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ORIGINAL PAPER

Studying lump waves in a (2+1)-dimensional spatially symmetric generalized Korteweg-de Vries model

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Abstract: We aim to investigate lump waves arising from the combined effects of nonlinearity and dispersion in a newly proposed (2+1)-dimensional spatially symmetric generalized Korteweg-de Vries model. Using the bilinear method, we employ a practical ansatz based on quadratic function solutions to construct lump waves through symbolic computation in Maple. The nonlinear and dispersive terms in the model are utilized to control the characteristics of the resulting lump waves. The stationary points of these waves lie along a straight line in the spatial plane, with coordinates that vary linearly in time. The corresponding optimal values are explicitly derived and found to be time-independent.

Keywords: Hirota bilinear form; Lump wave; Nonlinearity; Dispersion; Symbolic computation

1. Introduction

There is generally no universal method for obtaining closed-form solutions to nonlinear wave equations. However, within soliton theory, many nonlinear problems can be effectively approached through trial-and-error strategies, where symbolic computation serves as a powerful tool for testing various ansätze and identifying feasible solutions. Multiple types of nonlinear wave solutions, such as soliton and lump solutions, can thus be constructed through computer-assisted searches. These nonlinear dispersive waves arise from the intricate interplay between nonlinearity and dispersion.

The Hirota direct technique [1] and the inverse scattering method [2] are two of the most powerful approaches to finding soliton solutions. The inverse scattering method generalizes the Fourier transform as a nonlinear analog, targeting the solution of initial-value problems for nonlinear equations with Lax pairs [3] and exploring the long-time asymptotics of solitonless solutions [4]. The Hirota direct technique is a straightforward yet powerful method for obtaining soliton and lump wave solutions, especially in (2+1)-dimensional and (3+1)-dimensional nonlinear wave equations [5–9].

Let us start from a multivariate polynomial $S = S(t, x, y)$, where t is time and x, y are two space variables. A (2+1)-dimensional Hirota bilinear differential equation is defined by

$$S(D_t, D_x, D_y)g \cdot g = 0, \tag{1}$$

where D_t, D_x and D_y are Hirota bilinear derivatives [1]:

$$D_t^{n_1} D_x^{n_2} D_y^{n_3} g \cdot g = \left(\frac{\partial}{\partial t'} - \frac{\partial}{\partial t}\right)^{n_1} \left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial x}\right)^{n_2} \left(\frac{\partial}{\partial y'} - \frac{\partial}{\partial y}\right)^{n_3} g(t', x', y') g(t, x, y)\big|_{t'=t, x'=x, y'=y},$$

n_1, n_2, n_3 being nonnegative integers. Corresponding to Hirota bilinear forms, nonlinear partial differential equations that a dependent variable u needs to satisfy are usually generated by matching some of the logarithmic type transformations

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$$u = 2(\ln g)_x, u = 2(\ln g)_{xx}, u = 2(\ln g)_{xy},\\ u = 2(\ln g)_y, u = 2(\ln g)_{yy}. \tag{2}$$

The Hirota bilinear theory tells that an N-soliton solution (see, e.g., [5, 10]) can be stated as

$$g = \sum_{\nu=0,1} \exp\left(\sum_{i=1}^{N} \nu_i \zeta_i + \sum_{i<j} \nu_i \nu_j c_{ij}\right), \tag{3}$$

with $\sum_{\nu=0,1}$ denoting the sum over all possibilities for $\nu_1, \nu_2, \cdots, \nu_N$ being either 0 or 1, and the phase shifts c_{ij} and the wave variables ζ_i being given by

$$\exp(c_{ij}) = -\frac{S(\omega_j - \omega_i, \gamma_i - \gamma_j, \delta_i - \delta_j)}{S(\omega_j + \omega_i, \gamma_i + \gamma_j, \delta_i + \delta_j)},\ 1 \le i < j \le N, \tag{4}$$

and

$$\zeta_i = \gamma_i x + \delta_i y - \omega_i t + \zeta_{i,0},\ 1 \le i \le N. \tag{5}$$

To determine this N-soliton solution,
we need the dispersion relations

$$S(-\omega_i, \gamma_i, \delta_i) = 0,\ 1 \le i \le N, \tag{6}$$

but the constant phase shifts $\zeta_{i,0}$ would be arbitrary. Beginning with the dispersion relations (6), an algorithm to show if an analytical function g in (3) solves the Hirota bilinear equation (1) is carefully presented, together with concrete examples, in [10].

