



Wronskian and Grammian solutions to a $(3 + 1)$ -dimensional generalized KP equation

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ABSTRACT

Wronskian and Grammian formulations are established for a $(3 + 1)$ -dimensional generalized KP equation, based on the Plücker relation and the Jacobi identity for determinants. Generating functions for matrix entries satisfy a linear system of partial differential equations involving a free parameter. Examples of Wronskian and Grammian solutions are computed and a few particular solutions are plotted.

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1. Introduction

It is known that soliton equations such as the KdV equation, the Boussinesq equation and the KP equation possess multi-soliton solutions generated from a combination of exponential waves. Hirota bilinear form plays a crucial role in constructing soliton solutions [1,2]. The multiple exp-function method, oriented towards the ease of use and capability of computer algebra systems, provides a direct and efficient way to search for generic multi-exponential wave solutions to nonlinear equations including bilinear equations [3]. Besides soliton solutions, another class of interesting multi-exponential wave solutions is a linear combination of exponential waves. It is shown that a kind of nonlinear equations can possess such a linear superposition principle, and a sufficient criterion for its existence was furnished for Hirota bilinear equations in [4].

Solitons, and positons (a kind of periodic solutions), can be expressed as Wronskian determinants [5,6]. Particular solutions combining exponential functions and trigonometrical functions are presented and called complexiton (or briefly complexitons) [7], and lattice soliton equations have a similar situation [8]. Complexitons are also shown to exist for source solution equations [9] and soliton equations with sources [10]. For higher-dimensional soliton equations, there exist Grammian solutions and Pfaffian solutions [1]. Grammian solutions to the KP equation were constructed by Nakamura [11] and Pfaffian solutions to the BKP equation were presented by Hirota [12].

In this paper, we would like to study a $(3 + 1)$ -dimensional generalized KP equation

$$u_{xxxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0,$$

which can be written in Hirota bilinear form but does not belong to a class of generalized KP and Boussinesq equations [13]

$$(u_{x_1 x_1 x_1} - 6u u_{x_1})_{x_1} + \sum_{i,j=1}^M a_{ij} u_{x_i x_j} = 0, \quad a_{ij} = \text{constant}, \quad M \in \mathbb{N}.$$

We will show this generalized KP equation has a class of Wronskian solutions and a class of Grammian solutions, with all generating functions for matrix entries satisfying a linear system of partial differential equations involving a free parameter. The Plücker relation and the Jacobi identity for determinants are the key to establish the Wronskian and Grammian

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formulations [1]. Examples of Wronskian solutions and Grammian solutions are computed, and a few plots of particular solutions are made.

2. Wronskian formulation

Let us introduce the following helpful notation

$$|N - \widehat{j} - 1, i_1, \dots, i_j| = |\Phi^{(0)}, \Phi^{(1)}, \dots, \Phi^{(N-j-1)}, \Phi^{(i_1)}, \dots, \Phi^{(i_j)}| = \det(\Phi^{(0)}, \Phi^{(1)}, \dots, \Phi^{(N-j-1)}, \Phi^{(i_1)}, \dots, \Phi^{(i_j)}), \quad 1 \leq j \leq N-1, \quad (2.1)$$

where $i_1, \dots, i_j$ are non-negative integers, and the vectors of functions $\Phi^{(j)}$ are defined by

$$\Phi^{(j)} = (\phi_1^{(j)}, \phi_2^{(j)}, \dots, \phi_N^{(j)})^T, \quad \phi_i^{(j)} = \frac{\partial^j}{\partial x^j} \phi_i. \quad (2.2)$$

A Wronskian determinant is given by

$$W(\phi_1, \phi_2, \dots, \phi_N) = |N - \widehat{1}|. \quad (2.3)$$

We also use the assumption for convenience that if $i < 0$, the column vector $\Phi^{(i)}$ does not appear in the determinant $\det(\dots, \Phi^{(i)}, \dots)$.

We consider the following $(3+1)$ -dimensional nonlinear equation:

$$u_{xxx} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0. \quad (2.4)$$

When $y = x$, this equation reduces the KP equation, and so we call it a generalized KP equation. The KP equation was also generalized by constructing decomposition of $(2+1)$ -dimensional equations into $(1+1)$ -dimensional equations [14].

