



A bilinear Bäcklund transformation of a $(3 + 1)$ -dimensional generalized KP equation

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ABSTRACT

A bilinear Bäcklund transformation is presented for a $(3 + 1)$ -dimensional generalized KP equation, which consists of six bilinear equations and involves nine arbitrary parameters. Two classes of exponential and rational traveling wave solutions with arbitrary wave numbers are computed, based on the proposed bilinear Bäcklund transformation.

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1. Introduction

It is significantly important to search for exact solutions to nonlinear equations of mathematical physics [1,2]. The transformed rational function method [3] and the multiple exp-function method [4] provide two generic approaches for constructing traveling wave solutions and multiple wave solutions, respectively. If an equation possesses a Hirota bilinear form, then a perturbation expansion often generates a specific class of multiple wave solutions including N -soliton solutions [5]. Moreover, the linear superposition principle may apply to Hirota bilinear equations, and in particular, this presents linear subspaces of solutions for nonlinear equations [6].

Bäcklund transformations are another powerful approach to solutions of nonlinear equations, and they can be written in the Hirota bilinear form when an equation under consideration has a bilinear form [7,8]. For example, the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \quad (1.1)$$

which can be written as

$$D_x(D_t + D_x^3)f \cdot f = 0, \quad (1.2)$$

under $u = 2(\ln f)_{xx}$, D_r being Hirota's bilinear operator [2], has the bilinear Bäcklund transformation [7]:

$$\begin{cases} (D_x^2 - \lambda)f' \cdot f = 0, \\ (D_t + 3\lambda D_x + D_x^3)f' \cdot f = 0. \end{cases} \quad (1.3)$$

This means that f solves the bilinear KdV equation (1.2) if and only if f' solves the bilinear KdV equation (1.2). The $(2 + 1)$ -dimensional generalized KdV equation, i.e., the KP equation

$$(-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0, \quad (1.4)$$

which can be written as

$$(-4D_x D_t + 3D_y^2 + D_x^4)f \cdot f = 0, \quad (1.5)$$

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under $u = 2(\ln f)_{xx}$, has the bilinear Bäcklund transformation [2,9]:

$$\begin{cases} (D_y - D_x^2)f' \cdot f = 0, \\ (3D_y D_x - 4D_t + D_x^3)f' \cdot f = 0. \end{cases} \quad (1.6)$$

Such bilinear Bäcklund transformations also connect with Lax pairs and generate the modified soliton equations [9,10].

In this paper, we would like to study a $(3 + 1)$ -dimensional generalized KP equation

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0,$$

which can be written in the Hirota bilinear form

$$(D_x^3 D_y + D_t D_x + D_t D_y - D_z^2)f \cdot f = 0,$$

under $u = 2(\ln f)_x$. This equation was presented for the first time in a study on the linear superposition principle for exponential waves [6], and it is similar to a generalized $(3 + 1)$ -dimensional BKP equation [3,11]

$$(3D_x D_z - 2D_y D_t - D_y D_x^3)f \cdot f = 0,$$

whose bilinear Bäcklund transformation was presented in [12].

We would like to construct a bilinear Bäcklund transformation for the above $(3 + 1)$ -dimensional generalized KP equation, which consists of six equations and contains nine arbitrary parameters. The exchange formula for Hirota's bilinear operators are the basis for carrying out the necessary interchanges in deriving the bilinear Bäcklund transformation. Exponential and rational traveling wave solutions with arbitrary wave numbers are computed by applying the proposed bilinear Bäcklund transformation.

2. Bilinear Bäcklund transformation and traveling wave solutions

We consider the following $(3 + 1)$ -dimensional nonlinear equation:

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0. \quad (2.1)$$

This equation is different from the $(3 + 1)$ -dimensional KP equation [13]; but when $y = x$, the equation is reduced to the KP equation, and so it is called a generalized KP equation [6]. The KP equation was also generalized by constructing decomposition of $(2 + 1)$ -dimensional equations into $(1 + 1)$ -dimensional equations [14].

Under the dependent variable transformation

$$u = 2(\ln f)_x, \quad (2.2)$$

the above $(3 + 1)$ -dimensional generalized KP equation is put into a Hirota bilinear equation

$$(D_x^3 D_y + D_t D_x + D_t D_y - D_z^2)f \cdot f = 0, \quad (2.3)$$

where D_x, D_y, D_z and D_t are Hirota's bilinear differential operators [2,7]. This is equivalent to

$$(f_{xxxy} + f_{tx} + f_{ty} - f_{zz})f - 3f_{xy}f_x + 3f_{xy}f_{xx} - f_y f_{xxx} - f_t f_x - f_t f_y + (f_z)^2 = 0.$$

Its Wronskian and Grammian solutions, Pfaffianized generalized KP system, and non-singular and singular soliton solutions were presented in [15–17], respectively.