A wide variety of research studies exploits that there is some similarity between soliton waves and lump waves (and rogue waves) in nonlinear integrable models, both of which describe important nonlinear wave phenomena [11]. Such waves are formulated via rational functions, and they are localized in all directions in space and analytic everywhere (see, e.g., [11, 12]). The Kadomtsev-Petviashvili-I (KP-I) equation possesses various lump waves (see, e.g., [7]), and some of its specific lump waves are constructed from its soliton solutions by taking long wave limits [13]. On the other hand, there exist lump waves in nonlinear nonintegrable models, which include a few generalized (2+1)-dimensional KP, B-KP and KP-Boussinesq equations [14, 15].

Moreover, linear models in higher dimensions can possess lump waves as well (see, e.g., [16, 17]).

A successful and efficient scheme for constructing lump waves begins with quadratic function solutions of Hirota bilinear equations [7, 11]. Lump waves of nonlinear model equations can then be obtained through certain logarithmic-type transformations. In this paper, we aim to introduce a generalized Korteweg–de Vries (KdV)-type spatially symmetric nonlinear model in (2+1)-dimensions and to explore its lump wave solutions using this scheme based on quadratic polynomials. The Hirota bilinear form serves as the starting point of our analysis. The proposed (2+1)-dimensional spatially symmetric generalized KdV model contains five second-order dispersive terms and two sets of nonlinear terms. The balance between the dispersion and nonlinear effects generates the desired lump wave solutions. Symbolic computation with Maple is employed as the primary tool for deriving these solutions. Furthermore, the characteristic properties of the obtained lump waves, including their stationary points and corresponding optimal values, are examined in detail. Finally, some concluding remarks are provided in the last section.

2. A spatially symmetric generalized KdV model

Let α_1, α_2 and $\beta_i,\ 1 \le i \le 5$, be real constants. To look for lump waves created jointly by nonlinearity and dispersion, we consider a spatially symmetric generalized (2+1)-dimensional KdV model equation:

$$\begin{aligned} E(u) = {} & \alpha_1(3u_{xx}v_y + 3u_x v_{xy} + 3u_{xy}p + 3u_y p_x + u_{xxxy}) \\ & + \alpha_2(3u_{yy}w_x + 3u_y w_{xy} + 3u_{xy}q + 3u_x q_y + u_{xyyy}) \\ & + \beta_1 u_{ty} + \beta_2 u_{tx} + \beta_3 u_{xx} + \beta_4 u_{yy} + \beta_5 u_{xy} = 0, \end{aligned} \tag{7}$$

with $u_x = p_y, u_y = q_x, p = v_x, q = w_y$. The first example with $\alpha_2 = \beta_2 = 0$ or the second example with $\alpha_1 = \beta_1 = 0$ of this nonlinear model contains the (2+1)-dimensional KdV model equation [18]:

$$3u_{xx}u + 3u_x^2 + 3u_{xy}p + 3u_y p_x + u_{xxxy} + u_{ty} = 0,\ p_y = u_x, \tag{8}$$

up to an exchange of spatial variables for the second example. Both those two reduced models have N-soliton solutions, but the general model doesn't [10].

We take the logarithmic type transformations

$$u = 2(\ln g)_{xy},\ p = 2(\ln g)_{xx},\ q = 2(\ln g)_{yy},\\ v = 2(\ln g)_x,\ w = 2(\ln g)_y, \tag{9}$$

and then the above (2+1)-dimensional spatially symmetric generalized KdV model equation (7) is changed into the Hirota bilinear form:

$$
\begin{aligned}
S(g) &= (\alpha_1 D_x^3 D_y + \alpha_2 D_y^3 D_x + \beta_1 D_y D_t + \beta_2 D_x D_t \\
&\quad + \beta_3 D_x^2 + \beta_4 D_y^2 + \beta_5 D_x D_y) g \cdot g \\
&= 2[\alpha_1 (g_{xxxy} g - 3 g_{xxy} g_x + 3 g_{xy} g_{xx} - g_y g_{xxx}) \\
&\quad + \alpha_2 (g_{xyyy} g - 3 g_{xyy} g_y + 3 g_{xy} g_{yy} - g_x g_{yyy}) \\
&\quad + \beta_1 (g_{ty} g - g_t g_y) + \beta_2 (g_{tx} g - g_t g_x) \\
&\quad + \beta_3 (g_{xx} g - g_x^2) + \beta_4 (g_{yy} g - g_y^2) + \beta_5 (g_{xy} g - g_x g_y)] \\
&= 0,
\end{aligned}
\tag{10}
$$

in which D_t, D_x and D_y denote the traditional Hirota derivatives [1].