Under the dependent variable transformation

$$u = 2(\ln f)_x, \quad (2.5)$$

the above $(3+1)$ -dimensional generalized KP equation is mapped into a Hirota bilinear equation

$$(D_x^3 D_y + D_t D_x + D_t D_y - D_z^2) f \cdot f = 0, \quad (2.6)$$

where D_x, D_y, D_z and D_t are Hirota bilinear differential operators [1,15]. Equivalently, we have

$$(f_{xxx} + f_{tx} + f_{ty} - f_{zz})f - 3f_{xy}f_x + 3f_{xy}f_{xx} - f_y f_{xxx} - f_t f_x - f_t f_y + (f_z)^2 = 0. \quad (2.7)$$

The equation was presented for the first time in a study on the linear superposition principle for exponential waves [4], and it is similar to a generalized $(3+1)$ -dimensional BKP equation

$$(3D_x D_z - 2D_y D_t - D_y D_x^3) f \cdot f = 0,$$

which has Wronskian and Grammian solutions [16,17]. We would like to present a sufficient condition under which the Wronskian determinant solves the generalized Hirota bilinear KP Eq. (2.6).

Theorem 2.1. Let a set of functions $\phi_i = \phi_i(x, y, z, t)$ satisfy the following condition:

$$\phi_{i,y} = \gamma_1 \phi_{i,x}, \quad \phi_{i,z} = \gamma_2 \phi_{i,xx}, \quad \phi_{i,t} = \gamma_3 \phi_{i,xxx}, \quad 1 \leq i \leq N \quad (2.8)$$

with

$$\gamma_1 = -\frac{a^2}{3}, \quad \gamma_2 = a, \quad \gamma_3 = \frac{4a^2}{3-a^2}, \quad (2.9)$$

where $a \neq \pm\sqrt{3}$ is a free parameter. Then the Wronskian determinant $f_N = |N - \widehat{1}|$ defined by (2.3) solves the $(3+1)$ -dimensional generalized bilinear KP Eq. (2.6).

Proof. Under the condition (2.8), we can compute various derivatives of the Wronskian determinant $f_N = |N - \widehat{1}|$ with respect to the variables x, y, z, t . It is not difficult to obtain, using (2.8), that

$$\begin{aligned} f_{N,x} &= |N - \widehat{2}, N|, \\ f_{N,xx} &= |N - \widehat{3}, N - 1, N| + |N - \widehat{2}, N + 1|, \\ f_{N,xxx} &= |N - \widehat{4}, N - 2, N - 1, N| + 2|N - \widehat{3}, N - 1, N + 1| + |N - \widehat{2}, N + 2|, \\ f_{N,y} &= \gamma_1 |N - \widehat{2}, N|, \end{aligned}$$

$$\begin{aligned}
f_{N,xy} &= \gamma_1(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|), \\
f_{N,xy} &= \gamma_1(|\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|), \\
f_{N,xxx} &= \gamma_1(|\widehat{N-5}, N-3, N-2, N-1, N| + 3|\widehat{N-4}, N-2, N-1, N+1| \\
&\quad + 2|\widehat{N-3}, N, N+1| + 3|\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3|), \\
f_{N,z} &= \gamma_2(|\widehat{N-2}, N+1| - |\widehat{N-3}, N-1, N|), \\
f_{N,zz} &= \gamma_2^2(-|\widehat{N-4}, N-2, N-1, N+1| + 2|\widehat{N-3}, N, N+1| + |\widehat{N-2}, N+3| \\
&\quad + |\widehat{N-5}, N-3, N-2, N-1, N| - |\widehat{N-3}, N-1, N+2|), \\
f_{N,t} &= \gamma_3(|\widehat{N-4}, N-2, N-1, N| - |\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|), \\
f_{N,tx} &= \gamma_3(|\widehat{N-5}, N-3, N-2, N-1, N| - |\widehat{N-3}, N, N+1| + |\widehat{N-2}, N+3|), \\
f_{N,ty} &= \gamma_1\gamma_3(|\widehat{N-5}, N-3, N-2, N-1, N| - |\widehat{N-3}, N, N+1| + |\widehat{N-2}, N+3|).
\end{aligned}$$

