

Generalized Bilinear Differential Equations

Wen-Xiu Ma

Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
 mawx@cas.usf.edu

Abstract: We introduce a kind of bilinear differential equations by generalizing Hirota bilinear operators, and explore when the linear superposition principle can apply to the resulting generalized bilinear differential equations. Together with an algorithm using weights, two examples of generalized bilinear differential equations are computed to shed light on the presented general scheme for constructing bilinear differential equations which possess linear subspaces of solutions.

Key words: Bilinear differential equation, Linear superposition principle, Subspace of solutions

INTRODUCTION

It is known that the Hirota bilinear form provides an efficient tool to solve nonlinear differential equations of mathematical physics [1]. Particularly, based on the Hirota bilinear form, soliton solutions can be generated by the Hirota perturbation technique [1] or the multiple expansion algorithm [2]. It is interesting to note that the linear superposition principle can also apply to Hirota bilinear equations and present their resonant soliton solutions [3]. Such solutions would be dense in the solution set of a function space, appropriately equipped with an inner product, and play an important role in formulating basic approximate solutions to initial value problems.

Many important equations of mathematical physics are rewritten in the Hirota bilinear form through dependent variable transformations [1,4]. For instance, the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

the Boussinesq equation

$$u_{tt} + (u^2)_{xx} + u_{xxxx} = 0,$$

and the KP equation

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0,$$

which can be expressed as

$$(D_x D_t + D_x^4) f \cdot f = 0$$

under the transformation $u = 2(\ln f)_{xx}$,

$$(D_t^2 + D_x^4) f \cdot f = 0$$

under the transformation $u = 6(\ln f)_{xx}$, and

$$(D_t D_x + D_x^4 + D_y^2) f \cdot f = 0$$

under the transformation $u = 2(\ln f)_{xx}$, respectively. In the above equations, D_x , D_y and D_t are Hirota bilinear operators, and generally, we have

$$D_x^m D_y^n D_t^k f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n \times \\ \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k f(x, y, t) g(x', y', t') \Big|_{x'=x, y'=y, t'=t},$$

for nonnegative integers m, n and k . Wronskian solutions, including solitons, positons and complexitons [5]-[8], can be presented precisely, on the basis of the Hirota bilinear form.

In this letter, we would like to introduce a kind of generalized bilinear differential operators and explore when the linear superposition principle applies to the corresponding bilinear differential equations. The resulting theory paves a way to construct a new kind of bilinear differential equations which possess linear subspaces of solutions. The considered solutions are linear combinations of exponential traveling wave solutions, and the involved exponential wave solutions may and may not satisfy the corresponding dispersion relations. All the obtained results will exhibit that there are bilinear differential equations different from Hirota bilinear equations, which share some common features with linear differential equations.

The letter is structured as follows. In the 2nd section, we will generalize Hirota bilinear operators and introduce a kind of generalized bilinear differential equations. In the 3rd section, we will analyze the linear superposition principle for exponential traveling waves and establish a criterion for guaranteeing the existence of linear subspaces of exponential traveling wave solutions to generalized bilinear differential equations. In the 4th section, we will present two examples of newly introduced bilinear differential equations, together with an algorithm using weights to compute. In the final section, we will make a few concluding remarks.

BILINEAR DIFFERENTIAL OPERATORS AND EQUATIONS

Let $M, p \in \mathbb{N}$ be given. We introduce a kind of bilinear differential operators:

$$\prod_{i=1}^M D_{p,x_i}^{n_i} f \cdot g = \prod_{i=1}^M \left(\frac{\partial}{\partial x_i} + \alpha \frac{\partial}{\partial x'_i} \right)^{n_i} f(x) g(x') \Big|_{x'=x}, \quad (1)$$

where $x = (x_1, \dots, x_M)$, $x' = (x'_1, \dots, x'_M)$, $n_1, \dots, n_M$ are arbitrary nonnegative integers, and for an integer m , the m th power of α is defined by

$$\alpha^m = (-1)^{r(m)}, \text{ if } m \equiv r(m) \pmod{p} \quad (2)$$

with $0 \leq r(m) < p$.

