$.

**Theorem 3.4.** *Let $E_{ij}^{[m]}$ and $F_{ij}^{[m]}$ be defined by (3.22) and (3.23), respectively. If $W_{z_k}^{[n]}=[V^{(k)},W^{[n]}]$, where $V^{(k)}=(V_{ij}^{(k)})_{3\times 3}$, $1\le k\le 2$, then we have*

$$\begin{aligned}(V_{ji}^{(1)}W_{jk}^{[n]}-V_{jk}^{(1)}W_{ji}^{[n]})E_{ij,z_2}^{[m]}-(V_{ji}^{(2)}W_{jk}^{[n]}-V_{jk}^{(2)}W_{ji}^{[n]})E_{ij,z_1}^{[m]}\\ =E_{ij}^{[m]}[(2V_{jj}^{(2)}-V_{ii}^{(2)}-V_{kk}^{(2)})(V_{ji}^{(1)}W_{jk}^{[n]}-V_{jk}^{(1)}W_{ji}^{[n]})\\ -(2V_{jj}^{(1)}-V_{ii}^{(1)}-V_{kk}^{(1)})(V_{ji}^{(2)}W_{jk}^{[n]}-V_{jk}^{(2)}W_{ji}^{[n]})+3(V_{jk}^{(1)}V_{ji}^{(2)}-V_{jk}^{(2)}V_{ji}^{(1)})W_{jj}^{[n]}]\end{aligned}\tag{3.39}$$

*and*

$$\begin{aligned}(V_{ij}^{(1)}W_{ik}^{[n]}-V_{ik}^{(1)}W_{ij}^{[n]})F_{ij,z_2}^{[m]}-(V_{ij}^{(2)}W_{ik}^{[n]}-V_{ik}^{(2)}W_{ij}^{[n]})F_{ij,z_1}^{[m]}\\ =F_{ij}^{[m]}[(2V_{ii}^{(2)}-V_{jj}^{(2)}-V_{kk}^{(2)})(V_{ij}^{(1)}W_{ik}^{[n]}-V_{ik}^{(1)}W_{ij}^{[n]})\\ -(2V_{ii}^{(1)}-V_{jj}^{(1)}-V_{kk}^{(1)})(V_{ij}^{(2)}W_{ik}^{[n]}-V_{ik}^{(2)}W_{ij}^{[n]})+3(V_{ik}^{(1)}V_{ij}^{(2)}-V_{ik}^{(2)}V_{ij}^{(1)})W_{ii}^{[n]}],\end{aligned}\tag{3.40}$$

*where $\{i,j,k\}=\{1,2,3\}$.*

Note that (3.39) and (3.40) tell us that the weighted differences between two derivatives are multiples of $E_{ij}^{[m]}$ and $F_{ij}^{[m]}$, respectively.

**Theorem 3.5.** *Let $P=(\lambda,y(P))\in\mathcal{K}_g\backslash\{P_{\infty_1},P_{\infty_2},P_{\infty_3}\}$ and (3.13) hold. If $W_z^{[n]}=[V,W^{[n]}]$, where $V=(V_{ij})_{3\times 3}$, then the meromorphic functions $\phi_{ij}$, $1\le i,j\le 3$, defined by (3.18), satisfy*

$$\begin{aligned}\phi_{ij}(P)+\phi_{ij}(P^*)+\phi_{ij}(P^{**})&=\frac{3B_{ij}^{[m]}-2W_{jk}^{[n]}S_m}{E_{ij}^{[m]}}\\ &=\frac{1}{V_{ji}W_{jk}^{[n]}-V_{jk}W_{ji}^{[n]}}\left\{W_{jk}^{[n]}\left[\frac{E_{ij,z}^{[m]}}{E_{ij}^{[m]}}-(2V_{jj}-V_{ii}-V_{kk})\right]+3V_{jk}W_{jj}^{[n]}\right\},\end{aligned}\tag{3.41}$$

$$\phi_{ij}(P)\phi_{ij}(P^*)\phi_{ij}(P^{**})=\frac{F_{ij}^{[m]}}{E_{kj}^{[m]}},\tag{3.42}$$

$$W_{ij}^{[n]}[\phi_{ji}(P)+\phi_{ji}(P^*)+\phi_{ji}(P^{**})]+W_{ik}^{[n]}[\phi_{ki}(P)+\phi_{ki}(P^*)+\phi_{ki}(P^{**})]=-3W_{ii}^{[n]},\tag{3.43}$$

$$\begin{aligned}V_{ij}[\phi_{ji}(P)+\phi_{ji}(P^*)+\phi_{ji}(P^{**})]+V_{ik}[\phi_{ki}(P)+\phi_{ki}(P^*)+\phi_{ki}(P^{**})]\\ =\frac{E_{ji,z}^{[m]}}{E_{ji}^{[m]}}-(2V_{ii}-V_{jj}-V_{kk}),\end{aligned}\tag{3.44}$$

*where $\{i,j,k\}=\{1,2,3\}$.*

*Proof.* First, we start with the last equality in (3.18), and make use of (3.7). Then, we have

$$\begin{aligned}&\phi_{ij}(P)+\phi_{ij}(P^*)+\phi_{ij}(P^{**})\\ &\quad=\frac{(y_1^2+y_2^2+y_3^2)W_{jk}^{[n]}-(y_1+y_2+y_3)A_{ij}^{[m]}+3B_{ij}^{[m]}}{E_{ij}^{[m]}}=\frac{3B_{ij}^{[m]}-2W_{jk}^{[n]}S_m}{E_{ij}^{[m]}},\end{aligned}$$

which is exactly the first equality in (3.41). To prove the second equality in (3.41), we first note that, from (3.37), we have

$$2S_m=\frac{E_{ij,z}^{[m]}-(2V_{jj}-V_{ii}-V_{kk})E_{ij}^{[m]}+3(V_{jk}B_{kj}^{[m]}-V_{ji}B_{ij}^{[m]})}{V_{ji}W_{jk}^{[n]}-V_{jk}W_{ji}^{[n]}}.$$

Then making use of (3.35), we can directly verify the second equality in (3.41), starting from the first equality in (3.41).