In the above expressions, the column $\Phi^{(N-5)}$ does not appear if $N < 5$, as we assumed before since $N - 5 < 0$. Therefore, under the selection (2.9) of γ_1, γ_2 and γ_3 , we can now compute that

$$\begin{aligned}
f_{N,xxx} + f_{N,tx} + f_{N,ty} - f_{N,zz} &= -4a^2|\widehat{N-3}, N, N+1|, \\
-3f_{N,xy}f_{N,x} - f_{N,y}f_{N,xxx} - f_{N,t}f_{N,x} - f_{N,t}f_{N,y} &= 4a^2|\widehat{N-2}, N||\widehat{N-3}, N-1, N+1|, \\
3f_{N,xy}f_{N,xx} + (f_{N,z})^2 &= -4a^2|\widehat{N-3}, N-1, N||\widehat{N-2}, N+1|,
\end{aligned}$$

and further obtain that

$$\begin{aligned}
(D_x^3 D_y + D_t D_x + D_t D_y - D_z^2) f_N \cdot f_N &= (f_{N,xxx} + f_{N,tx} + f_{N,ty} - f_{N,zz}) f_N - 3f_{N,xy} f_{N,x} + 3f_{N,xy} f_{N,xx} - f_{N,y} f_{N,xxx} - f_{N,t} f_{N,x} - f_{N,t} f_{N,y} + (f_{N,z})^2 \\
&= -4a^2(|\widehat{N-1}, N||\widehat{N-3}, N, N+1| - |\widehat{N-2}, N||\widehat{N-3}, N-1, N+1| \\
&\quad + |\widehat{N-3}, N-1, N||\widehat{N-2}, N+1|) = 0.
\end{aligned}$$

This last equality is nothing but the Plücker relation for determinants:

$$|B, A_1, A_2||B, A_3, A_4| - |B, A_1, A_3||B, A_2, A_4| + |B, A_1, A_4||B, A_2, A_3| = 0,$$

where B denotes an $N \times (N-2)$ matrix, and A_i , $1 \leq i \leq 4$, are four N -dimensional column vectors. Therefore, we have shown that $f = f_N$ solves the $(3+1)$ -dimensional generalized Hirota bilinear KP Eq. (2.6), under the condition (2.8). $\square$

The condition (2.8) is a linear system of partial differential equations. It has an exponential-type function solution:

$$\phi_i = \sum_{j=1}^p d_{ij} e^{\eta_{ij}}, \quad \eta_{ij} = k_{ij}x + \gamma_1 k_{ij}y + \gamma_2 k_{ij}^2 z + \gamma_3 k_{ij}^3 t, \quad (2.10)$$

where d_{ij} and k_{ij} are free parameters and p is an arbitrary natural number.

In particular, we can have the following Wronskian solutions:

$$u = 2(\ln f_N)_x, \quad f_N = W(\phi_1, \phi_2, \dots, \phi_N), \quad (2.11)$$

where

$$\phi_i = e^{k_i x + \gamma_1 k_i y + \gamma_2 k_i^2 z + \gamma_3 k_i^3 t} + e^{l_i x + \gamma_1 l_i y + \gamma_2 l_i^2 z + \gamma_3 l_i^3 t}, \quad (2.12)$$

where k_i and l_i are free parameters. To illustrate Wronskian solutions, we set $N = 3$. The following three Figs. 2.1, 2.2 and 2.3 of three dimensional plots and two dimensional contour plots show the Wronskian solutions defined by (2.11) on the indicated specific regions, with specific values being chosen for the parameters. In the contour plots, we see the interaction regions and patterns of the involved solitons.

3. Grammian formulation

Let us now introduce the following Grammian determinant

$$f_N = \det(a_{ij})_{1 \leq i, j \leq N}, \quad a_{ij} = c_{ij} + \int^x \phi_i \psi_j dx, \quad c_{ij} = \text{constant}, \quad (3.1)$$