2.1. Bilinear Bäcklund transformation

We would like to present a bilinear Bäcklund transformation for the $(3 + 1)$ -dimensional generalized bilinear KP equation (2.3).

Let us assume that we have another solution f' to the generalized bilinear KP equation (2.3):

$$(D_x^3 D_y + D_t D_x + D_t D_y - D_z^2)f' \cdot f' = 0, \quad (2.4)$$

and introduce a key function

$$P = [(D_x^3 D_y + D_t D_x + D_t D_y - D_z^2)f' \cdot f']f^2 - [(D_x^3 D_y + D_t D_x + D_t D_y - D_z^2)f \cdot f]f'^2. \quad (2.5)$$

If $P = 0$, then f solves the generalized bilinear KP equation (2.3) if and only if f' solves the generalized bilinear KP equation (2.3). Therefore, if we can obtain, from $P = 0$ by interchanging the dependent variables f and f' , a system of bilinear equations that guarantees $P = 0$:

$$B_i(D_t, D_x, D_y, D_z)f' \cdot f = 0, \quad 1 \leq i \leq M,$$

where the B_i 's are polynomials in the indicated variables and M is a natural number depending on the complexity of the equation, then this system presents a bilinear Bäcklund transformation for the generalize bilinear KP equation (2.3).

Let us now start to explore what those bilinear equations could be. First we want to list three exchange identities for Hirota's bilinear operators:

$$(D_t D_x a \cdot a) b^2 - (D_t D_x b \cdot b) a^2 = 2D_x(D_t a \cdot b) \cdot ba, \quad (2.6)$$

$$(D_t D_y a \cdot a) b^2 - (D_t D_y b \cdot b) a^2 = 2D_y(D_t a \cdot b) \cdot ba, \quad (2.7)$$

$$\begin{aligned} & 2(D_x^3 D_y a \cdot a) b^2 - 2(D_x^3 D_y b \cdot b) a^2 \\ &= D_x[(3D_x^2 D_y a \cdot b) \cdot ba + (3D_x^2 a \cdot b) \cdot (D_y b \cdot a) + (6D_x D_y a \cdot b) \cdot (D_x b \cdot a)] \\ &+ D_y[(3D_x^3 a \cdot b) \cdot ba + (3D_x^3 a \cdot b) \cdot (D_x b \cdot a)]. \end{aligned} \quad (2.8)$$

The first and second identities can be found in [2], and the third one can be obtained from the coefficient of ε^1 , while taking the independent variable transformation $D_x \rightarrow D_x + \varepsilon D_y$ for

$$(D_x^4 a \cdot a) b^2 - (D_x^4 b \cdot b) a^2 = 2D_x[(D_x^3 a \cdot b) \cdot ba + (3D_x^2 a \cdot b) \cdot (D_x b \cdot a)],$$

which is the known identity in [2]. All these identities come from the general exchange formula (see [2] for details). Now from the first identity (2.6) or the second identity (2.7), we can easily obtain

$$(D_z^2 a \cdot a) b^2 - (D_z^2 b \cdot b) a^2 = 2D_z(D_z a \cdot b) \cdot ba, \quad (2.9)$$

$$D_r(D_s a \cdot b) \cdot ba = D_s(D_r a \cdot b) \cdot ba, \quad (2.10)$$

by taking $x = t = z$ and noting $D_r D_s g \cdot g = D_s D_r g \cdot g$.

Then, it can be proved that $P = 0$ if we take

$$\begin{cases} B_1 f' \cdot f \equiv (3D_x^2 D_y + 4D_t + \lambda_1 D_y + 4\lambda_8 D_z + \lambda_2) f' \cdot f = 0, \\ B_2 f' \cdot f \equiv (D_x^3 + 4D_t - \lambda_1 D_x + 4\lambda_9 D_z + \lambda_3) f' \cdot f = 0, \\ B_3 f' \cdot f \equiv (3D_x^2 + \lambda_4 D_y + \lambda_6) f' \cdot f = 0, \\ B_4 f' \cdot f \equiv (3D_x^2 + \lambda_5 D_x - \lambda_6) f' \cdot f = 0, \\ B_5 f' \cdot f \equiv (D_x D_y + \lambda_7 D_x) f' \cdot f = 0, \\ B_6 f' \cdot f \equiv (D_z + \lambda_8 D_x + \lambda_9 D_y) f' \cdot f = 0, \end{cases} \quad (2.11)$$

where nine arbitrary parameters have been introduced. This system provides a bilinear Bäcklund transformation for the $(3 + 1)$ -dimensional generalized KP equation (2.3).

