



Linear superposition principle applying to Hirota bilinear equations

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ABSTRACT

A linear superposition principle of exponential traveling waves is analyzed for Hirota bilinear equations, with an aim to construct a specific sub-class of N -soliton solutions formed by linear combinations of exponential traveling waves. Applications are made for the $3 + 1$ dimensional KP, Jimbo–Miwa and BKP equations, thereby presenting their particular N -wave solutions. An opposite question is also raised and discussed about generating Hirota bilinear equations possessing the indicated N -wave solutions, and a few illustrative examples are presented, together with an algorithm using weights.

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1. Introduction

It is significantly important in mathematical physics to search for exact solutions to nonlinear differential equations. Exact solutions play a vital role in understanding various qualitative and quantitative features of nonlinear phenomena. There are diverse classes of interesting exact solutions, such as traveling wave solutions and soliton solutions, but it often needs specific mathematical techniques to construct exact solutions due to the nonlinearity present in dynamics (see, e.g., [1,2]).

Among the existing theories, Hirota's bilinear technique [3] provides a direct powerful approach to nonlinear integrable equations [4], and it is widely used in constructing N -soliton solutions [5] and even for integrable couplings by perturbation [6]. The existence of N -soliton solutions often implies the integrability [7] of the considered differential equations. Interactions between solitons are elastic and nonlinear, but unfortunately, the linear superposition principle does not hold for solitons any more.

However, bilinear equations are the nearest neighbors to linear equations, and expected to have some features similar to those of linear equations. In this paper, we would like to explore a key feature of the linear superposition principle that linear equations possess, for Hirota bilinear equations, while aiming to construct a specific sub-class of N -soliton solutions formed by linear combinations of exponential traveling waves.

More specifically, we will prove that a linear superposition principle can apply to exponential traveling waves of Hirota bilinear equations. Applications will be made to show that the presented linear superposition principle is helpful in generating N -wave solutions to soliton equations, particularly those in higher dimensions. Illustrative examples include the $3 + 1$ dimensional KP, Jimbo–Miwa and BKP equations [8–12]. An opposite procedure is also proposed for generating Hirota bilinear equations possessing N -wave solutions of linear combinations of exponential waves, along with an algorithm using weights. A few new and general such Hirota bilinear equations are therefore computed.

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2. Linear superposition principle

We begin with a Hirota bilinear equation

$$P(D_{x_1}, D_{x_2}, \dots, D_{x_M})f \cdot f = 0, \quad (2.1)$$

where P is a polynomial in the indicated variables satisfying

$$P(\underbrace{0, 0, \dots, 0}_M) = 0, \quad (2.2)$$

and D_{x_i} , $1 \leq i \leq M$, are Hirota's differential operators defined by

$$D_y^p f(y) \cdot g(y) = (\partial_y - \partial_{y'})^p f(y)g(y') \Big|_{y'=y} = \partial_{y'}^p f(y+y')g(y-y') \Big|_{y'=0}, \quad p \geq 1.$$

Various nonlinear equations of mathematical physics are written as Hirota forms through dependent variable transformations [3,5].

Let us introduce N wave variables

$$\eta_i = k_{1,i}x_1 + k_{2,i}x_2 + \dots + k_{M,i}x_M, \quad 1 \leq i \leq N, \quad (2.3)$$

and N exponential wave functions

$$f_i = e^{\eta_i} = e^{k_{1,i}x_1 + k_{2,i}x_2 + \dots + k_{M,i}x_M}, \quad 1 \leq i \leq N, \quad (2.4)$$

where the $k_{i,j}$'s are constants. Observing that we have a bilinear identity [3]:

$$P(D_{x_1}, D_{x_2}, \dots, D_{x_M}) e^{\eta_i} \cdot e^{\eta_j} = P(k_{1,i} - k_{1,j}, k_{2,i} - k_{2,j}, \dots, k_{M,i} - k_{M,j}) e^{\eta_i + \eta_j}, \quad (2.5)$$

it follows immediately from (2.2) that all exponential wave functions f_i , $1 \leq i \leq N$, solve the Hirota bilinear equation (2.1).

Now consider an N -wave testing function

$$f = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \dots + \varepsilon_N f_N = \varepsilon_1 e^{\eta_1} + \varepsilon_2 e^{\eta_2} + \dots + \varepsilon_N e^{\eta_N}, \quad (2.6)$$

where ε_i , $1 \leq i \leq N$, are arbitrary constants. This is a general linear combination of N exponential traveling wave solutions. Naturally, we would like to ask if it will still present a solution to the Hirota bilinear equation (2.1) as each f_i does. The answer is positive. We will show that a linear superposition principle of those exponential waves will apply to Hirota bilinear equations, under some additional condition on the exponential waves and possibly on the polynomial P as well.