Secondly, we use (3.7) and the first equality in (3.18) to get

$$\begin{aligned}\phi_{ij}(P)\phi_{ij}(P^*)\phi_{ij}(P^{**})\\ &= \frac{y_1y_2y_3(W_{ik}^{[m]})^3+(y_1y_2+y_1y_3+y_2y_3)(W_{ik}^{[m]})^2C_{ij}^{[m]}+(y_1+y_2+y_3)W_{ik}^{[m]}(C_{ij}^{[m]})^2+(C_{ij}^{[m]})^3}{y_1y_2y_3(W_{jk}^{[m]})^3+(y_1y_2+y_1y_3+y_2y_3)(W_{jk}^{[m]})^2A_{ij}^{[m]}+(y_1+y_2+y_3)W_{jk}^{[m]}(A_{ij}^{[m]})^2+(A_{ij}^{[m]})^3}\\ &= \frac{T_m(W_{ik}^{[m]})^3+S_m(W_{ik}^{[m]})^2C_{ij}^{[m]}+(C_{ij}^{[m]})^3}{T_m(W_{jk}^{[m]})^3+S_m(W_{jk}^{[m]})^2A_{ij}^{[m]}+(A_{ij}^{[m]})^3}.\end{aligned}$$

Then, based on the properties in (3.21), (3.23) and (3.24), a direct application of (3.26) and (3.29) yields the equality (3.42).

Thirdly, using (3.13) in the definition of the Baker–Akhiezer functions, we have

$$\sum_{j=1}^{3} W_{ij}^{[n]}\phi_{ji}(P)=y_1,\quad \sum_{j=1}^{3} W_{ij}^{[n]}\phi_{ji}(P^*)=y_2,\quad \sum_{j=1}^{3} W_{ij}^{[n]}\phi_{ji}(P^{**})=y_3,$$

and then, based on (3.7), summing them up generates the equality (3.43).

Finally, note that the derivative formula (3.37) guarantees

$$\frac{E_{ji,z}^{[m]}}{E_{ji}^{[m]}}-(2V_{ii}-V_{jj}-V_{kk})=\frac{E_{ki,z}^{[m]}}{E_{ki}^{[m]}}-(2V_{ii}-V_{kk}-V_{jj}),$$

where $\{i,j,k\}=\{1,2,3\}$. Then, making use of the second equality in (3.41), we can arrive at the equality (3.44) by a direct computation. This completes the proof of the theorem. ■

When (3.11)–(3.13) in the definition of the Baker–Akhiezer functions hold, we have the two Lax equations in (3.15) and (3.16). Thus, upon noting $\mathrm{tr}(U)=\mathrm{tr}(V^{[r]})=0$, theorem 3.5 with $V=U$ and $V^{[r]}$ yields that

$$\begin{aligned}\phi_{ij}(P)+\phi_{ij}(P^*)+\phi_{ij}(P^{**})\\ &= \frac{1}{U_{ji}W_{jk}^{[n]}-U_{jk}W_{ji}^{[n]}}\left[W_{jk}^{[n]}\left(\frac{E_{ij,x}^{[m]}}{E_{ij}^{[m]}}-3U_{jj}\right)+3U_{jk}W_{jj}^{[n]}\right],\end{aligned}\tag{3.45}$$

$$\begin{aligned}\phi_{ij}(P)+\phi_{ij}(P^*)+\phi_{ij}(P^{**})\\ &= \frac{1}{V_{ji}^{[r]}W_{jk}^{[n]}-V_{jk}^{[r]}W_{ji}^{[n]}}\left[W_{jk}^{[n]}\left(\frac{E_{ij,t_r}^{[m]}}{E_{ij}^{[m]}}-3V_{jj}^{[r]}\right)+3V_{jk}^{[r]}W_{jj}^{[n]}\right],\end{aligned}\tag{3.46}$$

$$U_{ij}[\phi_{ji}(P)+\phi_{ji}(P^*)+\phi_{ji}(P^{**})]+U_{ik}[\phi_{ki}(P)+\phi_{ki}(P^*)+\phi_{ki}(P^{**})]=\frac{E_{ji,x}^{[m]}}{E_{ji}^{[m]}}-3U_{ii}\tag{3.47}$$

and

$$V_{ij}^{[r]}[\phi_{ji}(P)+\phi_{ji}(P^*)+\phi_{ji}(P^{**})]+V_{ik}^{[r]}[\phi_{ki}(P)+\phi_{ki}(P^*)+\phi_{ki}(P^{**})]=\frac{E_{ji,t_r}^{[m]}}{E_{ji}^{[m]}}-3V_{ii}^{[r]},\tag{3.48}$$

where $\{i,k,j\}=\{1,2,3\}$ (see [41] for the Kaup–Kupershmidt case and [43] for the coupled KdV case).