with ϕ_i and ψ_j satisfying

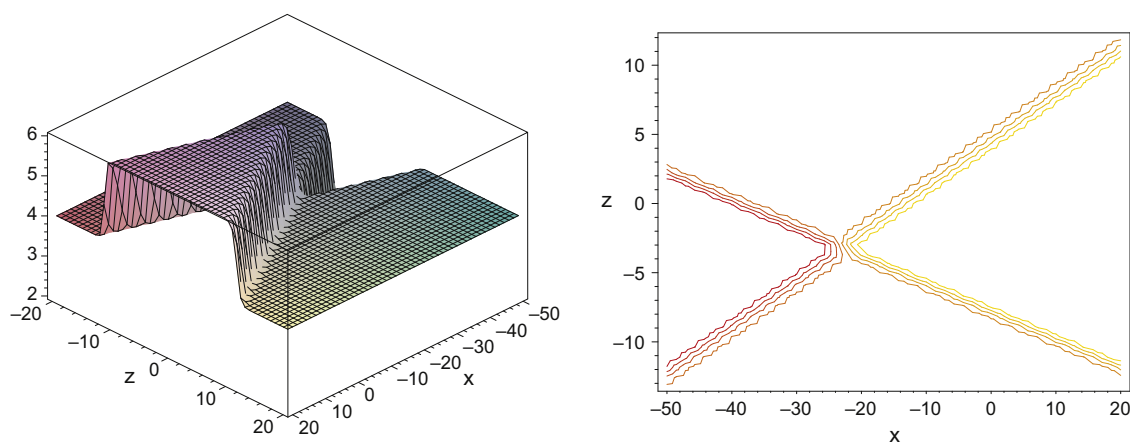


Fig. 2.1. $N = 3$ with $k_1 = 1$, $k_2 = 2$, $k_3 = -2$, $l_1 = 1$, $l_2 = 3$, $l_3 = -1$, $a = 1$, $y = 0$, $t = 1$.

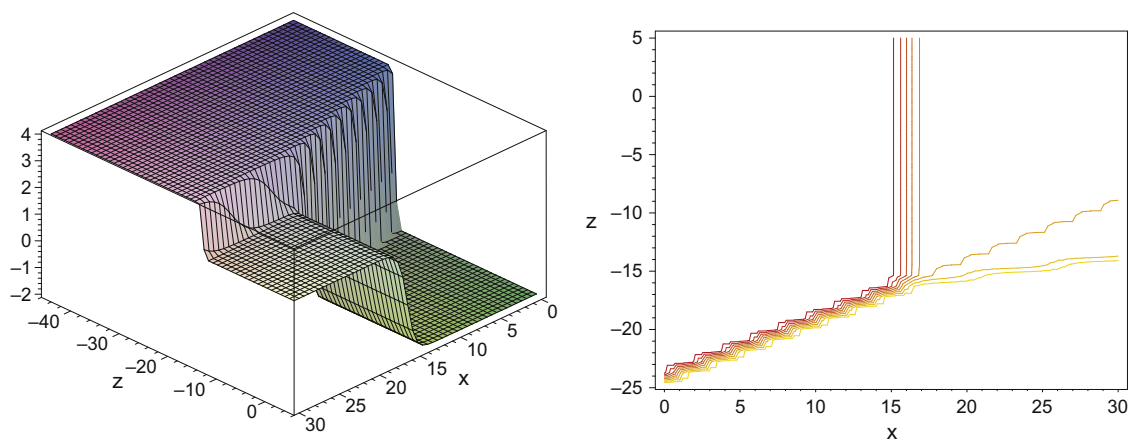


Fig. 2.2. $N = 3$ with $k_1 = -1$, $k_2 = 2$, $k_3 = -1$, $l_1 = -2$, $l_2 = 3$, $l_3 = 1$, $a = 2$, $y = 0$, $t = 1$.

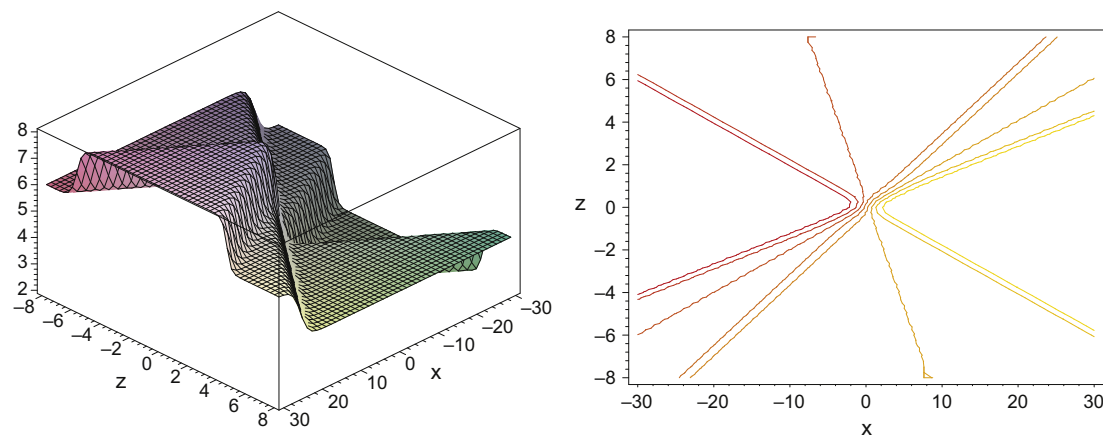


Fig. 2.3. $N = 3$ with $k_1 = 1$, $k_2 = 3$, $k_3 = -2$, $l_1 = 2$, $l_2 = 4$, $l_3 = -3$, $a = -1$, $y = 0$, $t = 0$.