Following (2.5), we can compute that

$$\begin{aligned} P(D_{x_1}, D_{x_2}, \dots, D_{x_M})f \cdot f &= \sum_{i,j=1}^N \varepsilon_i \varepsilon_j P(D_{x_1}, D_{x_2}, \dots, D_{x_M}) e^{\eta_i} \cdot e^{\eta_j} \\ &= \sum_{i,j=1}^N \varepsilon_i \varepsilon_j P(k_{1,i} - k_{1,j}, k_{2,i} - k_{2,j}, \dots, k_{M,i} - k_{M,j}) e^{\eta_i + \eta_j}. \end{aligned} \quad (2.7)$$

This bilinear property will play a prominent role in establishing the linear superposition principle for the exponential waves e^{η_i} , $1 \leq i \leq N$, and it is also a key tool in constructing quasi-periodic wave solutions (see, e.g., [13–15]).

It now follows directly from (2.7) that any linear combination of the N exponential wave solutions e^{η_i} , $1 \leq i \leq N$, solves the Hirota bilinear equation (2.1) if the following condition

$$P(k_{1,i} - k_{1,j}, k_{2,i} - k_{2,j}, \dots, k_{M,i} - k_{M,j}) = 0, \quad 1 \leq i \neq j \leq N, \quad (2.8)$$

is satisfied. In this condition (2.8), we excluded the case of $i = j$, since that case is just a consequence of (2.2). The condition (2.8) gives us a system of nonlinear algebraic equations on the wave related numbers $k_{i,j}$'s, when the polynomial P is fixed. We will see that there is a better possibility of existence of solutions for the variables $k_{i,j}$'s in higher dimensional cases, because there are more variables to solve for in the resulting system of algebraic equations.

We conclude the above analysis in the following theorem.

Theorem 2.1 (Linear Superposition Principle). *Let $P(x_1, x_2, \dots, x_M)$ be a multivariate polynomial satisfying (2.2) and the wave variables η_i , $1 \leq i \leq N$, be defined by (2.3). Then any linear combination of the exponential waves e^{η_i} , $1 \leq i \leq N$, solves the Hirota bilinear equation (2.1) if the condition (2.8) is satisfied.*

This shows a linear superposition principle of exponential wave solutions that applies to Hirota bilinear equations, and paves a way of constructing N -wave solutions from linear combinations of exponential waves within the Hirota bilinear formulation. The system (2.8) is a key condition we need to handle. Once we get a solution of the wave related numbers $k_{i,j}$'s by solving the system (2.8), we can present an N -wave solution, formed by (2.6), to the considered nonlinear equation.

Taking one of the wave variables η_i , $1 \leq i \leq N$, to be a constant, for example, taking

$$\eta_{i_0} = \varepsilon_{i_0}, \quad \text{i.e., } k_{i,i_0} = 0, \quad 1 \leq i \leq M, \quad (2.9)$$

where $1 \leq i_0 \leq N$ is fixed, the N -wave solution condition (2.8) subsequently requires all other wave related numbers to satisfy the dispersion relation

$$P(k_{1,i}, k_{2,i}, \dots, k_{M,i}) = 0, \quad 1 \leq i \leq N, \quad i \neq i_0. \quad (2.10)$$

This corresponds to a specific case of N -soliton solutions by the Hirota perturbation technique [3], truncated at the second-order perturbation term. But generally, we want to emphasize that it is not necessary to satisfy the dispersion relation.

3. Applications to soliton equations

Let us shed light on the linear superposition principle in Theorem 2.1 by three application examples of constructing N -wave solutions.

3.1. 3 + 1 dimensional KP equation

The first example is the 3 + 1 dimensional KP equation [8,9]:

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} + 3u_{zz} = 0. \quad (3.1)$$

Through the dependent variable transformation $u = -2(\ln f)_{xx}$, the 3 + 1 dimensional KP equation (3.1) is written as

$$(D_x^4 + D_t D_x + 3D_y^2 + 3D_z^2)f \cdot f = 0, \quad (3.2)$$

which is equivalent to

$$f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2 + f_{tx}f - f_{tx}f_x + 3(f_{yy}f - f_y^2 + f_{zz}f - f_z^2) = 0.$$

Assume that the N wave variables (2.3) are determined by

$$\eta_i = k_i x + l_i y + m_i z + \omega_i t, \quad 1 \leq i \leq N, \quad (3.3)$$

and then the N -wave solution condition (2.8) becomes

$$\begin{aligned} k_i^4 - 4k_i^3 k_j + 6k_i^2 k_j^2 - 4k_i k_j^3 + k_j^4 + k_i \omega_i - k_i \omega_j - k_j \omega_i + k_j \omega_j + 3l_i^2 - 6l_i l_j + 3l_j^2 \\ + 3m_i^2 - 6m_i m_j + 3m_j^2 = 0, \quad 1 \leq i \neq j \leq N. \end{aligned} \quad (3.4)$$