In view of the relations in (3.23) and (3.24), we only need to explore properties of the three sums $E_{21}^{[m]}, F_{21}^{[m]}$ and $F_{31}^{[m]}$, to determine the dynamics of zeros and poles of the meromorphic functions $\phi_{ij}$, $1\le i,j\le 3$. For all other sums, we can generate similar results. For example, the relations

$$E_{32}^{[m]}=-E_{12}^{[m]}=-F_{21}^{[m]}\quad\text{and}\quad F_{13}^{[m]}=E_{31}^{[m]}=-E_{21}^{[m]}$$

permit one to draw analogies for $E_{32}^{[m]}$ and $F_{13}^{[m]}$.

Taking $V=U$ and $V^{[r]}$, and noting $\mathrm{tr}(U)=\mathrm{tr}(V^{[r]})=0$, directly from theorem 3.3, we can obtain the following derivative formulae in the four-component AKNS case.

**Theorem 3.6.** *Let $E_{21}^{[m]}, F_{21}^{[m]}$ and $F_{31}^{[m]}$ be defined by (3.22) and (3.23), and $(\lambda, x, t_r)\in\mathbb{C}^3$. Suppose that (3.11)–(3.13) hold. Then, we have*

$$E_{21,x}^{[m]} = -6\lambda E_{21}^{[m]} - p_1(2W_{13}^{[n]}S_m - 3B_{21}^{[m]}) + p_2(2W_{12}^{[n]}S_m - 3B_{31}^{[m]}), \tag{3.49}$$

$$F_{21,x}^{[m]} = 3\lambda F_{21}^{[m]} - q_1(2W_{23}^{[n]}S_m - 3D_{21}^{[m]}), \tag{3.50}$$

$$F_{31,x}^{[m]} = 3\lambda E_{31}^{[m]} - q_2(2W_{32}^{[n]}S_m - 3D_{31}^{[m]}) \tag{3.51}$$

*and*

$$E_{21,t_r}^{[m]} = 3V_{11}^{[r]}E_{21}^{[m]} - V_{12}^{[r]}(2W_{13}^{[n]}S_m - 3B_{21}^{[m]}) + V_{13}^{[r]}(2W_{12}^{[n]}S_m - 3B_{31}^{[m]}), \tag{3.52}$$

$$F_{21,t_r}^{[m]} = 3V_{22}^{[r]}F_{21}^{[m]} - V_{21}^{[r]}(2W_{23}^{[n]}S_m - 3D_{21}^{[m]}) + V_{23}^{[r]}(2W_{21}^{[n]}S_m - 3D_{23}^{[m]}), \tag{3.53}$$

$$F_{31,t_r}^{[m]} = 3V_{33}^{[r]}F_{31}^{[m]} - V_{31}^{[r]}(2W_{32}^{[n]}S_m - 3D_{31}^{[m]}) + V_{32}^{[r]}(2W_{31}^{[n]}S_m - 3D_{32}^{[m]}). \tag{3.54}$$

We can further present the derivatives of $E_{21}^{[m]}$, $F_{21}^{[m]}$ and $F_{31}^{[m]}$ with respect to $t_r$ in terms of $E_{21}^{[m]}$, $F_{21}^{[m]}$ and $F_{31}^{[m]}$, and their derivatives with respect to $x$.

**Theorem 3.7.** *Let $E_{21}^{[m]}, F_{21}^{[m]}$ and $F_{31}^{[m]}$ be defined by (3.22) and (3.23), and $(\lambda, x, t_r)\in\mathbb{C}^3$. Suppose that (3.11)–(3.13) hold. Then, we have*

$$\begin{aligned} E_{21,t_r}^{[m]} &= E_{21,x}^{[m]} \frac{W_{13}^{[n]}V_{12}^{[r]} - W_{12}^{[n]}V_{13}^{[r]}}{p_1W_{13}^{[n]} - p_2W_{12}^{[n]}} \\ &\quad + E_{21}^{[m]}\left[3\left(V_{11}^{[r]} - \frac{p_1V_{13}^{[r]} - p_2V_{12}^{[r]}}{p_1W_{13}^{[n]} - p_2W_{12}^{[n]}}W_{11}^{[n]}\right) + 6\lambda \frac{W_{13}^{[n]}V_{12}^{[r]} - W_{12}^{[n]}V_{13}^{[r]}}{p_1W_{13}^{[n]} - p_2W_{12}^{[n]}}\right], \end{aligned} \tag{3.55}$$

$$\begin{aligned} F_{21,t_r}^{[m]} &= F_{21,x}^{[m]} \frac{W_{23}^{[n]}V_{21}^{[r]} - W_{21}^{[n]}V_{23}^{[r]}}{q_1W_{23}^{[n]}} \\ &\quad + F_{21}^{[m]}\left[3(V_{22}^{[r]} - \frac{W_{22}^{[n]}}{W_{23}^{[n]}}V_{23}^{[r]}) + 3\lambda \frac{W_{21}^{[n]}V_{23}^{[r]} - W_{23}^{[n]}V_{21}^{[r]}}{q_1W_{23}^{[n]}}\right] \end{aligned} \tag{3.56}$$

and

$$\begin{aligned} F_{31,t_r}^{[m]} &= F_{31,x}^{[m]} \frac{W_{32}^{[n]}V_{31}^{[r]} - W_{31}^{[n]}V_{32}^{[r]}}{q_2W_{32}^{[n]}} \\ &\quad + F_{31}^{[m]}\left[3\left(V_{33}^{[r]} - \frac{W_{33}^{[n]}}{W_{32}^{[n]}}V_{32}^{[r]}\right) + 3\lambda \frac{W_{31}^{[n]}V_{32}^{[r]} - W_{32}^{[n]}V_{31}^{[r]}}{q_2W_{32}^{[n]}}\right]. \end{aligned} \tag{3.57}$$

*Proof.* Note that (3.11)–(3.13) imply the Lax equations (3.15) and (3.16). Upon taking $V^{(1)}=U$ and $z=x$, and $V^{(2)}=V^{[r]}$ and $z=t_r$, theorem 3.4 immediately leads to the three derivative relations in (3.55)–(3.57). The proof is completed.

