



Direct search for exact solutions to the nonlinear Schrödinger equation

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ABSTRACT

A five-dimensional symmetry algebra consisting of Lie point symmetries is firstly computed for the nonlinear Schrödinger equation, which, together with a reflection invariance, generates two five-parameter solution groups. Three ansätze of transformations are secondly analyzed and used to construct exact solutions to the nonlinear Schrödinger equation. Various examples of exact solutions with constant, trigonometric function type, exponential function type and rational function amplitude are given upon careful analysis. A bifurcation phenomenon in the nonlinear Schrödinger equation is clearly exhibited during the solution process.

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1. Introduction

We are concerned with the cubic nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} + \mu|u|^2u = 0, \quad (1.1)$$

where $u = u(x, t)$ is a complex-valued function of two real variables x, t and μ is a non-zero real parameter. The physical model of the NLS equation (1.1) and its generalized ones occur in various areas of physics such as nonlinear optics, water waves, plasma physics, quantum mechanics, superconductivity and Bose–Einstein condensate theory [1,2]. In optics, the NLS equation (1.1) models many nonlinearity effects in a fiber, including but not limited to self-phase modulation, four-wave mixing, second harmonic generation, stimulated Raman scattering, etc. For water waves, the NLS equation (1.1) describes the evolution of the envelope of modulated nonlinear wave groups. All these physical phenomena can be better understood with the help of exact solutions when they exist for particular values of the parameter μ .

It is well known that the NLS equation (1.1) admits the bright soliton solution [3]:

$$u(x, t) = k\sqrt{\frac{2}{\mu}}\text{sech}(k(x - 2\alpha t))e^{i[\alpha x - (\alpha^2 - k^2)t]}, \quad (1.2)$$

where α and k are arbitrary real constants, for the self-focusing case $\mu > 0$, and the dark soliton solution [4]:

$$u(x, t) = k\sqrt{-\frac{2}{\mu}}\tanh(k(x - 2\alpha t))e^{i[\alpha x - (\alpha^2 + 2k^2)t]}, \quad (1.3)$$

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where α and k are arbitrary real constants, for the de-focusing case $\mu < 0$. These solutions are valid under the localized traveling wave assumption. The n -soliton solutions to both the self-focusing NLS equation and the de-focusing NLS equation can be computed by the inverse scattering transform, the Darboux transformation and the Hirota bilinear method (see, say, [5–7]). Moreover, $\tilde{u}(x, t) = u(ix, -t)$ offers a Bäcklund transformation between the self-focusing NLS equation and the de-focusing NLS equation.

A considerable amount of research has been devoted to the study of exact solutions including traveling wave solutions of the NLS equation (see, say, [8–25]). Both numerical and analytical methods have been used in dealing with the related problems. Generally, exact solutions to nonlinear equations are hard to come by, but it is significantly important in mathematical physics to find new ideals or approaches to discover solitary wave solutions of nonlinear equations. Recently, several interesting studies have been published to show that the NLS equation (1.1) has many new types of exact solutions (see, for instance, [19–25]).

In this paper, we would like to present some direct search approaches to exact solutions of the NLS equation (1.1) and construct its exact solutions over some region of $\mathbb{R}^2$, including analytical solutions on the whole plane of x and t . In what follows, on one hand, a five-dimensional symmetry algebra is presented and used to generate the corresponding five one-parameter solution groups. On the other hand, three ansätze of transformations are analyzed, and various examples of exact solutions with constant, trigonometric function type, exponential function type and rational function amplitude are calculated in detail, covering many known exact solutions in the literature. The presented ansätze are direct but powerful, particularly in getting traveling wave type solutions. A few concluding remarks are given in the final section.