Actually, by using the exchange identities (2.6)–(2.9), we can make the following conversion:

$$\begin{aligned} 2P &= [2(D_x^3 D_y f' \cdot f') f^2 - 2(D_x^3 D_y f \cdot f) f'^2] + [2(D_t D_x f' \cdot f') f^2 - 2(D_t D_x f \cdot f) f'^2] \\ &+ [2(D_t D_y f' \cdot f') f^2 - 2(D_t D_y f \cdot f) f'^2] - [2(D_z^2 f' \cdot f') f^2 - 2(D_z^2 f \cdot f) f'^2] \\ &= \{D_x[(3D_x^2 D_y f' \cdot f) \cdot ff' + (3D_x^2 f' \cdot f) \cdot (D_y f \cdot f') + (6D_x D_y f' \cdot f) \cdot (D_x f \cdot f')] \\ &+ D_y[(3D_x^3 f' \cdot f) \cdot ff' + (3D_x^3 f' \cdot f) \cdot (D_x f \cdot f')]\} \\ &+ 4D_x(D_t f' \cdot f) \cdot ff' + 4D_y(D_t f' \cdot f) \cdot ff' - 4D_z(D_z f' \cdot f) \cdot ff' \\ &= D_x(3D_x^2 D_y f' \cdot f + \lambda_1 D_y f' \cdot f + \lambda_2 f' f) \cdot ff' \\ &+ D_x(3D_x^2 f' \cdot f + \lambda_4 D_y f' \cdot f + \lambda_6 f' f) \cdot (D_y f \cdot f') \\ &+ D_x(6D_x D_y f' \cdot f + 6\lambda_7 D_x f' \cdot f) \cdot (D_x f \cdot f') \\ &+ D_y(D_x^3 f' \cdot f - \lambda_1 D_x f' \cdot f + \lambda_3 f' f) \cdot ff' \\ &+ D_y(3D_x^2 f' \cdot f + \lambda_5 D_x f' \cdot f - \lambda_6 f' f) \cdot (D_x f \cdot f') \\ &+ 4D_x(D_t f' \cdot f) \cdot ff' + 4D_y(D_t f' \cdot f) \cdot ff' \\ &- 4D_z(D_z f' \cdot f + \lambda_8 D_x f' \cdot f + \lambda_9 D_y f' \cdot f) \cdot ff' \\ &+ 4D_x(\lambda_8 D_z f' \cdot f) \cdot ff' + 4D_y(\lambda_9 D_z f' \cdot f) \cdot ff' \\ &= D_x(B_1 f' \cdot f) \cdot ff' + D_y(B_2 f' \cdot f) \cdot ff' + D_x(B_3 f' \cdot f) \cdot (D_y f \cdot f') \\ &+ D_y(B_4 f' \cdot f) \cdot (D_x f \cdot f') + 6D_x(B_5 f' \cdot f) \cdot (D_x f \cdot f') - 4D_z(B_6 f' \cdot f) \cdot ff'. \end{aligned}$$

In the above deduction, the coefficients of λ_2 , λ_3 , λ_4 , λ_5 and λ_7 are zero because of $D_r g \cdot g = 0$, and the coefficients of λ_1 , λ_6 , λ_8 and λ_9 are zero because of (2.10). This shows that (2.11) presents a Bäcklund transformation for the $(3 + 1)$ -dimensional generalized bilinear KP equation (2.3).

2.2. Traveling wave solutions

Let us take a simple solution $f = 1$ to the $(3 + 1)$ -dimensional generalized KP equation (2.3), which is transformed into the original variable u as $u = 2(\ln f)_x = 0$. Noting that

$$D_r^n g \cdot 1 = \frac{\partial^n}{\partial r^n} g, \quad n \geq 1,$$

the bilinear Bäcklund transformation (2.11) associated with $f = 1$ becomes a system of linear partial differential equations

$$\begin{cases} 3f'_{xy} + 4f'_t + \lambda_1 f'_y + 4\lambda_8 f'_z + \lambda_2 f' = 0, \\ f'_{xx} + 4f'_t - \lambda_1 f'_x + 4\lambda_9 f'_z + \lambda_3 f' = 0, \\ 3f'_{xx} + \lambda_4 f'_y + \lambda_6 f' = 0, \\ 3f'_{xx} + \lambda_5 f'_x - \lambda_6 f' = 0, \\ f'_{xy} + \lambda_7 f'_x = 0, \\ f'_z + \lambda_8 f'_x + \lambda_9 f'_y = 0. \end{cases} \quad (2.12)$$