For example, if $p = 2k$ ($k \in \mathbb{N}$), all above bilinear differential operators are Hirota bilinear operators, since $D_{2k,x} = D_x$ [1,9]. If $p = 3$, we have

$\alpha = -1$, $\alpha^2 = \alpha^3 = 1$, $\alpha^4 = -1$, $\alpha^5 = \alpha^6 = 1$, $\dots$, which tells the pattern of symbols

$$-, +, +, -, +, +, \dots \quad (p=3);$$

and if $p = 5$, we have

$$\alpha = -1, \alpha^2 = 1, \alpha^3 = -1, \alpha^4 = \alpha^5 = 1, \\ \alpha^6 = -1, \alpha^7 = 1, \alpha^8 = -1, \alpha^9 = \alpha^{10} = 1, \dots,$$

which tells the pattern of symbols

$$-, +, -, +, +, -, +, -, +, +, \dots \quad (p=5).$$

Note that the pattern of symbols for defining Hirota bilinear operators is

$$-, +, -, +, -, +, \dots \quad (p=2),$$

and similarly when $p = 7$, the pattern of symbols is

$$-, +, -, +, -, +, +, \\ -, +, -, +, -, +, +, \dots \quad (p=7).$$

Following those patterns of symbols, some new bilinear differential operators can be computed:

$$\begin{aligned} D_{3,x} f \cdot g &= f_x g - f g_x, \\ D_{3,x} D_{3,t} f \cdot g &= f_{xt} g - f_x g_t - f_t g_x + f g_{xt}, \\ D_{3,x}^3 f \cdot g &= f_{xxx} g - 3f_{xx} g_x + 3f_x g_{xx} + f g_{xxx}, \\ D_{3,x}^2 D_{3,t} f \cdot g &= f_{xxt} g - f_{xx} g_t - 2f_{xt} g_x + 2f_x g_{xt} \\ &\quad + f_t g_{xx} + f g_{xxt}, \\ D_{3,x}^3 D_{3,t} f \cdot g &= f_{xxx} g - f_{xxx} g_t - 3f_{xxt} g_x + 3f_{xt} g_{xx} \\ &\quad + 3f_{xx} g_{xt} + 3f_x g_{xxt} + f_t g_{xxx} - f g_{xxt}, \\ D_{3,x}^4 f \cdot g &= f_{xxxx} g - 4f_{xxx} g_x + 6f_{xx} g_{xx} \\ &\quad + 4f_x g_{xxx} - f g_{xxx}, \\ D_{3,x}^5 f \cdot g &= f_{xxxxx} g - 5f_{xxxx} g_x + 10f_{xxx} g_{xx} \\ &\quad + 10f_{xx} g_{xxx} - 5f_x g_{xxxx} + f g_{xxxx}; \end{aligned}$$

and

$$\begin{aligned} D_{5,x} f \cdot g &= f_x g - f g_x, \\ D_{5,x} D_{5,t} f \cdot g &= f_{xt} g - f_x g_t - f_t g_x + f g_{xt}, \\ D_{5,x}^3 f \cdot g &= f_{xxx} g - 3f_{xx} g_x + 3f_x g_{xx} - f g_{xxx}, \\ D_{5,x}^2 D_{5,t} f \cdot g &= f_{xxt} g - f_{xx} g_t - 2f_{xt} g_x + 2f_x g_{xt} \\ &\quad + f_t g_{xx} - f g_{xxt}, \\ D_{5,x}^3 D_{5,t} f \cdot g &= f_{xxx} g - f_{xxx} g_t - 3f_{xxt} g_x + 3f_{xt} g_{xx} \\ &\quad + 3f_{xx} g_{xt} - 3f_x g_{xxt} - f_t g_{xxx} + f g_{xxt}, \\ D_{5,x}^4 f \cdot g &= f_{xxxx} g - 4f_{xxx} g_x + 6f_{xx} g_{xx} \\ &\quad - 4f_x g_{xxx} + f g_{xxx}, \\ D_{5,x}^5 f \cdot g &= f_{xxxxx} g - 5f_{xxxx} g_x + 10f_{xxx} g_{xx} \\ &\quad - 10f_{xx} g_{xxx} + 5f_x g_{xxxx} + f g_{xxxx}. \end{aligned}$$

Immediately from those formulas, we see that except $D_{5,x}^3 f \cdot f = 0$, $D_{3,x}^3 f \cdot f$, $D_{3,x}^5 f \cdot f$ and $D_{5,x}^5 f \cdot f$ do not equal to zero, which is different from the Hirota case: $D_x^3 f \cdot f = D_x^5 f \cdot f = 0$.