By inspection, a solution to this equation is

$$l_i = a k_i^2, \quad m_i = b k_i^2, \quad \omega_i = -4k_i^3, \quad 1 \leq i \leq N, \quad (3.5)$$

where $a^2 + b^2 = 1$. Therefore by the linear superposition principle in Theorem 2.1, the 3 + 1 dimensional KP equation (3.1) has the following N -wave solution

$$u = -2(\ln f)_{xx}, \quad f = \sum_{i=1}^N \varepsilon_i f_i = \sum_{i=1}^N \varepsilon_i e^{k_i x + a k_i^2 y + b k_i^2 z - 4k_i^3 t}, \quad (3.6)$$

where $a^2 + b^2 = 1$, and the k_i 's and ε_i 's are arbitrary constants. In this solution f , each exponential wave f_i satisfies the corresponding nonlinear dispersion relation.

3.2. 3 + 1 dimensional Jimbo–Miwa equation

The second example is the 3 + 1 dimensional Jimbo–Miwa equation [10]:

$$u_{xxx} + 3(u_x u_y)_x + 2u_{yt} - 3u_{zz} = 0. \quad (3.7)$$

Through the dependent variable transformation $u = 2(\ln f)_x$, the 3 + 1 dimensional Jimbo–Miwa equation (3.7) is written as

$$(D_x^3 D_y + 2D_t D_y - 3D_z^2)f \cdot f = 0, \quad (3.8)$$

which equivalently reads

$$(f_{xxx} + 2f_{ty} - 3f_{zz})f - 3f_{xy}f_x + 3f_{xy}f_{xx} - f_{y}f_{xxx} - 2f_{t}f_y + 3f_z^2 = 0.$$

Assume that the N wave variables (2.3) are determined by (3.3), and then the N -wave solution condition (2.8) becomes

$$\begin{aligned} k_i^3 l_i - k_i^3 l_j - 3k_i^2 k_j l_i + 3k_i^2 k_j l_j + 3k_i k_j^2 l_i - 3k_i k_j^2 l_j - k_j^3 l_i + k_j^3 l_j \\ + 2\omega_i l_i - 2\omega_i l_j - 2\omega_j l_i + 2\omega_j l_j - 3m_i^2 + 6m_i m_j - 3m_j^2 = 0, \quad 1 \leq i \neq j \leq N. \end{aligned} \quad (3.9)$$

Similarly by inspection, a solution to this equation is

$$l_i = -a^2 k_i, \quad m_i = a k_i^2, \quad \omega_i = -2k_i^3, \quad 1 \leq i \leq N, \quad (3.10)$$

where a is an arbitrary constant. Therefore by the linear superposition principle in Theorem 2.1, the $3 + 1$ dimensional Jimbo–Miwa equation (3.7) has the following N -wave solution

$$u = 2(\ln f)_x, \quad f = \sum_{i=1}^N \varepsilon_i f_i = \sum_{i=1}^N \varepsilon_i e^{k_i x - a^2 k_i y + a k_i^2 z - 2k_i^3 t}, \quad (3.11)$$

where a , and the k_i 's and ε_i 's are arbitrary constants. In this solution f , each exponential wave f_i satisfies the corresponding nonlinear dispersion relation.

3.3. $3 + 1$ dimensional BKP equation

As the third example, let us form and consider a $3 + 1$ dimensional generalization of the BKP equation:

$$u_{zt} - u_{xxx} - 3(u_x u_y)_x + 3u_{xx} = 0. \quad (3.12)$$

If $z = y$, this $3 + 1$ dimensional BKP equation reduces to the BKP equation [11,12]:

$$u_{yt} - u_{xxx} - 3(u_x u_y)_x + 3u_{xx} = 0. \quad (3.13)$$

Through the dependent variable transformation $u = 2(\ln f)_x$, the $3 + 1$ dimensional BKP equation (3.12) is written as

$$(D_t D_z - D_x^3 D_y + 3D_x^2) f \cdot f = 0, \quad (3.14)$$

which is equivalent to

$$(f_{tz} - f_{xxx} + 3f_{xx})f - f_t f_z + f_{xxx} f_y + 3f_{xy} f_x - 3f_{xx} f_y - 3f_x^2 = 0.$$

Assume that the N wave variables (2.3) are determined by (3.3), and then the N -wave solution condition (2.8) becomes

$$\begin{aligned} \omega_i m_i - \omega_i m_j - \omega_j m_i + \omega_j m_j - k_i^3 l_i + k_i^3 l_j + 3k_i^2 k_j l_i - 3k_i^2 k_j l_j \\ - 3k_i k_j^2 l_i + 3k_i k_j^2 l_j + k_j^3 l_i - k_j^3 l_j + 3k_i^2 - 6k_i k_j + 3k_j^2 = 0, \quad 1 \leq i \neq j \leq N. \end{aligned} \quad (3.15)$$