$$\phi_{i,y} = \gamma_1 \phi_{i,x}, \quad \phi_{i,z} = \gamma_2 \phi_{i,xx}, \quad \phi_{i,t} = \gamma_3 \phi_{i,xxx}, \quad 1 \leq i \leq N, \quad (3.2)$$

$$\psi_{i,y} = \gamma_1 \psi_{i,x}, \quad \psi_{i,z} = -\gamma_2 \psi_{i,xx}, \quad \psi_{i,t} = \gamma_3 \psi_{i,xxx}, \quad 1 \leq j \leq N, \quad (3.3)$$

where the constants γ_i are defined by (2.9).

Theorem 3.1. Let ϕ_i and ψ_j satisfy (3.2) and (3.3), respectively. Then the Grammian determinant $f_N = \det(a_{ij})_{1 \leq i,j \leq N}$ defined by (3.1) solves the $(3+1)$ -dimensional generalized bilinear KP Eq. (2.6).

Proof. Let us express the Grammian determinant f_N by means of a Pfaffian as

$$f_N = (1, 2, \dots, N, N^*, \dots, 2^*, 1^*), \quad (3.4)$$

where $(i, j^*) = a_{ij}$ and $(i, j) = (i^*, j^*) = 0$. To compute derivatives of the entries a_{ij} and the Grammian f_N , we introduce new Pfaffian entries

$$(d_n, j^*) = \frac{\partial^n}{\partial x^n} \psi_j, \quad (d_n^*, i) = \frac{\partial^n}{\partial x^n} \phi_i, \quad (d_m, d_n^*) = (d_n, i) = (d_m^*, j^*) = 0, \quad m, n \geq 0, \quad (3.5)$$

as usual. In terms of these new entries, derivatives of the entries $a_{ij} = (i, j^*)$ are given, upon using (3.2) and (3.3), by

$$\begin{aligned} \frac{\partial}{\partial x} a_{ij} &= \phi_i \psi_j = (d_0, d_0^*, i, j^*), \\ \frac{\partial}{\partial y} a_{ij} &= \int^x (\phi_{i,y} \psi_j + \phi_i \psi_{j,y}) dx = \gamma_1 \int^x (\phi_{i,x} \psi_j + \phi_i \psi_{j,x}) dx = \gamma_1 \phi_i \psi_j = \gamma_1 (d_0, d_0^*, i, j^*), \\ \frac{\partial}{\partial z} a_{ij} &= \int^x (\phi_{i,z} \psi_j + \phi_i \psi_{j,z}) dx = \gamma_2 \int^x (\phi_{i,xx} \psi_j - \phi_i \psi_{j,xx}) dx = \gamma_2 (\phi_{i,x} \psi_j - \phi_i \psi_{j,x}) = \gamma_2 [-(d_1, d_0^*, i, j^*) + (d_0, d_1^*, i, j^*)], \\ \frac{\partial}{\partial t} a_{ij} &= \int^x (\phi_{i,t} \psi_j + \phi_i \psi_{j,t}) dx = \gamma_3 \int^x (\phi_{i,xxx} \psi_j + \phi_i \psi_{j,xxx}) dx = \gamma_3 (\phi_{i,xx} \psi_j - \phi_{i,x} \psi_{j,x} + \phi_i \psi_{j,xx}) \\ &= \gamma_3 [(d_2, d_0^*, i, j^*) - (d_1, d_1^*, i, j^*) + (d_0, d_2^*, i, j^*)]. \end{aligned}$$