2. Symmetry algebra and solution groups

We would like to present a five-dimensional symmetry algebra and its corresponding one-parameter solution groups. Obviously, the linearized equation of the NLS equation (1.1) is given by

$$i\sigma_t + \sigma_{xx} + 2\mu|u|^2\sigma + \mu u^2\bar{\sigma} = 0, \quad (2.1)$$

where $\bar{\sigma}$ is the complex conjugate of σ . It is direct to check that there are five local Lie-point symmetries:

$$\sigma_1 = iu, \quad \sigma_2 = u_x, \quad \sigma_3 = u_t, \quad \sigma_4 = ixu - 2tu_x, \quad \sigma_5 = u + xu_x + 2tu_t, \quad (2.2)$$

namely, the five functions σ_i , $1 \leq i \leq 5$, satisfy the linearized equation (2.1) when u solves the NLS equation (1.1). These are special reductions of symmetries of the general AKNS systems [26].

Let us recall that the commutator of vector fields is defined by

$$[K_1, K_2] = K'_1(u)[K_2] - K'_2(u)[K_1], \quad (2.3)$$

where $K'(u)[S]$ denotes the Gateaux derivative $K'(u)[S] = \frac{\partial}{\partial \varepsilon} \big|_{\varepsilon=0} K(u + \varepsilon S)$. The symmetries σ_i , $1 \leq i \leq 5$, constitute a five-dimensional Lie algebra over the complex field under the commutator of vector fields, and the non-zero commutators among $[\sigma_r, \sigma_s]$, $1 \leq r < s \leq 5$, are as follows:

$$[\sigma_2, \sigma_4] = \sigma_1, \quad [\sigma_2, \sigma_5] = \sigma_2, \quad [\sigma_3, \sigma_4] = -2\sigma_2, \quad [\sigma_3, \sigma_5] = 2\sigma_3, \quad [\sigma_4, \sigma_5] = -\sigma_4. \quad (2.4)$$

The symmetries $\sigma_1, \sigma_2, \sigma_3$ correspond to the u -scale invariance, the x -translational invariance and the t -translational invariance, respectively. The symmetries σ_4 and σ_5 are two of so-called τ -symmetries [27], obtained from Galilean invariance and general scale invariance. The general symmetry algebra of the NLS equation (1.1) contains Lie Bäcklund symmetries and other but non-local τ -symmetries.

It is evident to see that the symmetries σ_i , $1 \leq i \leq 5$, generate five one-parameter solution groups as follows:

$$\begin{cases} \tilde{u}_1(x, t) = e^{i\varepsilon} u(x, t), \\ \tilde{u}_2(x, t) = u(x + \varepsilon, t), \\ \tilde{u}_3(x, t) = u(x, t + \varepsilon), \\ \tilde{u}_4(x, t) = e^{i(\varepsilon x - \varepsilon^2 t)} u(x - 2\varepsilon t, t), \\ \tilde{u}_5(x, t) = e^\varepsilon u(xe^\varepsilon, te^{2\varepsilon}), \end{cases} \quad (2.5)$$

where u solves the NLS equation (1.1) and ε is a free real group parameter. Taking $\varepsilon = \pi, \frac{\pi}{2}$, the first solution group $\tilde{u}_1$ yields two special solutions $-u$ and iu , respectively.

Note that the NLS equation (1.1) also has a reflection symmetry. A coordinate reflection about the t -axis generates a new solution $u(-x, t)$ from a known one $u(x, t)$. Therefore, we can conclude the following two five-parameter solution groups:

$$\tilde{u}_\delta(x, t) = e^{i(\alpha x e^\beta - \alpha^2 t e^{2\beta} + \eta_0) + \beta} u(\delta x e^\beta - 2\delta \alpha t e^{2\beta} + \zeta_0, t e^{2\beta} + \zeta_0), \quad (2.6)$$

where $\delta = \pm 1$, and the five free parameters $\eta_0, \zeta_0, \zeta_0, \alpha, \beta$ correspond to the five symmetries $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$, respectively. These two solution groups can be used to construct various new solutions from known ones.

3. Transformations and exact solutions

We would, in this section, like to discuss three ansätze to transform the NLS equation (1.1) into real simplified systems of differential equations and construct exact solutions through those transformed NLS equations resulting from the three ansätze.