Now let P be a polynomial in M variables and introduce a generalized bilinear differential equation:

$$P(D_{p,x_1}, \dots, D_{p,x_M}) f \cdot f = 0. \quad (3)$$

Particularly, when $p = 3$, we have the generalized bilinear KdV equation

$$(D_{3,x} D_{3,t} + D_{3,x}^4) f \cdot f = 2 f_{xt} f - 2 f_x f_t + 6 f_{xx}^2 = 0, \quad (4)$$

the generalized bilinear Boussinesq equation

$$(D_{3,t}^2 + D_{3,x}^4) f \cdot f = 2 f_{tt} f - 2 f_t^2 + 6 f_{xx}^2 = 0, \quad (5)$$

and the generalized bilinear KP equation

$$(D_{3,t} D_{3,x} + D_{3,x}^4 + D_{3,y}^2) f \cdot f = 2 f_{xt} f - 2 f_x f_t \\ + 6 f_{xx}^2 + 2 f_{yy} f - 2 f_y^2 = 0. \quad (6)$$

We would like to discuss linear subspaces of solutions to the generalized bilinear differential equations defined by (3). More exactly, as in the Hirota case [3], we want to explore when the linear superposition principle will apply to the generalized bilinear differential equations (3).

LINEAR SUPERPOSITION PRINCIPLE

Let us now fix $N \in \mathbb{N}$ and introduce N wave variables

$$\eta_i = k_{1,i} x_1 + \dots + k_{M,i} x_M, \quad 1 \leq i \leq N, \quad (7)$$

and N exponential wave functions

$$f_i = e^{\eta_i} = e^{k_{1,i} x_1 + \dots + k_{M,i} x_M}, \quad 1 \leq i \leq N, \quad (8)$$

where the $k_{j,i}$'s are constants. Note that we have a bilinear identity:

$$P(D_{p,x_1}, \dots, D_{p,x_M}) e^{\eta_i} \cdot e^{\eta_j} = P(k_{1,i} + \alpha k_{1,j}, \dots, k_{M,i} + \alpha k_{M,j}) e^{\eta_i + \eta_j}, \quad (9)$$

where the powers of α obey the rule (2). Then we consider a linear combination solution to the bilinear differential equation (3):

$$f = \varepsilon_1 f_1 + \dots + \varepsilon_N f_N = \varepsilon_1 e^{\eta_1} + \dots + \varepsilon_N e^{\eta_N}, \quad (10)$$

where ε_i , $1 \leq i \leq N$, are arbitrary constants. Thanks to (9), we can compute that

$$\begin{aligned} & P(D_{p,x_1}, \dots, D_{p,x_M}) f \cdot f \\ &= \sum_{i,j=1}^N \varepsilon_i \varepsilon_j P(D_{p,x_1}, \dots, D_{p,x_M}) e^{\eta_i} \cdot e^{\eta_j} \\ &= \sum_{i,j=1}^N \varepsilon_i \varepsilon_j P(k_{1,i} + \alpha k_{1,j}, \dots, k_{M,i} + \alpha k_{M,j}) e^{\eta_i + \eta_j} \\ &= \sum_{i=1}^N \varepsilon_i^2 [P(k_{1,i} + \alpha k_{1,i}, \dots, k_{M,i} + \alpha k_{M,i}) e^{2\eta_i} \\ &+ \sum_{1 \leq i < j \leq N} \varepsilon_i \varepsilon_j [P(k_{1,i} + \alpha k_{1,j}, \dots, k_{M,i} + \alpha k_{M,j}) \\ &+ P(k_{1,j} + \alpha k_{1,i}, \dots, k_{M,j} + \alpha k_{M,i})] e^{\eta_i + \eta_j}]. \quad (11) \end{aligned}$$