Directly applying the following three equalities:

$$W_{11}^{[n]}E_{21}^{[m]} + W_{12}^{[n]}B_{21}^{[m]} - W_{13}^{[n]}B_{31}^{[m]} = 0, \tag{3.58}$$

$$W_{21}^{[n]}D_{21}^{[m]} + W_{22}^{[n]}F_{21}^{[m]} - W_{23}^{[n]}D_{23}^{[m]} = 0 \tag{3.59}$$

and

$$W_{31}^{[n]}D_{31}^{[m]} + W_{33}^{[n]}F_{31}^{[m]} - W_{32}^{[n]}D_{32}^{[m]} = 0, \tag{3.60}$$

which are consequences of (3.35) and (3.36), we can represent all terms on the right-hand side of each equation in (3.52)–(3.54), in terms of $E_{21}^{[m]}$, $F_{21}^{[m]}$ and $F_{31}^{[m]}$ and their derivatives with respect to $x$ in (3.49)–(3.51), which also presents the three derivative relations in (3.55)–(3.57), precisely. ■

## 4. Characteristic variables and Dubrovin-type equations

It is direct to see that the degrees of $E_{21}^{[m]}, F_{21}^{[m]}$ and $F_{31}^{[m]}$ are $g$, $g+1$ and $g+1$, respectively. Thus, we can assume that

$$E_{21}^{[m]}(x,t_r) = e_{21}^{[m]}(\alpha,u)\prod_{j=1}^{g}(\lambda-\mu_j(x,t_r)), \tag{4.1}$$

$$F_{21}^{[m]}(x,t_r) = f_{21}^{[m]}(\alpha,u)\prod_{j=0}^{g}(\lambda-\nu_j(x,t_r)) \tag{4.2}$$

and

$$F_{31}^{[m]}(x,t_r) = f_{31}^{[m]}(\alpha,u)\prod_{j=0}^{g}(\lambda-\xi_j(x,t_r)), \tag{4.3}$$

where $e_{21}^{[m]}$, $f_{21}^{[m]}$ and $f_{31}^{[m]}$ are three non-zero functions depending on $\alpha=(\alpha_0,\alpha_1,\dots,\alpha_n)$ and $u=(p_1,p_2,q_1,q_2)^{\mathrm{T}}$. We call those roots characteristic variables associated with the Baker–Akhiezer functions.

In the light of (3.18), we can then introduce the following three sets of particular points in $\mathcal{K}_g$:

$$\begin{aligned}\hat{\mu}_j(x,t_r) &= (\mu_j(x,t_r), y(\mu_j(x,t_r))) = \left(\mu_j(x,t_r), -\left.\frac{A_{21}^{[m]}(x,t_r)}{W_{13}^{[n]}(x,t_r)}\right|_{\lambda=\mu_j(x,t_r)}\right)\\ &= \left(\mu_j(x,t_r), -\left.\frac{A_{31}^{[m]}(x,t_r)}{W_{12}^{[n]}(x,t_r)}\right|_{\lambda=\mu_j(x,t_r)}\right), \quad 1\le j\le g,\end{aligned} \tag{4.4}$$

$$\hat{\nu}_j(x,t_r) = (\nu_j(x,t_r), y(\nu_j(x,t_r))) = \left(\nu_j(x,t_r), -\left.\frac{C_{21}^{[m]}(x,t_r)}{W_{23}^{[n]}(x,t_r)}\right|_{\lambda=\nu_j(x,t_r)}\right), \quad 0\le j\le g \tag{4.5}$$

and

$$\hat{\xi}_j(x,t_r) = (\xi_j(x,t_r), y(\xi_j(x,t_r))) = \left(\xi_j(x,t_r), -\left.\frac{C_{31}^{[m]}(x,t_r)}{W_{32}^{[n]}(x,t_r)}\right|_{\lambda=\xi_j(x,t_r)}\right), \quad 0\le j\le g, \tag{4.6}$$

where $(x,t_r)\in\mathbb{C}^2$. To determine zeros and poles of the Baker–Akhiezer functions $\psi_i$, $1\le i\le 3$, we set

$$J_r^{(i)} = U_{i1}\phi_{1i}+U_{i2}\phi_{2i}+U_{i3}\phi_{3i} \quad\text{and}\quad I_r^{(i)} = V_{i1}^{[r]}\phi_{1i}+V_{i2}^{[r]}\phi_{2i}+V_{i3}^{[r]}\phi_{3i}, \quad 1\le i\le 3. \tag{4.7}$$

Note that (3.11) and (3.12) give

$$\frac{\psi_{i,x}(P,x,x_0,t_r,t_{0,r})}{\psi_i(P,x,x_0,t_r,t_{0,r})} = J_r^{(i)}(P,x,t_r), \quad 1\le i\le 3 \tag{4.8}$$

and

$$\frac{\psi_{i,t_r}(P,x,x_0,t_r,t_{0,r})}{\psi_i(P,x,x_0,t_r,t_{0,r})} = I_r^{(i)}(P,x,t_r), \quad 1\le i\le 3, \tag{4.9}$$

respectively. It follows that the basic conservation laws associated with Lax pairs hold, i.e.