Then we can develop differential rules for Pfaffians as in [1], and compute various derivatives of the Grammian determinant $f_N = \det(a_{ij})$ with respect to the variables x, y, z, t as follows:

$$\begin{aligned} f_{N,x} &= (d_0, d_0^*, \bullet), \\ f_{N,xx} &= (d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet), \\ f_{N,xxx} &= (d_2, d_0^*, \bullet) + 2(d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet), \\ f_{N,y} &= \gamma_1 (d_0, d_0^*, \bullet), \\ f_{N,xy} &= \gamma_1 [(d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet)], \\ f_{N,xyy} &= \gamma_1 [(d_2, d_0^*, \bullet) + 2(d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet)], \\ f_{N,xyxy} &= \gamma_1 [(d_3, d_0^*, \bullet) + 3(d_2, d_1^*, \bullet) + 2(d_0, d_0^*, d_1, d_1^*, \bullet) + 3(d_1, d_2^*, \bullet) + (d_0, d_3^*, \bullet)], \\ f_{N,z} &= \gamma_2 [-(d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet)], \\ f_{N,zz} &= \gamma_2^2 [(d_3, d_0^*, \bullet) - (d_2, d_1^*, \bullet) + 2(d_0, d_0^*, d_1, d_1^*, \bullet) - (d_1, d_2^*, \bullet) + (d_0, d_3^*, \bullet)], \\ f_{N,t} &= \gamma_3 [(d_2, d_0^*, \bullet) - (d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet)], \\ f_{N,tx} &= \gamma_3 [(d_3, d_0^*, \bullet) - (d_0, d_0^*, d_1, d_1^*, \bullet) + (d_0, d_3^*, \bullet)], \\ f_{N,ty} &= \gamma_1 \gamma_3 [(d_3, d_0^*, \bullet) - (d_0, d_0^*, d_1, d_1^*, \bullet) + (d_0, d_3^*, \bullet)], \end{aligned}$$

where the abbreviated notation $\bullet$ denotes the list of indices $1, 2, \dots, N, N^*, \dots, 2^*, 1^*$ common to each Pfaffian.

Noting the selection (2.9) of γ_1, γ_2 and γ_3 , we can now compute that

$$\begin{aligned} f_{N,xyxy} + f_{N,tx} + f_{N,ty} - f_{N,zz} &= -4a^2 (d_0, d_0^*, d_1, d_1^*, \bullet), \\ -3f_{N,xyy} f_{N,x} - f_{N,y} f_{N,xxx} - f_{N,t} f_{N,x} - f_{N,t} f_{N,y} &= 4a^2 (d_0, d_0^*, \bullet) (d_1, d_1^*, \bullet), \\ 3f_{N,xy} f_{N,xx} + (f_{N,z})^2 &= -4a^2 (d_1, d_0^*, \bullet) (d_0, d_1^*, \bullet), \end{aligned}$$

and further obtain that

$$\begin{aligned} (D_x^3 D_y + D_t D_x + D_t D_y - D_z^2) f_N \cdot f_N &= (f_{N,xyxy} + f_{N,tx} + f_{N,ty} - f_{N,zz}) f_N - 3f_{N,xyy} f_{N,x} + 3f_{N,xy} f_{N,xx} - f_{N,y} f_{N,xxx} - f_{N,t} f_{N,x} - f_{N,t} f_{N,y} + (f_{N,z})^2 \\ &= -4a^2 [(\bullet) (d_0, d_0^*, d_1, d_1^*, \bullet) - (d_0, d_0^*, \bullet) (d_1, d_1^*, \bullet) + (d_1, d_0^*, \bullet) (d_0, d_1^*, \bullet)] = 0. \end{aligned}$$

This last equality is nothing but the Jacobi identity for determinants. Therefore, we have shown that $f_N = \det(a_{ij})_{1 \leq i,j \leq N}$ defined by (3.1) solves the $(3+1)$ -dimensional generalized Hirota bilinear KP Eq. (2.6) under the condition of (3.2) and (3.3). $\square$

The systems (3.2) and (3.3) have solutions

$$\phi_i = \sum_{j=1}^p d_{ij} e^{\eta_{ij}}, \quad \eta_{ij} = k_{ij}x + \gamma_1 k_{ij}y + \gamma_2 k_{ij}^2 z + \gamma_3 k_{ij}^3 t, \quad (3.6)$$

$$\psi_j = \sum_{i=1}^q e_{ji} e^{\zeta_{ji}}, \quad \zeta_{ji} = l_{ji}x + \gamma_1 l_{ji}y - \gamma_2 l_{ji}^2 z + \gamma_3 l_{ji}^3 t, \quad (3.7)$$

where d_{ij} , e_{ji} , k_{ij} and l_{ji} are free parameters and p, q are two arbitrary natural numbers. In particular, we have the following specific solutions

$$\phi_i = e^{\eta_i}, \quad \eta_i = k_i x + \gamma_1 k_i y + \gamma_2 k_i^2 z + \gamma_3 k_i^3 t, \quad (3.8)$$