By symbolic computation, we can determine a concrete relationship between the bilinear model and the nonlinear model, which is stated as follows:

$$
E(u) = [\frac{S(g)}{g^2}]_{xy}, \tag{11}
$$

where u, p, q, v, w are determined through g in (9). Such links also work for two generalized spatially symmetric KP and HSI models [19]. A further observation tells that u, p, q, v, w determined by (9) solve the newly introduced spatially symmetric generalized KdV model equation (7), when g solves the corresponding bilinear model equation (10). In what follows, we will exploit a type of lump waves for this nonlinear model.

3. Lump waves created by dispersion and nonlinearity

Now, we start to look for lump waves in the newly introduced spatially symmetric model equation in (2+1)-dimensions, (7), through the medium of symbolic computations. It is worth pointing out that the above generic (2+1)-dimensional nonlinear model equation does not pass the three-soliton test (see, e.g., [10] for the three-soliton test).

Using a general ansatz on lump waves in (2+1)-dimensions [7], we start to look for positive quadratic function solutions

$$
\begin{aligned}
&g = \zeta_1^2 + \zeta_2^2 + a_9, \ \zeta_1 = a_1 x + a_2 y + a_3 t + a_4, \\
&\zeta_2 = a_5 x + a_6 y + a_7 t + a_8,
\end{aligned}
\tag{12}
$$

to the corresponding Hirota bilinear equation (10), where the parameters $a_i, \ 1 \le i \le 9$, are real numbers to be determined (see, e.g., [7, 11] for illustrative examples). A general analysis shows that lower-order lump wave solutions in (2+1)-dimensions must take this form [11]. A crucial task is now to make symbolic computations to determine the involved constant parameters $a_i, \ 1 \le i \le 9$.

Simply substituting this g by (12) into the associated bilinear equation (10), we reach a consistent system of algebraic equations that the parameters need to satisfy. A straightforward computation with Maple to solve the system for a_3, a_7 and a_9 can yield the solution for the indicated parameters:

$$
\left\{
\begin{aligned}
a_3 &= -\frac{[(a_1^2 a_2 + 2a_1 a_5 a_6 - a_2 a_5^2)\beta_3 + (a_2^2 + a_6^2) a_2 \beta_4 + (a_2^2 + a_6^2) a_1 \beta_5]\beta_1}{(a_1\beta_2 + a_2\beta_1)^2 + (a_5\beta_2 + a_6\beta_1)^2} \\
&\quad - \frac{[(a_1^2 + a_5^2) a_1 \beta_3 + (a_1 a_2^2 + 2a_2 a_5 a_6 - a_1 a_6^2)\beta_4 + (a_1^2 + a_5^2) a_2 \beta_5]\beta_2}{(a_1\beta_2 + a_2\beta_1)^2 + (a_5\beta_2 + a_6\beta_1)^2}, \\
a_7 &= -\frac{[(a_5^2 a_6 + 2a_1 a_2 a_5 - a_1^2 a_6)\beta_3 + (a_2^2 + a_6^2) a_6 \beta_4 + (a_2^2 + a_6^2) a_5 \beta_5]\beta_1}{(a_1\beta_2 + a_2\beta_1)^2 + (a_5\beta_2 + a_6\beta_1)^2} \\
&\quad - \frac{[(a_1^2 + a_5^2) a_5 \beta_3 + (a_5 a_6^2 + 2a_1 a_2 a_6 - a_2^2 a_5)\beta_4 + (a_1^2 + a_5^2) a_6 \beta_5]\beta_2}{(a_1\beta_2 + a_2\beta_1)^2 + (a_5\beta_2 + a_6\beta_1)^2}, \\
a_9 &= -\frac{3(a_1 a_2 + a_5 a_6)[(a_1\beta_2 + a_2\beta_1)^2 + (a_5\beta_2 + a_6\beta_1)^2][(a_1^2 + a_5^2)\alpha_1 + (a_2^2 + a_6^2)\alpha_2]}{(a_1 a_6 - a_2 a_5)^2(\beta_1^2\beta_3 - \beta_1\beta_2\beta_5 + \beta_2^2\beta_4)},
\end{aligned}
\right.
\tag{13}
$$

and all other parameters are arbitrary.

Note that the above solutions for the two frequency parameters, a_3 and a_7, are crucial and they provide a type of dispersion relations determining nonlinear dispersive waves in (2+1)-dimensions. The solution for the constant term parameter, a_9, is complicated and determined through the coefficients of the dispersion terms and the wave numbers, playing a crucial role in creating lump waves, within the Hirota bilinear formulation. A type of dispersion relations of higher order that lump waves exhibit have also been explored, while studying lump wave solutions for the second model from the KP integrable hierarchy.