3.1. First ansatz

We look for solutions by appending a phase factor to a real-valued function. More precisely, we make an ansatz:

$$u(x, t) = v(x, t)e^{i\eta}, \quad \eta = \alpha x + \gamma t, \quad (3.1)$$

where v is a real-valued function, and α, γ are two real constants. This way, the NLS equation (1.1) becomes a real system:

$$v_t + 2\alpha v_x = 0, \quad v_{xx} - (\gamma + \alpha^2)v + \mu v^3 = 0. \quad (3.2)$$

If we set $v = g/f$, then the system (3.2) is put into

$$\begin{cases} fg_t - f_t g + 2\alpha f g_x - 2\alpha f_x g = 0, \\ f^2 g_{xx} - 2ff_x g_x + 2f_x^2 g - ff_{xx} g - (\gamma + \alpha^2)f^2 g + \mu g^3 = 0. \end{cases} \quad (3.3)$$

The first equation is bilinear while the second one is trilinear. This setup brings us a direct approach to searching for exact solutions. We can guess the type of functions f and g first, and then look for exact solutions, especially by computer algebra systems. For instance, a direct computation with Maple can show that there is no multiple traveling wave type solutions among the set of functions $u = (g/f)e^{i\eta}$ with

$$f = \sum_{m=0}^n a_m e^{k_m x + \omega_m t}, \quad g = \sum_{m=0}^n b_m e^{k_m x + \omega_m t},$$

where a_m, b_m, k_m, ω_m $1 \leq m \leq n$, are real constants.

Let us remark that if we look for solutions $v = v(\xi)$ with $\xi = k(x - 2\alpha t)$, where $k \neq 0$ and α are real constants, the transformed NLS equation (3.2) reduces to

$$k^2 v_{\xi\xi} - (\gamma + \alpha^2)v + \mu v^3 = 0. \quad (3.4)$$

This equation is integrable, and it can be integrated as follows:

$$\int_{v(\xi_0)}^{v(\xi)} \frac{1}{\sqrt{2(\gamma + \alpha^2)v^2 - \mu v^4 + C}} dv = \frac{\sqrt{2}}{2k} (\xi - \xi_0), \quad (3.5)$$

where C and ξ_0 are arbitrary real constants. The resulting solutions contain both elementary function and Jacobi elliptic function solutions.

In what follows, we would like to find exact solutions of the NLS equation (1.1) with elementary function amplitude.

3.1.1. Solutions with constant amplitude

A constant solution $v = v_0$ of (3.4) leads to a class of exact uniform solutions of the NLS equation (1.1):

$$u(x, t) = v_0 e^{i[\alpha x + (\mu v_0^2 - \alpha^2)t]}, \quad (3.6)$$

where v_0 and α are arbitrary real constants. This solution corresponds to a plane wave in water waves and α corresponds to a simple shift of carrier-wave wave number [10]. The special case of (3.6) with $v_0 = 1$ was also analyzed by other ansätze [19,23].

3.1.2. Solutions with trigonometric function type amplitude

Among the functions

$$u(x, t) = (c \sec \xi + d \csc \xi) e^{i\eta}, \quad \xi = k(x - 2\alpha t), \quad \eta = \alpha x + \gamma t,$$

where c, d, k, α, γ are real constants, we have the following solutions with sec- and csc-function amplitude:

$$u(x, t) = k \sqrt{-\frac{2}{\mu}} (\sec \xi) e^{i\eta}, \quad u(x, t) = k \sqrt{-\frac{2}{\mu}} (\csc \xi) e^{i\eta}, \quad (3.7)$$

where $\xi = k(x - 2\alpha t)$, $\eta = \alpha x - (\alpha^2 + k^2)t$, and k and α are arbitrary real constants.

Among the functions

$$u(x, t) = \frac{b_0 + b_1 \tan \xi + b_2 \tan^2 \xi}{a_0 + a_1 \tan \xi} e^{i\eta}, \quad \xi = k(x - 2\alpha t), \quad \eta = \alpha x + \gamma t,$$

where $a_i, b_i, k, \alpha, \gamma$ are real constants, we have the following three solutions with tan-function type amplitude. The first solution is

$$u(x, t) = k \sqrt{-\frac{2}{\mu} \frac{a_0 \tan \xi - a_1}{a_0 + a_1 \tan \xi}} e^{i\eta}, \quad (3.8)$$