Similarly by inspection, a solution to this equation is

$$l_i = k_i^{-1}, \quad m_i = a k_i^{-1}, \quad \omega_i = \frac{1}{a} k_i^3, \quad 1 \leq i \leq N, \quad (3.16)$$

where a is an arbitrary non-zero constant. It then follows from the linear superposition principle in Theorem 2.1 that the $3 + 1$ dimensional BKP equation (3.12) has the following N -wave solution

$$u = 2(\ln f)_x, \quad f = \sum_{i=1}^N \varepsilon_i f_i = \sum_{i=1}^N \varepsilon_i e^{k_i x + k_i^{-1} y + a k_i^{-1} z + (1/a) k_i^3 t}, \quad (3.17)$$

where a and the k_i 's are arbitrary non-zero constants and the ε_i 's are arbitrary constants. However, none of the N exponential waves f_i , $1 \leq i \leq N$, in the solution f satisfies the corresponding nonlinear dispersion relation. The case $a = 1$ produces $z = y$, and so it gives an N -wave solution to the BKP equation (3.13):

$$u = 2(\ln f)_x, \quad f = \sum_{i=1}^N \varepsilon_i e^{k_i x + k_i^{-1} y + k_i^3 t}. \quad (3.18)$$

4. An opposite question

We would like to propose an opposite procedure for conversely constructing Hirota bilinear equations that possess N -wave solutions formed by linear combinations of exponential waves. This is an opposite question on applying the linear superposition principle in Theorem 2.1.

We first find a multivariate polynomial $P(x_1, x_2, \dots, x_M)$ with no constant term such that

$$P(k_{1,1} - k_{2,1}, k_{1,2} - k_{2,2}, \dots, k_{1,M} - k_{2,M}) = 0, \quad (4.1)$$

for two sets of parameters $k_{i,1}, k_{i,2}, \dots, k_{i,M}, i = 1, 2$, each of which would better contain at least one free parameter. Then formulate a Hirota bilinear equation through (2.1) using the polynomial P . Theorem 2.1 tells that the resulting Hirota bilinear equation possesses multiple wave solutions of linear combinations of exponential traveling waves. Such a multivariate polynomial P can be normally found by balancing the involved free parameters in (4.1), and often upon assuming that two sets of parameters satisfy the dispersion relation

$$P(k_{i,1}, k_{i,2}, \dots, k_{i,M}) = 0, \quad i = 1, 2. \quad (4.2)$$

This construction procedure also brings us an interesting problem: how do we construct multivariate polynomials which satisfy the property (4.1) guaranteeing the linear superposition principle for exponential wave solutions, when (4.2) holds?

4.1. Examples of equations expressed in u

One example is the following polynomial

$$P(x, y, z, t) = x^3y + tx + ty - z^2, \quad (4.3)$$

whose corresponding condition (4.1) is given by

$$\begin{aligned} P(k_1 - k_2, l_1 - l_2, m_1 - m_2, \omega_1 - \omega_2) &= k_1^3l_1 - k_1^3l_2 - 3k_1^2k_2l_1 + 3k_1^2k_2l_2 + 3k_1k_2^2l_1 - 3k_1k_2^2l_2 - k_2^3l_1 + k_2^3l_2 \\ &\quad + \omega_1l_1 - \omega_1l_2 - \omega_2l_1 + \omega_2l_2 + \omega_1k_1 - \omega_1k_2 - \omega_2k_1 + \omega_2k_2 \\ &\quad - m_1^2 + 2m_1m_2 - m_2^2 = 0. \end{aligned}$$

Obviously, the corresponding Hirota bilinear equation reads

$$(D_x^3D_y + D_tD_x + D_tD_y - D_z^2)f \cdot f = 0, \quad (4.4)$$

that is,

$$(f_{xxxxy} + f_{tx} + f_{ty} - f_{zz})f - 3f_{xy}f_{xx} - 3f_{xy}f_{xx} - f_yf_{xxx} - f_tf_x - f_tf_y + f_z^2 = 0.$$

Under the dependent variable transformation $u = 2(\ln f)_x$, this equation is transformed into

$$u_{xxxxy} + 3(u_xu_y)_x + u_{tx} + u_{ty} - u_{zz} = 0. \quad (4.5)$$

Following the linear superposition principle of exponential waves, we have its N -wave solution

$$u = 2(\ln f)_x, \quad f = \sum_{i=1}^N \varepsilon_i f_i = \sum_{i=1}^N \varepsilon_i e^{k_i x - (1/3)a^2 k_i y + a k_i^2 z + 4[a^2/(3-a^2)]k_i^3 t}, \quad (4.6)$$

where $a \neq \pm\sqrt{3}$, and the ε_i 's and k_i 's are arbitrary parameters. Each exponential wave f_i in the solution f satisfies the corresponding nonlinear dispersion relation.