$$(I_r^{(i)})_x = \left(\frac{\psi_{i,t_r}}{\psi_i}\right)_x = \left(\frac{\psi_{i,x}}{\psi_i}\right)_{t_r} = (J_r^{(i)})_{t_r}, \quad 1\le i\le 3, \tag{4.10}$$

from which we can also generate infinitely many conservation laws by observing Laurent series of the conserved quantities $J_r^{(i)}$, $1\le i\le 3$, and the conserved fluxes $I_r^{(i)}$, $1\le i\le 3$, at $\lambda=\infty$ (or $\zeta=\lambda^{-1}=0$). Furthermore, (4.8) and (4.9) imply the expressions for the Baker–Akhiezer functions $\psi_i$,

$1 \le i \le 3$,

$$\psi_i(P, x, x_0, t_r, t_{0,r}) = \exp\left(\int_{x_0}^{x} J_r^{(i)}(P, x', t_r)\,\mathrm{d}x' + \int_{t_{0,r}}^{t_r} I_r^{(i)}(P, x_0, t')\,\mathrm{d}t'\right), \quad 1 \le i \le 3, \tag{4.11}$$

upon taking advantage of the basic conservation laws in (4.10).

Let us first determine the general dynamics of zeros of $E_{21}^{[m]}, F_{21}^{[m]}$ and $F_{31}^{[m]}$.

**Theorem 4.1.** *Let $W_z^{[n]} = [V, W^{[n]}]$, where $V = (V_{ij})_{3\times 3}$. If $\mu_i \neq \mu_j$, $\nu_i \neq \nu_j$ and $\xi_i \neq \xi_j$ for $i \neq j$, then the zeros of $E_{21}^{[m]}, F_{21}^{[m]}$ and $F_{31}^{[m]}$ satisfy the Dubrovin-type equations*

$$\mu_{j,z} = -\frac{[(V_{12}W_{13}^{[n]} - V_{13}W_{12}^{[n]})(3y^2 + S_m)]|_{\lambda=\mu_j}}{e_{21}^{[m]}\prod_{k=1,k\neq j}^{g}(\mu_j - \mu_k)}, \quad 1 \le j \le g, \tag{4.12}$$

$$\nu_{j,z} = -\frac{[(V_{21}W_{23}^{[n]} - V_{23}W_{21}^{[n]})(3y^2 + S_m)]|_{\lambda=\nu_j}}{f_{21}^{[m]}\prod_{k=0,k\neq j}^{g}(\nu_j - \nu_k)}, \quad 0 \le j \le g \tag{4.13}$$

*and*

$$\xi_{j,z} = -\frac{[(V_{31}W_{32}^{[n]} - V_{32}W_{31}^{[n]})(3y^2 + S_m)]|_{\lambda=\xi_j}}{f_{31}^{[m]}\prod_{k=0,k\neq j}^{g}(\xi_j - \xi_k)}, \quad 0 \le j \le g. \tag{4.14}$$

*Proof.* We first prove the Dubrovin-type equation (4.12). Using (3.26) and (3.58), we have

$$(y^2 + S_m)|_{\lambda=\mu_j} = \left[\left(-\frac{A_{21}^{[m]}}{W_{13}^{[n]}}\right)^2 + S_m\right]\Bigg|_{\lambda=\mu_j} = \frac{(A_{21}^{[m]})^2 + (W_{13}^{[n]})^2 S_m}{(W_{13}^{[n]})^2}\Bigg|_{\lambda=\mu_j}$$
$$= \frac{W_{13}^{[n]}B_{21}^{[m]}}{(W_{13}^{[n]})^2}\Bigg|_{\lambda=\mu_j} = \frac{B_{21}^{[m]}}{W_{13}^{[n]}}\Bigg|_{\lambda=\mu_j} = \frac{B_{31}^{[m]}}{W_{12}^{[n]}}\Bigg|_{\lambda=\mu_j}, \quad 1 \le j \le g.$$

Following these two expressions for $B_{21}^{[m]}$ and $B_{31}^{[m]}$, we have

$$(V_{12}B_{21}^{[m]} - V_{13}B_{31}^{[m]})|_{\lambda=\mu_j} = [(V_{12}W_{13}^{[n]} - V_{13}W_{12}^{[n]})(y^2 + S_m)]|_{\lambda=\mu_j}, \quad 1 \le j \le g$$

and thus, applying the derivative formula (3.37), we can get

$$E_{21,z}^{[m]}|_{\lambda=\mu_j} = [(V_{12}W_{13}^{[n]} - V_{13}W_{12}^{[n]})(3y^2 + S_m)]|_{\lambda=\mu_j}, \quad 1 \le j \le g. \tag{4.15}$$

Now, according to (4.1), this leads to the Dubrovin-type equation (4.12) for $\mu_j$, $1 \le j \le g$.