With Maple, the simplification becomes simple and the obtained formulas for the three constants, the constant term and the two wave frequencies, in (13) have been simplified by conducting symbolic computations. Obviously, we can see that if

$$a_1\beta_2 + a_2\beta_1 = a_5\beta_2 + a_6\beta_1 = 0, \tag{14}$$

then

$$(a_1a_6 - a_2a_5)(\beta_1^2 + \beta_2^2) = 0. \tag{15}$$

Accordingly, by means of the logarithmic type transformations, building lump wave solutions requires two crucial conditions:

$$a_1a_6 - a_2a_5 \neq 0, \tag{16}$$

and

$$\frac{(a_1a_2 + a_5a_6)[(a_1^2 + a_5^2)\alpha_1 + (a_2^2 + a_6^2)\alpha_2]}{\beta_1^2\beta_3 - \beta_1\beta_2\beta_5 + \beta_2^2\beta_4} < 0. \tag{17}$$

Those two critical conditions tell that the fundamental properties of lump waves can be guaranteed. Firstly, the condition (16) ensures that the resultant solutions of u, p, q are spatially localized. Secondly, $a_9 > 0$ ensured by (16) and (17) guarantees that the solutions are analytic everywhere in the whole space. Actually, the positiveness of a_9 is not only a sufficient condition, but also a necessary condition for u, p, q to be analytic, which will be explained later.

The second condition determined by (17) involves the coefficients, α_1, α_2, of the two nonlinear terms and the coefficients, β_i, $1 \leq i \leq 5$, of the five dispersion terms. If $a_1a_2 + a_5a_6 > 0$, then

$$\frac{(a_1^2 + a_5^2)\alpha_1 + (a_2^2 + a_6^2)\alpha_2}{\beta_1^2\beta_3 - \beta_1\beta_2\beta_5 + \beta_2^2\beta_4} < 0,$$

which implies that

$$\beta_1^2\beta_3 - \beta_1\beta_2\beta_5 + \beta_2^2\beta_4 < 0 \ (>0),$$

when $\alpha_1, \alpha_2 > 0$ (<0). If $a_1a_2 + a_5a_6 < 0$, then

$$\frac{(a_1^2 + a_5^2)\alpha_1 + (a_2^2 + a_6^2)\alpha_2}{\beta_1^2\beta_3 - \beta_1\beta_2\beta_5 + \beta_2^2\beta_4} > 0,$$

which implies that

$$\beta_1^2\beta_3 - \beta_1\beta_2\beta_5 + \beta_2^2\beta_4 > 0 \ (<0),$$

when α_1 and α_2 are positive (or negative). Therefore, the mutual effects of the dispersion and the nonlinearity play an essential role in formulating lump wave solutions for the considered model equation (7), though the nonlinearity involved has no effect on the traveling speeds of the both single waves in the presented lump solutions, in view of (13). Moreover, the condition on the dispersion terms

$$\beta_1^2\beta_3 - \beta_1\beta_2\beta_5 + \beta_2^2\beta_4 \neq 0 \tag{18}$$

is always required in the presentation of the lump waves.

Two special reduced cases can be worked out. When $\alpha_2 = \beta_2 = 0$, we obtain

$$\begin{cases} a_3 = -\dfrac{(a_5^2a_6 + 2a_1a_2a_5 - a_1^2a_6)\beta_3 + (a_2^2 + a_6^2)a_6\beta_4 + (a_2^2 + a_6^2)a_5\beta_5}{(a_2^2 + a_6^2)\beta_1}, \\ a_7 = -\dfrac{(a_5^2a_6 + 2a_1a_2a_5 - a_1^2a_6)\beta_3 + (a_2^2 + a_6^2)a_6\beta_4 + (a_2^2 + a_6^2)a_5\beta_5}{(a_2^2 + a_6^2)\beta_1}, \\ a_9 = -\dfrac{3(a_1a_2 + a_5a_6)(a_2^2 + a_6^2)(a_1^2 + a_5^2)\alpha_1}{(a_1a_6 - a_2a_5)^2\beta_3}. \end{cases} \tag{19}$$

The conditions for a_9 to be positive in this first reduced case are

$$a_1a_6 - a_2a_5 \neq 0, \ (a_1a_2 + a_5a_6)\alpha_1\beta_3 < 0. \tag{20}$$

Therefore, the linear dispersion term u_{xx} is essential in the creation of lump waves.