$$\psi_j = e^{\zeta_j}, \quad \zeta_j = l_j x + \gamma_1 l_j y - \gamma_2 l_j^2 z + \gamma_3 l_j^3 t, \quad (3.9)$$

where k_i and l_j are free parameters. To illustrate Grammian solutions, we further choose $(c_{ij})_{N \times N}$ as the identity matrix of order N . The following three Figs. 3.1, 3.2 and 3.3 of three dimensional plots and two dimensional contour plots show the corresponding Grammian solutions $u = 2(\ln f)_x$ on the indicated specific regions with $N = 2, 3, 4$, respectively, with specific values being chosen for the parameters. In the contour plots, we see the interaction regions and patterns of the involved solitons.

4. Conclusions and remarks

We have established one Wronskian formulation and one Grammian formulation for the $(3+1)$ -dimensional generalized KP equation

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0.$$

The facts used in our construction of Wronskian and Grammian solutions are the Plücker relation for determinants and the Jacobi identity for determinants, respectively. Theorems 2.1 and 3.1 present the main results on Wronskian and Grammian solutions, which say that

$$u = 2(\ln f)_x, \quad f = |\widehat{N-1}| = W(\phi_1, \phi_2, \dots, \phi_N),$$

with ϕ_i satisfying

$$\phi_{i,y} = -\frac{a^2}{3} \phi_{i,x}, \quad \phi_{i,z} = a \phi_{i,xx}, \quad \phi_{i,t} = \frac{4a^2}{3-a^2} \phi_{i,xxxx},$$

and

$$u = 2(\ln f)_x, \quad f = \det(a_{ij})_{1 \leq i,j \leq N}, \quad a_{ij} = c_{ij} + \int^x \phi_i \psi_j dx, \quad c_{ij} = \text{constant},$$

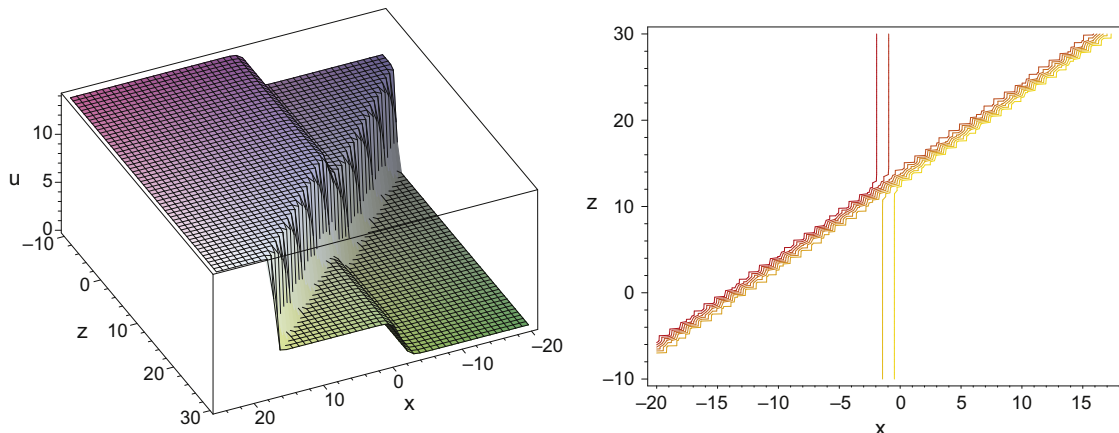


Fig. 3.1. $N = 2$ with $k_1 = 1$, $k_2 = 2$, $l_1 = 1$, $l_2 = 3$, $a = 1$, $y = 0$, $t = 1$.