It thus follows that a linear combination function f defined by (10) solves the generalized bilinear differential equation (3) if and only if for $1 \leq i \leq j \leq N$, all

$$P(k_{1,i} + \alpha k_{1,j}, \dots, k_{M,i} + \alpha k_{M,j}) + P(k_{1,j} + \alpha k_{1,i}, \dots, k_{M,j} + \alpha k_{M,i}) = 0 \quad (12)$$

are satisfied, where the powers of α obey the rule (2). The conditions in (12) present a system of nonlinear algebraic equations on the wave related numbers $k_{i,j}$'s and the coefficients of the polynomial P . Generally, it is not easy to solve (12). But in many cases, such systems have various sets of solutions. In the next section, we will present an idea to solve.

We now summarize our result as follows.

Thm 1. (Criterion for linear superposition principle)

Let $P(x_1, \dots, x_M)$ be a polynomial in the indicated variables and the N wave variables η_i , $1 \leq i \leq N$, be defined by $\eta_i = k_{1,i}x_1 + \dots + k_{M,i}x_M$, $1 \leq i \leq N$, where the $k_{i,j}$'s are all constants. Then any linear combination of the exponential waves e^{η_i} , $1 \leq i \leq N$, solves the generalized bilinear differential equation $P(D_{p,x_1}, \dots, D_{p,x_M}) f \cdot f = 0$ if and only if the conditions in (12) are satisfied.

This theorem tells us that the linear superposition principle can apply to generalized bilinear differential

equations defined by (3). It also paves a way of constructing N -wave solutions to generalized bilinear differential equations. The system (12) is a resonance condition we need to deal with (see [10] for resonance of 2-solitons for Hirota bilinear equations). As soon as we find a solution of the wave related numbers $k_{i,j}$'s to the system (12), we can present an N -wave solution, formed by (10), to the considered generalized bilinear differential equation (3).

APPLICATIONS

Let us now consider how to compute examples of generalized bilinear differential equations, defined by (3), with linear subspaces of solutions, by applying Theorem 1. The problem is how to construct a multivariate polynomial $P(x_1, \dots, x_M)$ such that

$$P(k_{1,1} + \alpha k_{1,2}, \dots, k_{M,1} + \alpha k_{M,2}) + P(k_{1,2} + \alpha k_{1,1}, \dots, k_{M,2} + \alpha k_{M,1}) = 0, \quad (13)$$

holds for two sets of constants $k_{1,i}, \dots, k_{M,i}$, $i = 1, 2$, where the powers of α obey the rule (2). Our basic idea to solve is to introduce weights for the independent variables and then use parameterizations of wave numbers and frequencies (see [11] for the case of Hirota bilinear equations).

Let us first introduce the weights for the independent variables:

$$(w(x_1), \dots, w(x_M)) = (n_1, \dots, n_M), \quad (14)$$

where each weight $w(x_i) = n_i$ is an integer, and then form a polynomial $P(x_1, \dots, x_M)$ being homogeneous in some weight. Second, for $i = 1, 2$, we parameterize the constants $k_{1,i}, \dots, k_{M,i}$, consisting of wave numbers and frequencies, using a free parameter k_i as follows:

$$k_{j,i} = b_j k_i^{n_i}, \quad 1 \leq j \leq M, \quad (15)$$

where the b_j 's are constants to be determined. This parameterization balances the degrees of the free parameters in the system (13). Then, plugging the parameterized constants (15) into (13), we collect terms by powers of the parameters k_1 and k_2 , and set the coefficient of each power to be zero, to obtain algebraic equations on the constants b_j 's and the coefficients of the polynomial P . Finally, solve the resulting algebraic equations to determine the polynomial P and the parameterization.

Now, the resulting solution obviously yields a generalized bilinear differential equation defined by (3) and a linear subspace of its solutions given by

$$f = \sum_{i=1}^N \varepsilon_i f_i = \sum_{i=1}^N \varepsilon_i e^{b_1 k_i^{n_1} x_1 + \dots + b_M k_i^{n_M} x_M}, \quad N \geq 1, \quad (16)$$

where the ε_i 's and k_i 's are arbitrary constants.