We secondly verify the Dubrovin-type equation (4.13). Now, using (3.28) and (3.59), we can compute that

$$(y^2 + S_m)|_{\lambda=\nu_j} = \left[\left(-\frac{C_{21}^{[m]}}{W_{23}^{[n]}}\right)^2 + S_m\right]\Bigg|_{\lambda=\nu_j} = \frac{(C_{21}^{[m]})^2 + (W_{23}^{[n]})^2 S_m}{(W_{23}^{[n]})^2}\Bigg|_{\lambda=\nu_j}$$
$$= \frac{W_{23}^{[n]}D_{21}^{[m]}}{(W_{23}^{[n]})^2}\Bigg|_{\lambda=\nu_j} = \frac{D_{21}^{[m]}}{W_{23}^{[n]}}\Bigg|_{\lambda=\nu_j} = \frac{D_{23}^{[m]}}{W_{21}^{[n]}}\Bigg|_{\lambda=\nu_j}, \quad 0 \le j \le g.$$

Based on these two expressions for $D_{21}^{[m]}$ and $D_{23}^{[m]}$, we get

$$(V_{21}D_{21}^{[m]} - V_{23}D_{23}^{[m]})|_{\lambda=\nu_j} = [(V_{21}W_{23}^{[n]} - V_{23}W_{21}^{[n]})(y^2 + S_m)]|_{\lambda=\nu_j}, \quad 0 \le j \le g$$

and thus, from the derivative formula (3.38), we can have

$$F_{21,z}^{[m]}|_{\lambda=\nu_j} = [(V_{21}W_{23}^{[n]} - V_{23}W_{21}^{[n]})(3y^2 + S_m)]|_{\lambda=\nu_j}, \quad 0 \le j \le g. \tag{4.16}$$

Then, based on (4.2), this yields the Dubrovin-type equation (4.13) for $\nu_j$, $0 \le j \le g$.

We thirdly prove the Dubrovin-type equation (4.14). Similarly, using (3.28) and (3.60), we have

$$(y^2+S_m)|_{\lambda=\xi_j}=\left[\left(-\frac{C_{31}^{[m]}}{W_{32}^{[n]}}\right)^2+S_m\right]\Bigg|_{\lambda=\xi_j}=\frac{(C_{31}^{[m]})^2+(W_{32}^{[n]})^2S_m}{(W_{32}^{[n]})^2}\Bigg|_{\lambda=\xi_j}$$
$$=\frac{W_{32}^{[n]}D_{31}^{[m]}}{(W_{32}^{[n]})^2}\Bigg|_{\lambda=\xi_j}=\frac{D_{31}^{[m]}}{W_{32}^{[n]}}\Bigg|_{\lambda=\xi_j}=\frac{D_{32}^{[m]}}{W_{31}^{[n]}}\Bigg|_{\lambda=\xi_j},\quad 0\le j\le g.$$

From these two expressions for $D_{31}^{[m]}$ and $D_{32}^{[m]}$, we obtain

$$(V_{31}D_{31}^{[m]}-V_{32}D_{32}^{[m]})|_{\lambda=\xi_j}=[(V_{31}W_{32}^{[n]}-V_{32}W_{31}^{[n]})(y^2+S_m)]|_{\lambda=\xi_j},\quad 0\le j\le g,$$

and then, applying the derivative formula (3.38), we can get

$$F_{31,z}^{[m]}|_{\lambda=\xi_j}=[(V_{31}W_{32}^{[n]}-V_{32}W_{31}^{[n]})(3y^2+S_m)]|_{\lambda=\xi_j},\quad 0\le j\le g. \tag{4.17}$$

Finally, according to (4.3), this equality generates the Dubrovin-type equation (4.14) for $\xi_j$, $0\le j\le g$. The proof is completed. ■

In order to determine zeros and poles of the Baker–Akhiezer functions $\psi_i$, $1\le i\le 3$, we verify the following statements.

**Theorem 4.2.** *Let* $W_z^{[n]}=[V,W^{[n]}]$, *where* $V=(V_{ij})_{3\times 3}$ *with* $\mathrm{tr}(V)=0$. *If* $\mu_i\neq\mu_j$, $\nu_i\neq\nu_j$ *and* $\xi_i\neq\xi_j$ *for* $i\neq j$, *then we have*

$$V_{11}+V_{12}\phi_{21}+V_{13}\phi_{31}\underset{\lambda\to\mu_j}{=}\partial_z\ln(\lambda-\mu_j)+\mathrm{O}(1),\quad 1\le j\le g, \tag{4.18}$$

$$V_{21}\phi_{12}+V_{22}+V_{23}\phi_{32}\underset{\lambda\to\nu_j}{=}\partial_z\ln(\lambda-\nu_j)+\mathrm{O}(1),\quad 0\le j\le g \tag{4.19}$$

*and*

$$V_{31}\phi_{13}+V_{22}\phi_{23}+V_{33}\underset{\lambda\to\xi_j}{=}\partial_z\ln(\lambda-\xi_j)+\mathrm{O}(1),\quad 0\le j\le g. \tag{4.20}$$

*Proof.* We only prove the first statement. The proofs for the other two statements are similar. Using (3.24) and noting $\mathrm{tr}(V)=0$, we can compute that

$$\begin{aligned}
&V_{11}+V_{12}\phi_{21}+V_{13}\phi_{31}\\
&=V_{11}+V_{12}\frac{y^2W_{13}^{[n]}-yA_{21}^{[m]}+B_{21}^{[m]}}{E_{21}^{[m]}}-V_{13}\frac{y^2W_{12}^{[n]}-yA_{31}^{[m]}+B_{31}^{[m]}}{E_{21}^{[m]}}\\
&=\frac{1}{3}\frac{E_{21,z}^{[m]}}{E_{21}^{[m]}}-\frac{2}{3}\frac{(V_{13}W_{12}^{[n]}-V_{12}W_{13}^{[n]})S_m}{E_{21}^{[m]}}+\frac{y^2(V_{12}W_{13}^{[n]}-V_{13}W_{12}^{[n]})-y(V_{12}A_{21}^{[m]}-V_{13}A_{31}^{[m]})}{E_{21}^{[m]}}\\
&=\frac{1}{3}\frac{E_{21,z}^{[m]}}{E_{21}^{[m]}}+\frac{2}{3}\frac{(V_{12}W_{13}^{[n]}-V_{13}W_{12}^{[n]})(3y^2+S_m)}{E_{21}^{[m]}}\\
&\quad-\frac{V_{12}W_{13}^{[n]}y(y+A_{21}^{[m]}/W_{13}^{[n]})-V_{13}W_{12}^{[n]}y(y+A_{31}^{[m]}/W_{21}^{[n]})}{E_{21}^{[m]}}\\
&\underset{\lambda\to\mu_j}{=}-\frac{\mu_{j,z}}{\lambda-\mu_j}+\mathrm{O}(1)\underset{\lambda\to\mu_j}{=}\partial_z\ln(\lambda-\mu_j)+\mathrm{O}(1),
\end{aligned}$$