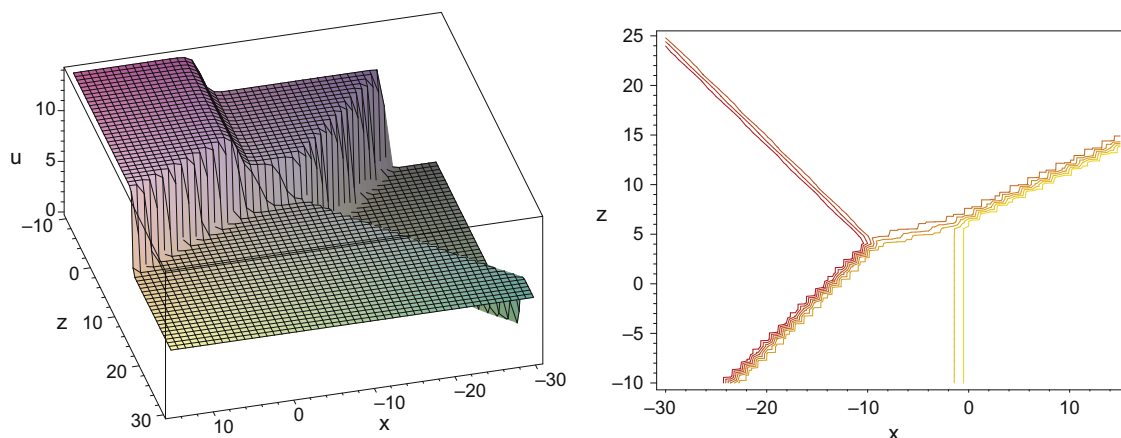


Fig. 3.2. $N = 3$ with $k_1 = 3$, $k_2 = 10$, $k_3 = 1$, $l_1 = 50$, $l_2 = 3$, $l_3 = 20$, $a = 5$, $y = 0$, $t = 1$.

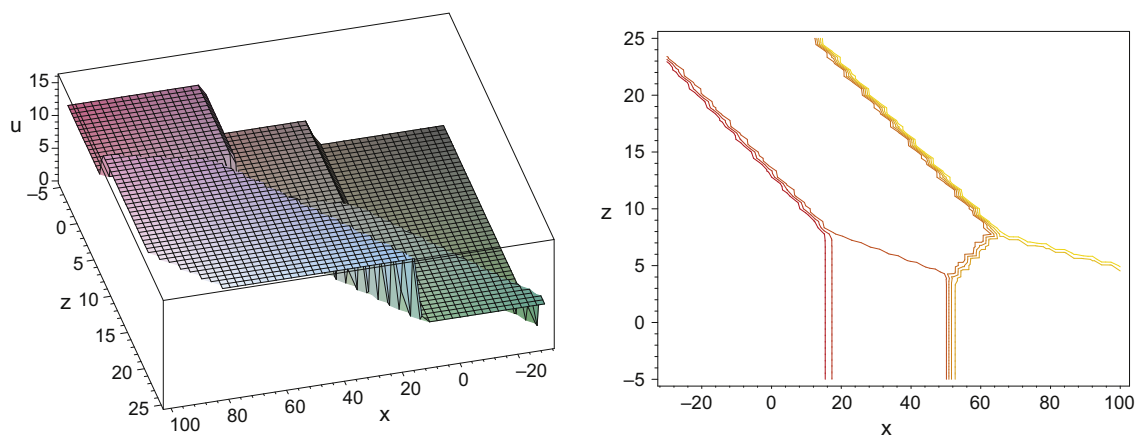


Fig. 3.3. $N = 4$ with $k_1 = 1$, $k_2 = 3$, $k_3 = 2$, $k_4 = 2$, $l_1 = 1$, $l_2 = 2$, $l_3 = 1$, $l_4 = 2$, $a = 3$, $y = 1$, $t = 2$.

with ϕ_i and ψ_j satisfying

$$\begin{aligned} \phi_{i,y} &= -\frac{a^2}{3} \phi_{i,x}, & \phi_{i,z} &= a \phi_{i,xx}, & \phi_{i,t} &= \frac{4a^2}{3-a^2} \phi_{i,xxx}, \\ \psi_{j,y} &= -\frac{a^2}{3} \psi_{j,x}, & \psi_{j,z} &= -a \psi_{j,xx}, & \psi_{j,t} &= \frac{4a^2}{3-a^2} \psi_{j,xxx}, \end{aligned}$$

solve the above $(3+1)$ -dimensional generalized KP equation. Here a can be any real number, except $\pm\sqrt{3}$. Examples of Wronskian and Grammian solutions were made, along with a few plots of particular solutions.

In Theorems 2.1 and 3.1, we considered only specific sufficient conditions: (2.8), (3.2) and (3.3), though there is a free parameter a in the conditions. There should exist more general conditions involving combined equations for Wronskian and Grammian solutions. Such conditions were presented for Wronskian solutions of the KdV equation [18], the Boussinesq equation [19,20] and the Toda lattices [8,21], and for Grammian solutions of the KP equation [22]. It should be interesting to find such a condition consisting of combined equations for the $(3+1)$ -dimensional generalized KP equation.

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