In what follows, we present two illustrative examples in 3+1 dimensions, which apply the above parameterization achieved by using one free parameter.

Example 1. Example with positive weights:

Let us introduce the weights of independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, 2, 3, 4). \quad (17)$$

Then, a general polynomial being homogeneous in weight 5 reads

$$P = c_1 x^5 + c_2 x^3 y + c_3 x^2 z + c_4 x t + c_5 y z. \quad (18)$$

Following the parameterization of wave numbers and frequency in (15), the wave variables read

$$\eta_i = k_i x + b_1 k_i^2 y + b_2 k_i^3 z + b_3 k_i^4 t, \quad 1 \leq i \leq N,$$

where k_i , $1 \leq i \leq N$, are arbitrary constants, but b_1 , b_2 and b_3 are constants to be determined. In this example, the corresponding generalized bilinear differential equation reads

$$\begin{aligned} & P(D_{3,x}, D_{3,y}, D_{3,z}, D_{3,t}) f \cdot f \\ &= 2 c_1 f_{xxxxx} f - 10 c_1 f_{xxx} f_x + 20 c_1 f_{xxx} f_{xx} \\ &+ 6 c_2 f_{xx} f_{xy} + 2 c_3 f_{xz} f + 2 c_4 f_{xt} f \\ &- 2 c_4 f_x f_t + 2 c_5 f_{yz} f - 2 c_5 f_y f_z = 0. \end{aligned} \quad (19)$$

The corresponding linear subspace of N -wave solutions is given by

$$f = \sum_{i=1}^N \varepsilon_i f_i = \sum_{i=1}^N \varepsilon_i e^{k_i x + b_1 k_i^2 y + b_2 k_i^3 z + b_3 k_i^4 t}, \quad (20)$$

where ε_i , $1 \leq i \leq N$, are arbitrary constants but b_1, b_2, b_3 need to satisfy

$$\begin{cases} c_5 b_1 b_2 + c_4 b_3 + c_1 + c_3 b_2 = 0, \\ -5 c_1 - c_4 b_3 = 0, \\ 10 c_1 - c_5 b_1 b_2 + 3 c_2 b_1 = 0. \end{cases} \quad (21)$$

A special solution to (21) is

$$b_1 = -\frac{5 c_3}{c_5}, \quad b_2 = -\frac{3 c_2}{c_5}, \quad b_3 = -\frac{15 c_2 c_3}{c_4 c_5}, \quad (22)$$

when the coefficients of the polynomial P satisfy

$$c_1 = \frac{3 c_2 c_3}{c_5}. \quad (23)$$

If $c_4 = 0$, then $c_1 = 0$, and further a non-trivial (e.g., $b_1 b_2 \neq 0$) solution of b_1 and b_2 is given by

$$b_1 = -\frac{c_3}{c_5}, \quad b_2 = \frac{3 c_2}{c_5}, \quad (24)$$

but b_3 is arbitrary.

If $c_4 \neq 0$, then

$$b_3 = -\frac{5 c_1}{c_4}, \quad (25)$$

and further two non-trivial (e.g., $b_1 b_2 \neq 0$) solutions of b_1 and b_2 are determined by

$$\begin{cases} c_3 b_2 - 4 c_1 = 0, \\ 3 c_2 b_1 + 10 c_1 = 0, \end{cases} \quad (26)$$

when $c_5 = 0$, and

$$\begin{cases} c_3 c_5 b_2^2 + (6 c_1 c_5 - 3 c_2 c_3) b_2 + 12 c_1 c_2 = 0, \\ c_5 b_1 b_2 + c_3 b_2 - 4 c_1 = 0, \end{cases} \quad (27)$$

when $c_5 \neq 0$. The formula (27) provides a large class of generalized bilinear equations which possess the discussed N -wave solutions.