where $\xi = k(x - 2\alpha t)$, $\eta = \alpha x - (\alpha^2 - 2k^2)t$, and a_0, a_1, k, α are all arbitrary real constants satisfying $a_0^2 + a_1^2 \neq 0$. This solution contains exact solutions with tan- and cot-function amplitude, which correspond to $a_1 = 0$ and $a_0 = 0$, respectively. The second and third solutions are

$$u = \frac{k\sqrt{-2\mu}}{\mu} (\cot \xi + \tan \xi) e^{i\eta} \quad (3.9)$$

with $\xi = k(x - 2\alpha t)$ and $\eta = \alpha x - (\alpha^2 + 4k^2)t$, and

$$u = \frac{k\sqrt{-2\mu}}{\mu} (-\cot \xi + \tan \xi) e^{i\eta} \quad (3.10)$$

with $\xi = k(x - 2\alpha t)$ and $\eta = \alpha x - (\alpha^2 - 8k^2)t$, where k and α are arbitrary real constants. Those two solutions (3.9) and (3.10) can be simplified to the solution (3.7) with csc-function amplitude and the solution (3.8) with $a_0 = 0$, respectively.

3.1.3. Solutions with exponential function type amplitude

If we focus on the set of functions $v = g/f$ with

$$f = \sum_{m=0}^n a_m e^{m\xi}, \quad g = \sum_{m=0}^n b_m e^{m\xi}, \quad \xi = k(x - 2\alpha t),$$

where $a_m, b_m, 1 \leq m \leq n$, are real constants, we obtain the following solutions with exponential function type amplitude.

The non-constant solution of first order (i.e., $n = 1$) is

$$v(x, t) = \frac{k}{2} \sqrt{-\frac{2}{\mu} \frac{e^\xi - a_0}{e^\xi + a_0}}, \quad (3.11)$$

where $\xi = k(x - 2\alpha t)$ and $\gamma = -(\alpha^2 + \frac{k^2}{2})$ in (3.4). It thus follows that the corresponding solution reads

$$u(x, t) = \frac{k}{2} \sqrt{-\frac{2}{\mu} \frac{e^\xi - a_0}{e^\xi + a_0}} e^{i\eta}, \quad (3.12)$$

where $\xi = k(x - 2\alpha t)$, $\eta = \alpha x - (\alpha^2 + \frac{k^2}{2})t$, and a_0, k and α are arbitrary real constants. This solution contains solutions with tanh- and coth-function amplitude, in particular, the dark soliton solution. It also gives the solutions presented in [25] since $\cos(2 \tan^{-1} e^\xi) = (1 - e^{2\xi})/(1 + e^{2\xi})$.

The first non-constant solution of second order (i.e., $n = 2$) is

$$v(x, t) = \frac{8a_0 b_1 k^2 e^\xi}{8a_0^2 k^2 + \mu b_1^2 e^{2\xi}} = \frac{8ck^2 e^\xi}{8k^2 + \mu c^2 e^{2\xi}}, \quad (3.13)$$

where $\xi = k(x - 2\alpha t)$ and $\gamma = -\alpha^2 + k^2$ in (3.4). It thus follows that the corresponding solution reads

$$u(x, t) = \frac{8ck^2 e^\xi}{8k^2 + \mu c^2 e^{2\xi}} e^{i\eta}, \quad (3.14)$$

where $\xi = k(x - 2\alpha t)$, $\eta = \alpha x - (\alpha^2 - k^2)t$, and c, k and α are arbitrary real constants. This solution contains a special solution with sech-function amplitude including the bright soliton solution if $\mu > 0$, and a special solution with csch-function amplitude if $\mu < 0$ [21]. It also suggests that we can show that the self-focusing NLS equation (1.1) does not have any exact solution with csch-function amplitude.