The other example is the following polynomial

$$P(x, y, z, t) = ty - x^3y + 3xz, \quad (4.7)$$

whose corresponding condition (4.1) is given by

$$\begin{aligned} P(k_1 - k_2, l_1 - l_2, m_1 - m_2, \omega_1 - \omega_2) &= \omega_1l_1 - \omega_1l_2 - \omega_2l_1 + \omega_2l_2 - k_1^3l_1 + k_1^3l_2 + 3k_1^2k_2l_1 - 3k_1^2k_2l_2 \\ &\quad - 3k_1k_2^2l_1 + 3k_1k_2^2l_2 + k_2^3l_1 - k_2^3l_2 + 3k_1m_1 - 3k_1m_2 \\ &\quad - 3k_2m_1 + 3k_2m_2 = 0. \end{aligned}$$

Obviously, the corresponding Hirota bilinear equation reads

$$(D_tD_y - D_x^3D_y + 3D_xD_z)f \cdot f = 0, \quad (4.8)$$

that is,

$$(f_{ty} - f_{xxxxy} + 3f_{xz})f - f_tf_y + f_{xx}f_y + 3f_{xy}f_x - 3f_{xx}f_y - 3f_xf_z = 0.$$

Under the dependent variable transformation $u = 2(\ln f)_x$, this equation is transformed into

$$u_{yt} - u_{xxxxy} - 3(u_xu_y)_x + 3u_{xz} = 0. \quad (4.9)$$

If $z = x$, the equation (4.9) reduces to the BKP equation (3.13). Following the linear superposition principle of exponential waves, we have an N -wave solution for (4.9):

$$u = 2(\ln f)_x, \quad f = \sum_{i=1}^N \varepsilon_i f_i = \sum_{i=1}^N \varepsilon_i e^{k_i x + (1/a)k_i^{-1}y + a k_i z + k_i^3 t}, \quad (4.10)$$

where $a \neq 0$, and the ε_i 's and k_i 's are arbitrary parameters. However, each exponential wave f_i in the solution f does not satisfy the corresponding nonlinear dispersion relation. The case of $a = 1$ produces $z = x$ and gives the same N -wave solution to the BKP equation (3.13) as in (3.18).

4.2. Examples of equations expressed in f

An algorithm can be given to use the concept of weights, to compute examples of Hirota bilinear equations that possess the linear superposition principle of exponential waves. Let us first define the weights of independent variables: $(w(x_1), w(x_2), \dots, w(x_M))$, where each weight $w(x_i)$ is an integer, and then form a homogeneous polynomial $P(x_1, x_2, \dots, x_M)$ of some weight to check if it will satisfy the condition (4.1). A nice idea to start our checking is to assume that the wave variables η_i 's involve arbitrary constants. This way, we can compare powers of those arbitrary constants in (4.1) to obtain algebraic equations on other constants and/or coefficients to solve. The following are a few examples which apply this algorithm using weights.

4.2.1. Examples with N -waves satisfying the dispersion relation

Example 1. *Weights* $(w(x), w(y), w(z), w(t)) = (1, 2, 3, 3)$:

Let us first introduce the weights of independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, 2, 3, 3). \quad (4.11)$$

Then, a general homogeneous polynomial of weight 4 reads

$$P = c_1 x^4 + c_2 x^2 y + c_3 x z + c_4 x t + c_5 y^2. \quad (4.12)$$

Assume that the wave variables are

$$\eta_i = k_i x + b_1 k_i^2 y + b_2 k_i^3 z + b_3 k_i^3 t, \quad 1 \leq i \leq N, \quad (4.13)$$

where k_i , $1 \leq i \leq N$, are arbitrary constants, but b_1 , b_2 and b_3 are constants to be determined.

This way, a direct computation tells that we must have $c_2 = 0$ to keep the non-triviality $b_1 b_2 b_3 \neq 0$, and b_1 , b_2 and b_3 need to satisfy

$$c_5 b_1^2 - 3c_1 = 0, \quad c_3 b_2 + c_4 b_3 + 4c_1 = 0. \quad (4.14)$$

It follows now that the corresponding Hirota bilinear equation reads

$$(c_1 D_x^4 + c_3 D_x D_z + c_4 D_x D_t + c_5 D_y^2) f \cdot f = 0, \quad (4.15)$$

and it possesses an N -wave solution

$$f = \sum_{i=1}^N \varepsilon_i f_i = \sum_{i=1}^N \varepsilon_i e^{k_i x + b_1 k_i^2 y + b_2 k_i^3 z + b_3 k_i^3 t}, \quad (4.16)$$

where the ε_i 's and k_i 's are arbitrary, but b_1 , b_2 and b_3 satisfy (4.14).