where we have used the derivative formula (3.37) and the Dubrovin-type equation (4.12). The proof is completed. ■

Taking $V=U$ and $V^{[r]}$ and noting $\mathrm{tr}(U)=\mathrm{tr}(V^{[r]})=0$, we can have the following two conclusions from theorem 4.1 and theorem 4.2.

**Theorem 4.3.** *Let $u=(p_1,p_2,q_1,q_2)^{\mathrm{T}}$ solve the rth four-component AKNS equations (2.27), and $\Omega_\mu$ be an open and connected set of $\mathbb{C}^2$. If*

$$\mu_i(x,t_r)\neq\mu_j(x,t_r),\quad \nu_i(x,t_r)\neq\nu_j(x,t_r)\quad\text{and}\quad \xi_i(x,t_r)\neq\xi_j(x,t_r) \tag{4.21}$$

*for $i\neq j$ and $(x,t_r)\in\Omega_\mu$, then the zeros of $E_{21}^{[m]}$, $F_{21}^{[m]}$ and $F_{31}^{[m]}$ satisfy the Dubrovin-type equations:*

$$\mu_{j,x}(x,t_r)=\frac{[(p_2(x,t_r)W_{12}^{[n]}-p_1(x,t_r)W_{13}^{[n]})(3y^2+S_m)]|_{\lambda=\mu_j(x,t_r)}}{e_{21}^{[m]}\prod_{k=1,k\neq j}^{g}(\mu_j(x,t_r)-\mu_k(x,t_r))},\quad 1\leq j\leq g, \tag{4.22}$$

$$\nu_{j,x}(x,t_r)=-\frac{[q_1(x,t_r)W_{23}^{[n]}(3y^2+S_m)]|_{\lambda=\nu_j(x,t_r)}}{f_{21}^{[m]}\prod_{k=0,k\neq j}^{g}(\nu_j(x,t_r)-\nu_k(x,t_r))},\quad 0\leq j\leq g, \tag{4.23}$$

$$\xi_{j,x}(x,t_r)=-\frac{[q_2(x,t_r)W_{32}^{[n]}(3y^2+S_m)]|_{\lambda=\xi_j(x,t_r)}}{f_{31}^{[m]}\prod_{k=0,k\neq j}^{g}(\xi_j(x,t_r)-\xi_k(x,t_r))},\quad 0\leq j\leq g \tag{4.24}$$

*and*

$$\mu_{j,t_r}(x,t_r)=\frac{[(V_{13}^{[r]}W_{12}^{[n]}-V_{12}^{[r]}W_{13}^{[n]})(3y^2+S_m)]|_{\lambda=\mu_j(x,t_r)}}{e_{21}^{[m]}\prod_{k=1,k\neq j}^{g}(\mu_j(x,t_r)-\mu_k(x,t_r))},\quad 1\leq j\leq g, \tag{4.25}$$

$$\nu_{j,t_r}(x,t_r)=\frac{[(V_{23}^{[r]}W_{21}^{[n]}-V_{21}^{[r]}W_{23}^{[n]})(3y^2+S_m)]|_{\lambda=\nu_j(x,t_r)}}{f_{21}^{[m]}\prod_{k=0,k\neq j}^{g}(\nu_j(x,t_r)-\nu_k(x,t_r))},\quad 0\leq j\leq g, \tag{4.26}$$

$$\xi_{j,t_r}(x,t_r)=\frac{[(V_{32}^{[r]}W_{31}^{[n]}-V_{31}^{[r]}W_{32}^{[n]})(3y^2+S_m)]|_{\lambda=\xi_j(x,t_r)}}{f_{31}^{[m]}\prod_{k=0,k\neq j}^{g}(\xi_j(x,t_r)-\xi_k(x,t_r))},\quad 0\leq j\leq g. \tag{4.27}$$

*Proof.* Note that now we have the Lax equations (3.15) and (3.16). Two immediate applications of theorem 4.1 to the case of $V=U$ and $z=x$, and the case of $V=V^{[r]}$ and $z=t_r$ yield the Dubrovin-type dynamical equations in (4.22)–(4.27), respectively. This completes the proof of the theorem. ∎

**Theorem 4.4.** *Let $P=(\lambda,y)\in\mathcal{K}_g\backslash\{P_{\infty_1},P_{\infty_2},P_{\infty_3}\}$, $(x,x_0,t_r,t_{0,r})\in\mathbb{C}^4$, and $\Omega_\mu$ be an open and connected set of $\mathbb{C}^2$. Suppose that $u=(p_1,p_2,q_1,q_2)^{\mathrm{T}}$ solves the rth four-component AKNS equations (2.27). If the conditions in (4.21) hold for $i\neq j$ and $(x,t_r)\in\Omega_\mu$, and*

$$\hat{\mu}_j(x,t_r)\neq\hat{\mu}_j(x_0,t_{0,r}),\quad \hat{\nu}_j(x,t_r)\neq\hat{\nu}_j(x_0,t_{0,r}),\quad \hat{\xi}_j(x,t_r)\neq\hat{\xi}_j(x_0,t_{0,r}) \tag{4.28}$$