Similarly, when $\alpha_1 = \beta_1 = 0$, we obtain

$$\begin{cases} a_3 = -\dfrac{(a_1^2 + a_5^2)a_5\beta_3 + (a_5a_6^2 + 2a_1a_2a_6 - a_2^2a_5)\beta_4 + (a_1^2 + a_5^2)a_6\beta_5}{(a_1^2 + a_5^2)\beta_2}, \\ a_7 = -\dfrac{(a_1^2 + a_5^2)a_5\beta_3 + (a_5a_6^2 + 2a_1a_2a_6 - a_2^2a_5)\beta_4 + (a_1^2 + a_5^2)a_6\beta_5}{(a_1^2 + a_5^2)\beta_2}, \\ a_9 = -\dfrac{3(a_1a_2 + a_5a_6)(a_1^2 + a_5^2)(a_2^2 + a_6^2)\alpha_2}{(a_1a_6 - a_2a_5)^2\beta_4}. \end{cases} \tag{21}$$

The conditions for a_9 to be positive in this second reduced case read

$$a_1a_6 - a_2a_5 \neq 0, \ (a_1a_2 + a_5a_6)\alpha_2\beta_4 < 0. \tag{22}$$

Accordingly, the linear dispersion term u_{yy} is essential for a lump wave to exist.

It would also be of great interest to exploit if there are lump waves in multiple component integrable modified Korteweg-de Vries type models or nonlinear Schrödinger type models (see, e.g., [20–22]).

4. Dynamical characteristics

Let us analyze characteristic dynamical behaviors that the lump waves, obtained in the previous section, possess.

Set of stationary points:

We first determine stationary points of the wave function g defined by (12). To the end, we consider and solve the derivative equations

$$\frac{\partial g}{\partial x}(t,x,y) = 0, \ \frac{\partial g}{\partial y}(t,x,y) = 0. \tag{23}$$

Because g is a quadratic polynomial, this exactly leads to

$$a_1\zeta_1 + a_5\zeta_2 = 0,\ a_2\zeta_1 + a_6\zeta_2 = 0. \tag{24}$$

Therefore, in view of the condition by (16), we have

$$\zeta_1 = a_1x + a_2y + a_3t + a_4 = 0,\ \zeta_2 = a_5x + a_6y + a_7t + a_8 = 0, \tag{25}$$

Solving this system of linear equations for x and y in terms of t, one gets the stationary points of the multivariate quadratic function g:

$$x(t) = \frac{[2(a_1a_2 + a_5a_6)\beta_3 + (a_2^2 + a_6^2)\beta_5]\beta_1 + [(a_1^2 + a_5^2)\beta_3 - (a_2^2 + a_6^2)\beta_4]\beta_2}{(a_1\beta_2 + a_2\beta_1)^2 + (a_5\beta_2 + a_6\beta_1)^2}t + \frac{a_2a_8 - a_4a_6}{a_1a_6 - a_2a_5}, \tag{26}$$

$$y(t) = -\frac{[(a_1^2 + a_5^2)\beta_3 - (a_2^2 + a_6^2)\beta_4]\beta_1 - [2(a_1a_2 + a_5a_6)\beta_4 + (a_1^2 + a_5^2)\beta_5]\beta_2}{(a_1\beta_2 + a_2\beta_1)^2 + (a_5\beta_2 + a_6\beta_1)^2}t - \frac{a_1a_8 - a_4a_5}{a_1a_6 - a_2a_5}, \tag{27}$$

at an arbitrarily fixed time t.

Clearly, we see that those stationary points move on a line, extending infinitely in both directions and changing at constant rates. Further, a straightforward calculation can show that all those points $(x(t), y(t))$ computed above are stationary points of the three wave functions u, p and q defined by (9), too.

Analyticity condition:

Based on (25), we find that the function $g - a_9$, namely, the sum of two square functions, vanishes at all stationary points defined by (26) and (27). Accordingly, we see that g is positive if and only if a_9 is positive. There is no question on the sufficiency, as exploited previously. The necessity can be seen in a way that when a_9 is zero, g vanishes at the stationary points; and when a_9 is negative, g vanishes at any point satisfying the equation $\zeta_1^2 + \zeta_2^2 = -a_9$.

Now, we observe that u, p, q defined by (9) become analytic functions in $\mathbb{R}^3$, while the parameter a_9 is positive. Then, in light of the discussion on the positiveness of a_9 in Section 3, the two critical conditions in (16) and (17) are necessary and sufficient, in order for u, p, q to be analytic in the whole space.