The first equation in (4.14) requires $c_1 c_5 > 0$, to guarantee a non-zero real solution for b_1 . The solution of the second equation of b_2 and b_3 in (4.14) contains an arbitrary constant if $c_3 c_4 \neq 0$. We point out that when both those conditions, $c_1 c_5 > 0$ and $c_3 c_4 \neq 0$, are satisfied, the resulting equation (4.15) actually can be transformed into the 2 + 1 KP equation.

Example 2. *Weights* $(w(x), w(y), w(z), w(t)) = (1, 2, 2, 3)$:

Let us second introduce the weights of independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, 2, 2, 3). \quad (4.17)$$

Then, a general homogeneous polynomial of weight 4 reads

$$P = c_1 x^4 + c_2 x^2 y + c_3 x^2 z + c_4 x t + c_5 y^2 + c_6 y z + c_7 z^2. \quad (4.18)$$

Assume that the wave variables are

$$\eta_i = k_i x + b_1 k_i^2 y + b_2 k_i^2 z + b_3 k_i^3 t, \quad 1 \leq i \leq N, \quad (4.19)$$

where k_i , $1 \leq i \leq N$, are arbitrary constants, but b_1 , b_2 and b_3 are constants to be determined.

This way, a direct computation tells that the corresponding Hirota bilinear equation $P(D_x, D_y, D_z, D_t) f \cdot f = 0$ possesses an N -wave solution

$$f = \sum_{i=1}^N \varepsilon_i f_i = \sum_{i=1}^N \varepsilon_i e^{k_i x + b_1 k_i^2 y + b_2 k_i^2 z + b_3 k_i^3 t}, \quad (4.20)$$

where the ε_i 's and k_i 's are arbitrary, but b_1 , b_2 and b_3 satisfy

$$\begin{cases} c_4 b_3 + 4c_1 = 0, \\ c_2 b_1 + c_3 b_2 = 0, \\ c_5 b_1^2 + c_7 b_2^2 + c_6 b_1 b_2 = 3c_1. \end{cases} \quad (4.21)$$

The first equation above determines b_3 uniquely if $c_4 \neq 0$ or does not present any condition on b_3 if $c_1 = c_4 = 0$. There are two cases to determine b_1 and b_2 , which are depicted as follows.

(a) The case of $c_2 c_3 \neq 0$:

Then we need

$$c_1(c_2^2 c_7 - c_2 c_3 c_6 + c_3^2 c_5) > 0, \quad (4.22)$$

to have non-zero b_1 and b_2 . Under this condition, the solutions for b_1 and b_2 are given by

$$(b_1, b_2) = \left(\frac{c_3 \sqrt{3c_1 d}}{d}, -\frac{c_2 \sqrt{3c_1 d}}{d} \right), \quad (b_1, b_2) = \left(-\frac{c_3 \sqrt{3c_1 d}}{d}, \frac{c_2 \sqrt{3c_1 d}}{d} \right), \quad (4.23)$$

where $d = c_2^2 c_7 - c_2 c_3 c_6 + c_3^2 c_5$.

(b) The case of $c_2 = c_3 = 0$:

Then if $c_5 \neq 0$, b_2 needs to satisfy

$$(c_6^2 - 4c_5 c_7) b_2^2 + 12c_1 c_5 \geq 0, \quad (4.24)$$

to have a real b_1 , and b_1 is determined in terms of b_2 by the third equation in (4.21). The value of b_2 may have lots of chooses. For example, if

$$c_1 c_5 \geq 0, \quad c_6^2 - 4c_5 c_7 \geq 0,$$

then b_2 is arbitrary; if

$$c_1 c_5 \geq 0, \quad c_6^2 - 4c_5 c_7 < 0,$$

then b_2 must be

$$-\frac{2\sqrt{3c_1 c_5 (4c_5 c_7 - c_6^2)}}{c_6^2 - 4c_5 c_7} \leq b_2 \leq \frac{2\sqrt{3c_1 c_5 (4c_5 c_7 - c_6^2)}}{c_6^2 - 4c_5 c_7}.$$

Symmetrically, if $c_7 \neq 0$, b_2 needs to satisfy

$$(c_6^2 - 4c_5 c_7) b_1^2 + 12c_1 c_7 \geq 0, \quad (4.25)$$

to have a real b_2 , and b_2 is determined in terms of b_1 by the third equation in (4.21). The value of b_1 may have lots of chooses. For example, if

$$c_1 c_7 \geq 0, \quad c_6^2 - 4c_5 c_7 \geq 0,$$

then b_1 is arbitrary; if

$$c_1 c_7 \leq 0, \quad c_6^2 - 4c_5 c_7 > 0,$$

then b_1 must be

$$b_1 \leq -\frac{2\sqrt{3c_1 c_7 (4c_5 c_7 - c_6^2)}}{c_6^2 - 4c_5 c_7} \quad \text{or} \quad b_1 \geq \frac{2\sqrt{3c_1 c_7 (4c_5 c_7 - c_6^2)}}{c_6^2 - 4c_5 c_7}.$$