The second non-constant solution of second order (i.e., $n = 2$) is

$$v(x, t) = \frac{k(-2\sqrt{-2\mu}\mu + 4ck\mu e^\xi + c^2 k^2 \sqrt{-2\mu} e^{2\xi})}{2\mu(2\mu + c^2 k^2 e^{2\xi})}, \quad (3.15)$$

where $\xi = k(x - 2\alpha t)$, $\gamma = -(\alpha^2 + \frac{k^2}{2})$ in (3.4) and $c = a_1/b_0$. It thus follows that the corresponding solution reads

$$u(x, t) = \frac{k(-2\sqrt{-2\mu}\mu + 4ck\mu e^\xi + c^2 k^2 \sqrt{-2\mu} e^{2\xi})}{2\mu(2\mu + c^2 k^2 e^{2\xi})} e^{i\eta}, \quad (3.16)$$

where $\xi = k(x - 2\alpha t)$, $\eta = \alpha x - (\alpha^2 + \frac{k^2}{2})t$, and c, k and α are arbitrary real constants.

3.1.4. Solutions with rational function amplitude

Among the set of functions $v = g/f$ with

$$f = \sum_{r,s=0}^2 a_{rs} x^r t^s, \quad g = \sum_{r,s=0}^2 b_{rs} x^r t^s,$$

where a_{rs}, b_{rs} are real constants, we have the following class of exact solutions with rational function amplitude:

$$v(x, t) = \frac{-2b_{0,0}\sqrt{-2\mu\alpha} - \sqrt{-2\mu\alpha} + 2\sqrt{-2\mu\alpha}t + 2a_{0,1}}{4b_{0,0}^2\mu\alpha^2 + 4b_{0,0}\mu\alpha x - 8b_{0,0}\mu\alpha^2 t + \mu x^2 - 4\mu\alpha x t + 4\mu\alpha^2 t^2 + 2a_{0,1}^2}, \quad (3.17)$$

where $b_{0,0}, a_{0,1}, \alpha$ are arbitrary constants and $\gamma = -\alpha^2$ in (3.2). It thus follows that the corresponding solution reads

$$u(x, t) = v(x, t)e^{i(\alpha x - \alpha^2 t)}, \quad (3.18)$$

with v being defined by (3.17). Taking $b_{0,0} = 0$ leads to

$$u(x, t) = \frac{-\sqrt{-2\mu\alpha} + 2\sqrt{-2\mu\alpha}t + 2a_{0,1}}{\mu x^2 - 4\mu\alpha x t + 4\mu\alpha^2 t^2 + 2a_{0,1}^2} e^{i(\alpha x - \alpha^2 t)}. \quad (3.19)$$

A further reduction with $a_{0,1} = 0$ yields a special solution:

$$u(x, t) = \sqrt{-\frac{2}{\mu x - 2\alpha t}} e^{i(\alpha x - \alpha^2 t)},$$

which, upon selecting $\mu = -2$ and $\alpha = -2\xi$, gives the solution presented by a limiting process in [15].

3.2. Second ansatz

We make the second ansatz

$$u(x, t) = [p(x, t) + iq(x, t)]e^{i\gamma t}, \quad (3.20)$$

where p and q are two real-valued functions and γ is a real constant. This way, the NLS equation (1.1) becomes

$$\begin{cases} p_t + q_{xx} - [\gamma - \mu(p^2 + q^2)]q = 0, \\ q_t - p_{xx} + [\gamma - \mu(p^2 + q^2)]p = 0. \end{cases} \quad (3.21)$$

It can be transformed into a system of two trilinear equations if taking $p = g/f$ and $q = h/f$. A solution with constant amplitude is

$$u(x, t) = (c + id)e^{i\mu(c^2 + d^2)t}, \quad (3.22)$$

where c and d are arbitrary real constants. Another special solution with $p = c$ (a constant) is

$$u(x, t) = \left[c + id \tan \left(\frac{\sqrt{-2\mu d}}{2} x + cd\mu t \right) \right] e^{i\mu(c^2 - d^2)t}, \quad (3.23)$$

where c and d are arbitrary real constants.