Optimum values:

We use the second partial derivative test in Calculus to determine optimum values. First, the lump waves by p and q attain maximum value at the stationary points $(x(t), y(t))$, because we have

$$\begin{cases} p_{xx} = -\dfrac{8(a_1^2 + a_5^2)^2(a_1a_6 - a_2a_5)^4(\beta_1^2\beta_3 - \beta_1\beta_2\beta_5 + \beta_2^2\beta_4)^2}{3(a_1a_2 + a_5a_6)^2[(a_1\beta_2 + a_2\beta_1)^2 + (a_5\beta_2 + a_6\beta_1)^2]^2a_{10}^2} < 0, \\ p_{xx}p_{yy} - p_{xy}^2 \\ = \dfrac{64(a_1^2 + a_5^2)^2(a_1a_6 - a_2a_5)^{10}(\beta_1^2\beta_3 - \beta_1\beta_2\beta_5 + \beta_2^2\beta_4)^4}{27(a_1a_2 + a_5a_6)^4[(a_1\beta_2 + a_2\beta_1)^2 + (a_5\beta_2 + a_6\beta_1)^2]^4a_{10}^4} > 0, \end{cases} \tag{28}$$

and

$$\begin{cases} q_{xx} = -\dfrac{8[(a_1a_6 + a_2a_5)^2 + 3(a_1^2a_2^2 + a_5^2a_6^2)](a_1a_6 - a_2a_5)^4(\beta_1^2\beta_3 - \beta_1\beta_2\beta_5 + \beta_2^2\beta_4)^2}{9(a_1a_2 + a_5a_6)^2[(a_1\beta_2 + a_2\beta_1)^2 + (a_5\beta_2 + a_6\beta_1)^2]^2a_{10}^2} \\ < 0, \\ q_{xx}q_{yy} - q_{xy}^2 \\ = \dfrac{64(a_2^2 + a_6^2)^2(a_1a_6 - a_2a_5)^{10}(\beta_1^2\beta_3 - \beta_1\beta_2\beta_5 + \beta_2^2\beta_4)^4}{27(a_1a_2 + a_5a_6)^4[(a_1\beta_2 + a_2\beta_1)^2 + (a_5\beta_2 + a_6\beta_1)^2]^4a_{10}^4} > 0, \end{cases} \tag{29}$$

where a_{10} is defined by

$$a_{10} = \alpha_1(a_1^2 + a_5^2) + \alpha_2(a_2^2 + a_6^2). \tag{30}$$

Similarly, we can find that

$$\begin{cases} u_{xx} = -\dfrac{8(a_1^2 + a_5^2)(a_1a_6 - a_2a_5)^4(\beta_1^2\beta_3 - \beta_1\beta_2\beta_5 + \beta_2^2\beta_4)^2}{3(a_1a_2 + a_5a_6)[(a_1\beta_2 + a_2\beta_1)^2 + (a_5\beta_2 + a_6\beta_1)^2]a_{10}^2}, \\ u_{xx}u_{yy} - u_{xy}^2 \\ = \dfrac{64a_{11}(a_1a_6 - a_2a_5)^{10}(\beta_1^2\beta_3 - \beta_1\beta_2\beta_5 + \beta_2^2\beta_4)^4}{81(a_1a_2 + a_5a_6)^4[(a_1\beta_2 + a_2\beta_1)^2 + (a_5\beta_2 + a_6\beta_1)^2]^4a_{10}^4}, \end{cases} \tag{31}$$

where a_{10} is given by (30) and a_{11} is defined by

$$a_{11} = 3(a_1a_2 + a_5a_6)^2 - (a_1a_6 - a_2a_5)^2. \tag{32}$$

Consequently, the wave function u attains maximum value (or minimum value) at the stationary points $(x(t), y(t))$, when $a_1a_2 + a_5a_6$ is positive (or negative) and a_{11} is