We point out that in this case, if $c_1 c_5 c_6 c_7 \neq 0$, then there will definitely be non-zero b_1 and b_2 . This is because we have

$$c_7 b_2^2 = 3c_1, \quad (c_6^2 - 4c_5 c_7) b_2^2 + 12c_1 c_5 = 0,$$

which leads to $c_6 = 0$, if, for example, b_1 has to be zero.

Example 3. Weights $(w(x), w(y), w(z), w(t)) = (1, 1, 2, 3)$:

Let us third introduce the weights of independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, 1, 2, 3). \quad (4.26)$$

Then, a general homogeneous polynomial of weight 4 reads

$$P = c_1 x^4 + c_2 y^4 + c_3 x^3 y + c_4 x y^3 + c_5 x^2 y^2 + c_6 x^2 z + c_7 y^2 z + c_8 x t + c_9 y t + c_{10} z^2. \quad (4.27)$$

Assume that the wave variables are

$$\eta_i = k_i x + b_1 k_i y + b_2 k_i^2 z + b_3 k_i^3 t, \quad 1 \leq i \leq N, \quad (4.28)$$

where k_i , $1 \leq i \leq N$, are arbitrary constants, but b_1 , b_2 and b_3 are constants to be determined. In this example, the corresponding Hirota bilinear equation $P(D_x, D_y, D_z, D_t)f \cdot f = 0$ has an associated N -wave solutions

$$f = \sum_{i=1}^N \varepsilon_i f_i = \sum_{i=1}^N \varepsilon_i e^{k_i x + b_1 k_i y + b_2 k_i^2 z + b_3 k_i^3 t}. \quad (4.29)$$

There are four cases to determine the involved constants b_1 , b_2 and b_3 as follows.

(a) The case of $c_7 \neq 0$ and $c_{10} \neq 0$:

In this case, a direct computation yields

$$b_1^2 = -\frac{c_6}{c_7}, \quad (4.30)$$

and

$$b_3 = -\frac{4(c_2 b_1^4 + c_4 b_1^3 + c_5 b_1^2 + c_3 b_1 + c_1)}{c_9 b_1 + c_8}, \quad b_2^2 = -\frac{3b_3(c_9 b_1 + c_8)}{4c_{10}}, \quad (4.31)$$

where

$$c_9 b_1 + c_8 \neq 0, \quad c_2 b_1^4 + c_4 b_1^3 + c_5 b_1^2 + c_3 b_1 + c_1 \neq 0, \quad (4.32)$$

which actually give two conditions on the Hirota bilinear equation considered.

(b) The case of $c_7 = 0$ but $c_{10} \neq 0$:

In this case, we automatically have $c_6 = 0$, to keep the non-triviality $b_1 b_2 b_3 \neq 0$. A similar direct computation shows that b_1 needs to satisfy (4.32), and that b_2 and b_3 are defined by (4.31).

(c) The case of $c_7 = c_{10} = 0$ but $c_9 \neq 0$:

In this case, we automatically have $c_6 = 0$ and $c_8 \neq 0$, to keep the non-triviality $b_1 b_2 b_3 \neq 0$. A similar direct computation tells that

$$b_1 = -\frac{c_8}{c_9} \quad (4.33)$$

and b_2 is arbitrary, and that

$$c_2 c_8^4 - c_4 c_8^3 c_9 + c_5 c_8^2 c_9^2 - c_3 c_8 c_9^3 + c_1 c_9^4 = 0. \quad (4.34)$$

This condition on the Hirota bilinear equation considered is equivalent to

$$c_2 b_1^4 + c_4 b_1^3 + c_5 b_1^2 + c_3 b_1 + c_1 = 0. \quad (4.35)$$

(d) The case of $c_7 = c_9 = c_{10} = 0$:

In this case, we automatically have $c_6 = c_8 = 0$, to keep the non-triviality $b_1 b_2 b_3 \neq 0$. This time, b_1 must satisfy (4.35), but b_2 can be arbitrary.