Example 2. Example with positive and negative weights:

Let us introduce the weights of independent variables:

$$(w(x), w(y), w(t)) = (1, -1, 3). \quad (28)$$

Then, a polynomial being homogeneous in weight 2 reads

$$P = c_1 x^2 + c_2 x^3 y + c_3 x^4 y^2 + c_4 y t. \quad (29)$$

Following the parameterization of wave numbers and frequency in (15), the wave variables read

$$\eta_i = k_i x + b_1 k_i^{-1} y + b_2 k_i^3 t, \quad 1 \leq i \leq N,$$

where k_i , $1 \leq i \leq N$, are arbitrary constants, but b_1 and b_2 are constants to be determined.

Now, a similar direct computation tells that the corresponding generalized bilinear differential equation

$$\begin{aligned} & (c_1 D_{3,x}^2 + c_2 D_{3,x}^3 D_{3,y} + c_3 D_{3,x}^4 D_{3,y}^2 \\ &+ c_4 D_{3,t} D_{3,y}) f \cdot f = 0 \end{aligned} \quad (30)$$

possesses the linear subspace of N -wave solutions determined by

$$f = \sum_{i=1}^N \varepsilon_i f_i = \sum_{i=1}^N \varepsilon_i e^{k_i x + b_1 k_i^{-1} y + b_2 k_i^3 t}, \quad (31)$$

where the ε_i 's and k_i 's are arbitrary, but b_1 and b_2 satisfy

$$\begin{cases} -c_1 + 6 c_3 b_1^2 = 0, \\ -c_4 b_1 b_2 + 4 c_3 b_1^2 = 0, \\ c_4 b_1 b_2 + c_3 b_1^2 + c_1 + 3 c_2 b_1 = 0. \end{cases} \quad (32)$$

Therefore, the coefficients of the polynomial P need satisfy

$$121 c_1 c_3 = 54 c_2^2, \quad (33)$$

and then the corresponding generalized bilinear differential equation

$$\begin{aligned} & P(D_{3,x}, D_{3,y}, D_{3,z}, D_{3,t}) f \cdot f \\ &= 2 c_1 f_{xx} f - 2 c_1 f_x^2 + 6 c_2 f_{xx} f_{xy} \\ &+ 2 c_3 f_{xxxx} f + 8 c_3 f_{xxx} f_{xyy} \\ &+ 12 c_3 f_{xxy}^2 + 2 c_4 f_{yt} f - 2 c_4 f_y f_t = 0 \end{aligned} \quad (34)$$

has the N -wave solution defined by (31) with

$$b_1 = -\frac{11 c_1}{18 c_2}, b_2 = -\frac{12 c_2}{11 c_4}. \quad (35)$$

CONCLUDING REMARKS

We introduced a new kind of bilinear differential operators and analyzed when the corresponding generalized bilinear equations possess the linear superposition principle. Particularly, we computed two examples by an algorithm using weights and their linear subspaces of exponential traveling wave solutions. The balance requirement of weights allows us to present a class of parameterizations of wave numbers and frequencies,

Theorem 1 presents a sufficient and necessary criterion for guaranteeing the applicability of the linear superposition principle to a kind of generalized bilinear differential equations. It also follows from Theorem 1 that if we begin with an arbitrary multivariate polynomial $P(x_1, \dots, x_M)$, then a sufficient and necessary condition to guarantee the applicability of the discussed linear superposition principle is

$$P(k_{1,i} + \alpha k_{1,j}, \dots, k_{M,i} + \alpha k_{M,j}) \\ + P(k_{1,j} + \alpha k_{1,i}, \dots, k_{M,j} + \alpha k_{M,i}) = 0,$$

where $1 \leq i \leq j \leq N$. This also serves as a starting point for us to search for generalized bilinear differential equations which possess linear subspaces of solutions.

Our results generalize Hirota bilinear operators and establish a bridge between bilinear differential equations and linear differential equations, which exhibits a kind of integrability of nonlinear differential equations [12]. The existence of linear subspaces of solutions amends diversity of exact solutions generated by various analytical methods (see, for example, [13]-[19]). The generalized bilinear operators (1) definitely bring more opportunities to generate non-trivial trilinear differential equations. It is also extremely interesting to explore other mathematical properties that the generalized bilinear differential equations (3) possess and how to construct their non-resonant soliton solutions.

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