*for every j, then*

(a) *$\psi_1(P,x,x_0,t_r,t_{0,r})$ on $\mathcal{K}_g\backslash\{P_{\infty_1},P_{\infty_2},P_{\infty_3}\}$ has g zeros, $\hat{\mu}_1(x,t_r),\ldots,\hat{\mu}_g(x,t_r)$ and g poles, $\hat{\mu}_1(x_0,t_{0,r}),\ldots,\hat{\mu}_g(x_0,t_{0,r})$;*
(b) *$\psi_2(P,x,x_0,t_r,t_{0,r})$ on $\mathcal{K}_g\backslash\{P_{\infty_1},P_{\infty_2},P_{\infty_3}\}$ has $g+1$ zeros, $\hat{\nu}_0(x,t_r),\ldots,\hat{\nu}_g(x,t_r)$ and $g+1$ poles, $\hat{\nu}_0(x_0,t_{0,r}),\ldots,\hat{\nu}_g(x_0,t_{0,r})$;*
(c) *$\psi_3(P,x,x_0,t_r,t_{0,r})$ on $\mathcal{K}_g\backslash\{P_{\infty_1},P_{\infty_2},P_{\infty_3}\}$ has $g+1$ zeros, $\hat{\xi}_0(x,t_r),\ldots,\hat{\xi}_g(x,t_r)$ and $g+1$ poles, $\hat{\xi}_0(x_0,t_{0,r}),\ldots,\hat{\xi}_g(x_0,t_{0,r})$.*

*Proof.* We only prove the statement (a), and the proofs for the other two statements can be given similarly.

Noting that $\mathrm{tr}(U)=\mathrm{tr}(V^{[r]})=0$ and considering two cases of theorem 4.2 with $V=U$ and $V=V^{[r]}$, we have

$$J_r^{(1)}\underset{\lambda\to\mu_j}{=}\partial_x\ln(\lambda-\mu_j)+\mathrm{O}(1)\quad\text{and}\quad I_r^{(1)}\underset{\lambda\to\mu_j}{=}\partial_{t_r}\ln(\lambda-\mu_j)+\mathrm{O}(1),$$

where $1 \le j \le g$. Consequently, for each $1 \le j \le g$, we can compute that

$$\begin{aligned}\psi_1(P,x,x_0,t_r,t_{0,r}) &= \exp\left(\int_{x_0}^{x} J_r^{(1)}(P,x',t_r)\,\mathrm{d}x' + \int_{t_{0,r}}^{t_r} I_r^{(1)}(P,x_0,t')\,\mathrm{d}t'\right)\\ &= \frac{\lambda-\mu_j(x,t_r)}{\lambda-\mu_j(x_0,t_r)}\,\frac{\lambda-\mu_j(x_0,t_r)}{\lambda-\mu_j(x_0,t_{0,r})}\mathrm{O}(1) = \frac{\lambda-\mu_j(x,t_r)}{\lambda-\mu_j(x_0,t_{0,r})}\mathrm{O}(1)\\ &= \begin{cases}(\lambda-\mu_j(x,t_r))\mathrm{O}(1) & \text{for } P \text{ near } \hat{\mu}_j(x,t_r) \ne \hat{\mu}_j(x_0,t_{0,r}),\\ \mathrm{O}(1) & \text{for } P \text{ near } \hat{\mu}_j(x,t_r) = \hat{\mu}_j(x_0,t_{0,r}),\\ (\lambda-\mu_j(x_0,t_{0,r}))^{-1}\mathrm{O}(1) & \text{for } P \text{ near } \hat{\mu}_j(x_0,t_{0,r}) \ne \hat{\mu}_j(x,t_r),\end{cases}\end{aligned}$$

where $\mathrm{O}(1) \ne 0$. Under the conditions in (4.28), this leads to the statement (a), which completes the proof. ∎

This theorem determines zeros and poles of the Baker–Akhiezer functions $\psi_i$, $1 \le i \le 3$, in $\mathcal{K}_g \backslash \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}\}$.

## 5. Concluding remarks

In this part of our study on Riemann theta function representations of algebro-geometric solutions to soliton hierarchies, we introduced a class of trigonal curves, based on linear combinations of Lax matrices in the zero curvature formulation, analysed general properties of their meromorphic functions, including derivative relations between derivatives of the characteristic variables with respect to time and space, and determined zeros and poles of the Baker–Akhiezer functions and their Dubrovin-type dynamical equations.

In the second part [60], we will explore asymptotic properties of the Baker–Akhiezer functions at the points at infinity, straighten out all soliton flows under the Abel–Jacobi coordinates and construct the Riemann theta function representations for algebro-geometric solutions to the four-component AKNS equations based on asymptotic behaviours of the Baker–Akhiezer functions.

Data accessibility. This paper has no data or associated material.
Competing interests. There are no competing interests.
Funding. This work was, in part, supported by National Natural Science Foundation of China grants 11371326, 11371086 and 11301331, NSF under the grant DMS-1664561 and the Distinguished Professorships by Shanghai University of Electric Power and Shanghai Second Polytechnic University.
Acknowledgements. The authors would like to thank S.T. Chen, X. Gu, S. Manukure, M. Mcanally, Y. Sun, F.D. Wang, J. Yu and Y. Zhou for their valuable discussions.

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