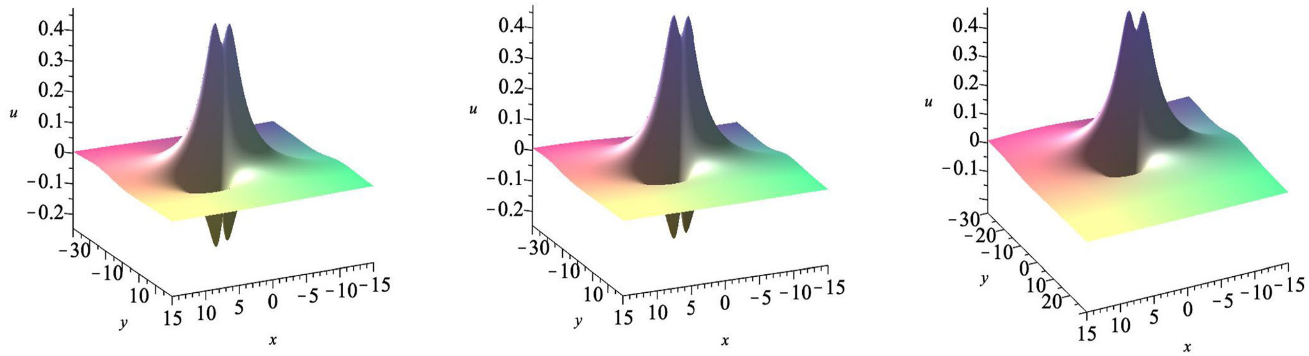

Fig. 1 3d-plots with $t=0$ (left), $t=3$ (middle) and $t=5$ (right)

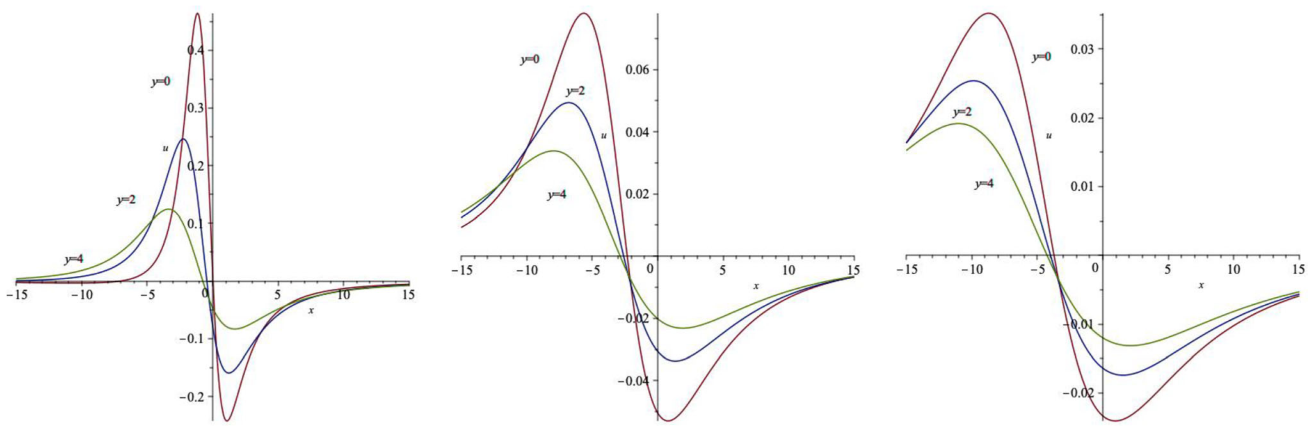

Fig. 2 x curves with $t=0$ (left), $t=3$ (middle) and $t=5$ (right)

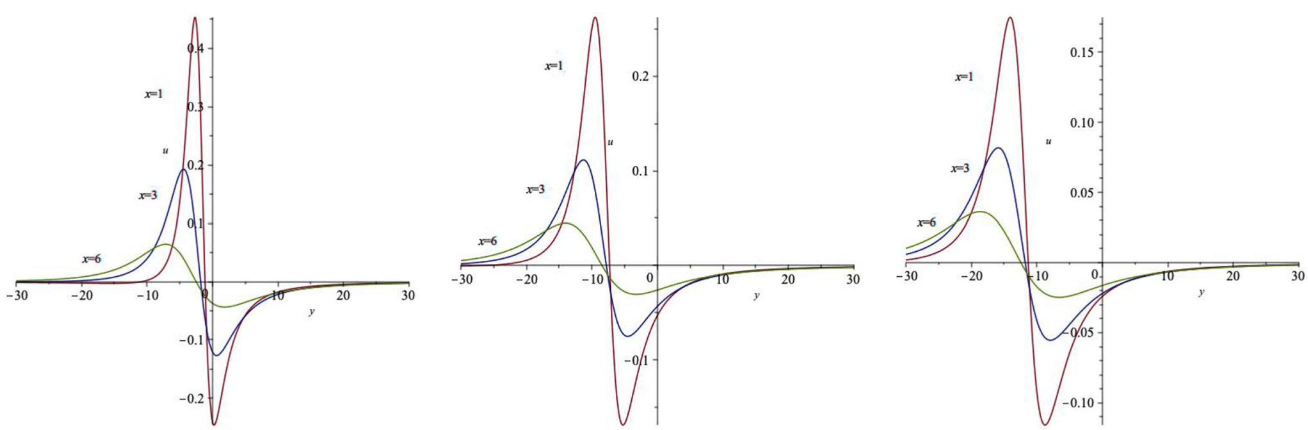

Fig. 3 y curves with $t=0$ (left), $t=3$ (middle) and $t=5$ (right)

positive; $(x(t), y(t))$ will be the saddle points for u, when a_{11} is negative; and there is no conclusion, when a_{11} is zero.