If we search for solutions with $q = 0$, then $p = p(x)$ and so, the transformed NLS equation (3.21) reduces to

$$p_{xx} - \gamma p + \mu p^3 = 0.$$

The other case with $p = 0$ can be transformed into this case by using the auto-Bäcklund transformation $u \mapsto iu$. The first ansatz with $\alpha = 0$ provides solutions for the above case. For example, the class of solutions (3.12) with $\alpha = 0$ leads to

$$u(x, t) = \frac{k}{2} \sqrt{-\frac{2e^{kx} - a_0}{\mu e^{kx} + a_0}} e^{-\frac{k^2}{2}t}. \quad (3.24)$$

This further gives the solutions with the tanh- and coth-function amplitude [21]:

$$u(x, t) = \frac{k}{2} \sqrt{-\frac{2}{\mu}} \tanh \left(\frac{k}{2} x \right) e^{-\frac{k^2}{2}t}, \quad u(x, t) = \frac{k}{2} \sqrt{-\frac{2}{\mu}} \coth \left(\frac{k}{2} x \right) e^{-\frac{k^2}{2}t},$$

which correspond to $a_0 = 1$ and $a_0 = -1$, respectively.

A special solution of the system (3.21) with non-constant p and q is given by

$$p(x, t) = \frac{ae^{k(x-2xt)} \cos \alpha x}{b + ce^{2k(x-2xt)}}, \quad q(x, t) = \frac{ae^{k(x-2xt)} \sin \alpha x}{b + ce^{2k(x-2xt)}}, \quad (3.25)$$

where $\mu a^2 = 8k^2 bc$ and $\gamma = -\alpha^2 + k^2$ in (3.21), and thus, a particular solution of the NLS equation (1.1) reads

$$u(x, t) = [p(x, t) + iq(x, t)]e^{i(-\alpha^2 + k^2)t}, \quad (3.26)$$

which includes two solutions $\tilde{u}(x, t) = u(ix, -t)$ with $u(x, t)$ defined by (1.2) and (1.3). Other two solutions of the NLS equation (1.1) within this ansatz are

$$u(x, t) = \sqrt{\frac{2}{\mu}} \left[\frac{2ab^2 \cosh(2a^2 bct) + 2iabc \sinh(2a^2 bct)}{2 \cosh(2a^2 bct) \pm \sqrt{2}c \cos(\sqrt{2}abx)} - a \right] e^{2ia^2 t}, \quad c = \sqrt{2 - b^2}, \quad (3.27)$$

where a and $b \leq \sqrt{2}$ are arbitrary real constants. Those solutions were generated by the inverse scattering transform in [9]. Taking a limiting reduction of $b \rightarrow 0$, the solution (3.27) with the minus sign gives an interesting solution [10,28]:

$$u(x, t) = \sqrt{\frac{2}{\mu}} \frac{3a - 16a^5 t^2 - 4a^3 x^2 + 16ia^3 t}{1 + 16a^4 t^2 + 4a^2 x^2} e^{2ia^2 t},$$

a special case of which was also analyzed by using the Adomian decomposition method [20].

3.3. Third ansatz

We make the third ansatz

$$u(x, t) = [p(x, t) + iq(x, t)]e^{i\alpha x}, \quad (3.28)$$

where p and q are two real-valued functions, and α is a real constant. This way, the NLS equation (1.1) becomes

$$\begin{cases} p_t + 2\alpha p_x + q_{xx} - [\alpha^2 - \mu(p^2 + q^2)]q = 0, \\ q_t + 2\alpha q_x - p_{xx} + [\alpha^2 - \mu(p^2 + q^2)]p = 0. \end{cases} \quad (3.29)$$

It also can be transformed into a system of two trilinear equations if taking $p = g/f$ and $q = h/f$. A solution with constant amplitude is

$$u(x, t) = (c + id)e^{i\sqrt{\mu(c^2 + d^2)}x}, \quad (3.30)$$

where c and d are arbitrary real constants.

If we search for solutions with $q = 0$, then the transformed NLS equation (3.29) reduces to

$$p_t + 2\alpha p_x = 0, \quad p_{xx} - \alpha^2 p + \mu p^3 = 0. \quad (3.31)$$

The other case with $p = 0$ can be put into this case by using the auto-Bäcklund transformation $u \mapsto iu$. A special solution of the NLS equation (1.1) through (3.31) is given by

$$u = p(\xi)e^{i\alpha x}, \quad \xi = k(x - 2\alpha t),$$

if the function p solves

$$k^2 p_{\xi\xi} - \alpha^2 p + \mu p^3 = 0,$$

where k and α are real constants. The first ansatz with $\gamma = 0$ provides solutions for this case.