Let us first consider a class of exponential wave solutions

$$f' = 1 + \varepsilon e^{kx + ly + mz - \omega t}, \quad (2.13)$$

where ε , k , l , m and ω are constants to be determined. Upon selecting

$$\lambda_2 = 0, \quad \lambda_3 = 0, \quad \lambda_6 = 0, \quad (2.14)$$

a direct computation tells

$$m = -(\lambda_8 k + \lambda_9 l), \quad \omega = \frac{k^3 l - (\lambda_8 k + \lambda_9 l)^2}{k + l}, \quad (2.15)$$

and

$$\lambda_1 = \frac{k^3 - 3k^2 l + 4\lambda_8^2 k - 4\lambda_8 \lambda_9 (k - l) - 4\lambda_9^2 l}{k + l}, \quad \lambda_4 = -\frac{3k^2}{l}, \quad \lambda_5 = -3k, \quad \lambda_7 = -l. \quad (2.16)$$

Therefore, we obtain a class of exponential wave solutions to the $(3 + 1)$ -dimensional generalized bilinear KP equation (2.3):

$$f' = 1 + \varepsilon \exp \left[kx + ly - (\lambda_8 k + \lambda_9 l)z - \frac{k^3 l - (\lambda_8 k + \lambda_9 l)^2}{k + l} t \right], \quad (2.17)$$

where ε , k , l , λ_8 and λ_9 are arbitrary constants; and $u = 2(\ln f')_x$ solves the $(3 + 1)$ -dimensional generalized KP equation (2.1).

Let us second consider a class of first-order polynomial solutions

$$f' = kx + ly + mz - \omega t, \quad (2.18)$$

where ε , k , l , m and ω are constants to be determined. Similarly upon selecting

$$\lambda_i = 0, \quad 2 \leq i \leq 7, \quad (2.19)$$

a direct computation shows that the system (2.12) becomes

$$\begin{cases} l\lambda_1 + 4m\lambda_8 - 4\omega = 0, \\ -k\lambda_1 + 4m\lambda_9 - 4\omega = 0, \\ k\lambda_8 + l\lambda_9 + m = 0. \end{cases} \quad (2.20)$$

Obviously, this system requires a necessary but not sufficient (see the last section for a counterexample) condition

$$(k + l)\omega + m^2 = 0 \quad (2.21)$$

for the existence of λ_1 , λ_8 and λ_9 . Under this condition (2.21), it is direct to check that f' defined by (2.18) solves the $(3 + 1)$ -dimensional generalized bilinear KP equation (2.3), and so,

$$u = 2(\ln f')_x = \frac{2k}{kx + ly + mz - \omega t} \quad (2.22)$$

produces a class of rational solutions to the $(3 + 1)$ -dimensional generalized KP equation (2.1).

3. Conclusions and remarks

We have computed a bilinear Bäcklund transformation for the $(3 + 1)$ -dimensional generalized KP equation

$$u_{xxx} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0.$$

The facts used in our construction are the exchange identities for Hirota's bilinear operators. The obtained bilinear Bäcklund transformation consists of six bilinear equations and involves nine arbitrary parameters. It is therefore a pretty large system, which in turn implies that the above $(3 + 1)$ -dimensional generalized KP equation should have diverse solutions. Indeed, two classes of exponential and rational traveling wave solutions with arbitrary wave numbers have been generated from the proposed bilinear Bäcklund transformation.

It is interesting to note that the condition (2.21) has a solution

$$k = l = m = 0, \quad \omega \neq 0,$$

but this makes it impossible to solve (2.20). Therefore, the corresponding function

$$f' = -\omega t$$

provides a solution for the generalized bilinear KP equation (2.3), but it is not generated from the bilinear Bäcklund transformation (2.11) associated with $f = 1$. It is actually a limit solution of the presented polynomial solutions.

We remark that the above $(3 + 1)$ -dimensional generalized KP equation possesses linear subspaces of exponential wave solutions [6]. This shows a nice integrability property that nonlinear equations normally do not possess. One can also get some nonlinear superposition formulas of solutions generated from the proposed bilinear Bäcklund transformation [9,18], but it is hard to prove that the resulting functions are solutions due to a large number of different equations involved in the Bäcklund transformation. To overcome this complexity, one should find a bilinear Bäcklund transformation consisting of a small number of bilinear equations. However, it is a very difficult challenge for us to get a bilinear Bäcklund transformation defined by a system of less than six equations, for example, two or three equations for the above $(3 + 1)$ -dimensional generalized KP equation. Some new specific exchange identities must be developed for use in merging terms resulted from $P = 0$. There might also be other equations different from $P = 0$ which one can begin with to formulate bilinear Bäcklund transformations.

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