Moreover, we can work out the optimum values for the lumps waves by u, p, q at the obtained stationary points $(x(t), y(t))$ as follows:

$$u_{optimum} = -\frac{4(a_1a_6 - a_2a_5)^2(\beta_1^2\beta_3 - \beta_1\beta_2\beta_5 + \beta_2^2\beta_4)}{3[(a_1\beta_2 + a_2\beta_1)^2 + (a_5\beta_2 + a_6\beta_1)^2]a_{10}}, \tag{33}$$

$$p_{maximum} = -\frac{4(a_1^2+a_5^2)(a_1a_6-a_2a_5)^2(\beta_1^2\beta_3-\beta_1\beta_2\beta_5+\beta_2^2\beta_4)}{3(a_1a_2+a_5a_6)[(a_1\beta_2+a_2\beta_1)^2+(a_5\beta_2+a_6\beta_1)^2]a_{10}}, \quad (34)$$

$$q_{maximum} = -\frac{4(a_2^2+a_6^2)(a_1a_6-a_2a_5)^2(\beta_1^2\beta_3-\beta_1\beta_2\beta_5+\beta_2^2\beta_4)}{3(a_1a_2+a_5a_6)[(a_1\beta_2+a_2\beta_1)^2+(a_5\beta_2+a_6\beta_1)^2]a_{10}}, \quad (35)$$

where a_{10} is given by (30). Consequently, we see that all presented optimum values are constants along the characteristic line of the above stationary points, being independent of time t (see also, e.g., [19] for more examples). Therefore, the peak and the valley of the lump waves continue to be the same, when time t changes. Moreover, in view of those three formulas for the optimum values, we can show that all those lump waves u, p, q may not start decaying, when the two directions (a_1, a_2) and (a_5, a_6) becomes parallel to each other, or equivalently when the determinant of the two vectors, $a_1a_6 - a_2a_5$, approaches zero (Figs. 1, 2 and 3).

The following figures display 3d- and 2d-plots of the lump wave $u = 2(\ln f)_{xy}$, generated under the parameter choices specified below:

$$\alpha_1 = \alpha_2 = -\beta_1 = \beta_2 = -\beta_3 = \beta_4 = -\beta_5 = -1,$$

and

$$a_1 = -2,\ a_2 = 1,\ a_4 = 1,\ a_5 = 2,\ a_6 = 2,\ a_8 = 3.$$

5. Conclusions

By performing symbolic computations using computer algebra systems such as Mathematica and MATLAB, lump waves have been analyzed for a newly introduced (2+1)-dimensional spatially symmetric generalized KdV model, generated by the combined effects of nonlinearity and dispersion. The resulting lump waves were obtained in exact form; in particular, the frequencies a_3, a_7, and the constant term a_9 were expressed as rational functions of the wave numbers. The dynamical characteristics of the obtained lump wave solutions, including their stationary points and corresponding optimal values, were explicitly derived. Furthermore, the mutual influence of the nonlinear and dispersive terms was discussed in detail.

Abundant studies have exhibited an enormous richness of lump waves in linear wave model equations [16, 17], and nonlinear wave model equations in (2+1)-dimensions (see, e.g., [23–28]) and (3+1)-dimensions (see, e.g., [29]). Besides the Hirota bilinear formulation, generalized bilinear formulations are also helpful in constructing lump waves [11, 30]. On the other hand, for (2+1)-dimensonal integrable model equations, diverse interaction wave solutions of lump wave solutions with other important nonlinear wave solutions, both homoclinic and heteroclinic, have been exploited (see, e.g., [15, 31–33]).

Riemann-Hilbert problems and Darbopux transformations have been established to study N-soliton solutions to integrable models, both local and nonlocal, obtained by groups reductions (see, e.g., [34–36]), exhibiting remarkable properties such as elastic scattering, fusion, and fission of solitons. It is very interesting to see if reduced integrable models (see, e.g., [37–40]) and multi-component integrable models (see, e.g., [41–45]) could possess lump waves, either in the local case or the nonlocal case. Both lump and soliton waves provide insights into the dynamics of nonlinear waves and have applications in various fields, including nonlinear optics, plasma physics, and mathematical physics.

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