The following several special solutions of the system (3.29) with non-constant p and q can be obtained immediately, based on our analysis within the first ansatz:

$$p = k \cos[(\mu k^2 - \alpha^2)t], \quad q = k \sin[(\mu k^2 - \alpha^2)t], \quad (3.32)$$

$$p = k \sqrt{-\frac{2}{\mu} f(\xi)} \cos[(\alpha^2 + k^2)t], \quad q = -k \sqrt{-\frac{2}{\mu} f(\xi)} \sin[(\alpha^2 + k^2)t], \quad f(\xi) = \sec \xi \quad \text{or} \quad \csc \xi, \quad (3.33)$$

$$p = k \sqrt{-\frac{2}{\mu} g(\xi)} \cos[(\alpha^2 - 2k^2)t], \quad q = -k \sqrt{-\frac{2}{\mu} g(\xi)} \sin[(\alpha^2 - 2k^2)t], \quad g(\xi) = \tan \xi \quad \text{or} \quad -\cot(\xi), \quad (3.34)$$

and

$$p = \frac{8ck^2 e^\xi}{8k^2 + \mu c^2 e^{2\xi}} \cos[(\alpha^2 - k^2)t], \quad q = -\frac{8ck^2 e^\xi}{8k^2 + \mu c^2 e^{2\xi}} \sin[(\alpha^2 - k^2)t], \quad (3.35)$$

where $\xi = k(x - 2\alpha t)$, and k , α and c are arbitrary real constants. Another interesting solution of the NLS equation (1.1) in the form (3.28) with non-constant p and q is given by

$$u(x, t) = (p + iq)e^{izx}, \quad p = \sqrt{\frac{\alpha^2 - 2k^2}{\mu}}, \quad q = k\sqrt{-\frac{2}{\mu}}\tan(kx - d(t)), \quad (3.36)$$

where $d(t) = [2k\alpha + k\sqrt{2(2k^2 - \alpha^2)}]t$, and k and α are arbitrary real constants satisfying $2k^2 \geq \alpha^2$.

4. Concluding remarks

We have analyzed a five-dimensional symmetry algebra and three ansätze of transformations for the NLS equation. Five one-parameter solution groups are explicitly presented, and various exact solutions with constant, trigonometric function type, exponential function type and rational function amplitude are computed in detail, covering many exact solutions generated in the literature. Our solution analysis also provides clear information about a bifurcation phenomenon between the self-focusing and the de-focusing NLS equations. There are several solutions, like (3.6), (3.14) and (3.26), which works for both the self-focusing case and the de-focusing case. Only a few solutions, like (3.27) and (3.30), are valid for the self-focusing case, and most of the presented solutions are valid for the de-focusing case.

We remark that the first ansatz can be put into the second ansatz [or the third ansatz], upon absorbing the phase factor e^{izx} [or e^{iyt}] into the amplitude function $p + iq$ in (3.20) [or (3.28)]. Therefore, all solutions in the first ansatz also provide solutions within the second ansatz and the third ansatz. But the later two ansätze are more general than the first one. Further applications of the three ansätze and combinations of the three ansätze with the solution groups in (2.6) can engender other exact solutions to the NLS equation (1.1). The three ansätze can also be used to construct exact solutions to other nonlinear equations in mathematical physics, for example, the Davey–Stewartson equation [29] and the generalized NLS equations [30,31].

As discussed for many typical integrable equations (see, for example, the KdV equation [32,33], the Toda lattice [34,35], the Boussinesq equation [36,37], and the 2D Toda lattice [38]), we will adopt the determinant techniques to construct different kinds of exact solutions to the NLS equation in a future publication. All the working methods of solutions, including the inverse scattering transform [1], Darboux transformation [39], the algebro-geometric method [40] and the Hirota method [41] and the determinant techniques, help us exploit the diversity of exact solutions of the NLS equation.

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