4.2.2. Examples with N -waves not satisfying the dispersion relation

Example 4. Weights $(w(x), w(y), w(z), w(t)) = (1, -1, -1, 3)$:

Let us now introduce the weights of independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, -1, -1, 3). \quad (4.36)$$

Then, a homogeneous polynomial of weight 2 reads

$$P = c_1 t y + c_2 t z + c_3 x^2 + c_4 x^3 y + c_5 x^3 z + c_6 x^4 y^2 + c_7 x^4 z^2. \quad (4.37)$$

Assume that the wave variables are

$$\eta_i = k_i x + b_1 k_i^{-1} y + b_2 k_i^{-1} z + b_3 k_i^3 t, \quad 1 \leq i \leq N, \quad (4.38)$$

where k_i , $1 \leq i \leq N$, are arbitrary constants, but b_1 , b_2 and b_3 are constants to be determined.

Now, a similar direct computation tells that the corresponding Hirota bilinear equation

$$(c_1 D_t D_y + c_2 D_t D_z + c_3 D_x^2 + c_4 D_x^3 D_y + c_5 D_x^3 D_z + c_6 D_x^4 D_y^2 + c_7 D_x^4 D_z^2) f \cdot f = 0 \quad (4.39)$$

possesses an N -wave solution

$$f = \sum_{i=1}^N \varepsilon_i f_i = \sum_{i=1}^N \varepsilon_i e^{k_i x + b_1 k_i^{-1} y + b_2 k_i^{-1} z + b_3 k_i^3 t}, \quad (4.40)$$

where the ε_i 's and k_i 's are arbitrary, but b_1 , b_2 and b_3 satisfy

$$\begin{cases} c_3 + 3c_4 b_1 + 3c_5 b_2 = 0, \\ c_3 - 3c_1 b_1 b_3 - 3c_2 b_2 b_3 = 0, \\ c_6 b_1^2 + c_7 b_2^2 = 0. \end{cases} \quad (4.41)$$

The BKP equation (4.8) is just one special example with $c_2 = 1$, $c_3 = 3$, $c_4 = -1$, and the other coefficients being zero. An analysis on the existence of non-zero real b_1 and b_2 can be given similarly.

Example 5. *Weights* $(w(x), w(y), w(z), w(t)) = (1, -1, -2, 3)$:

Let us finally introduce the weights of independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, -1, -2, 3). \quad (4.42)$$

Then, a homogeneous polynomial of weight 2 reads

$$P = c_1 t y + c_2 x^3 y + c_3 x^4 y^2 + c_5 x^4 z + c_6 x^2. \quad (4.43)$$

Assume that the wave variables are

$$\eta_i = k_i x + b_1 k_i^{-1} y + b_2 k_i^{-2} z + b_3 k_i^3 t, \quad 1 \leq i \leq N, \quad (4.44)$$

where k_i , $1 \leq i \leq N$, are arbitrary constants, but b_1 , b_2 and b_3 are constants to be determined.

Now, a similar direct computation tells that the corresponding Hirota bilinear equation

$$(c_1 D_t D_y + c_2 D_x^3 D_y + c_3 D_x^4 D_y^2 + c_5 D_x^4 D_z + c_6 D_x^2) f \cdot f = 0 \quad (4.45)$$

possesses an N -wave solution

$$f = \sum_{i=1}^N \varepsilon_i f_i = \sum_{i=1}^N \varepsilon_i e^{k_i x + b_1 k_i^{-1} y + b_2 k_i^{-2} z + b_3 k_i^3 t}, \quad (4.46)$$

where the ε_i 's and k_i 's are arbitrary, but b_1 , b_2 and b_3 satisfy

$$\begin{cases} 3c_2 b_1 + c_6 = 0, \\ c_1 b_1 b_3 + c_2 b_1 = 0, \\ c_3 b_1^2 = c_5 b_2 = 0. \end{cases} \quad (4.47)$$

Then, taking $c_3 = c_5 = 0$ tells that the Hirota bilinear equation

$$(c_1 D_t D_y + c_2 D_x^3 D_y + c_6 D_x^2) f \cdot f = 0, \quad (4.48)$$

has the N -wave solution defined by (4.46) with $b_1 = -\frac{c_6}{3c_2}$ and $b_3 = -\frac{c_2}{c_1}$.

5. Concluding remarks

To summarize, we analyzed a specific sub-class of N -soliton solutions, formed by linear combinations of exponential traveling waves, for Hirota bilinear equations. The starting point is to solve a system of nonlinear algebraic equations for the wave related numbers, called the N -wave solution condition. The resulting system tells what Hirota bilinear equations the linear superposition principle of exponential waves will apply to. Higher dimensional Hirota bilinear equations have a better opportunity to satisfy the N -wave solution condition since there are more parameters to choose from. Applications were made for the 3 + 1 dimensional KP, Jimbo–Miwa and BKP equations, thereby presenting their particular exact multiple wave solutions. Moreover, an opposite procedure was proposed for conversely generating Hirota bilinear equations possessing multiple wave solutions formed by linear combinations of exponential waves, and a few such examples were computed.

Generally speaking, the linear superposition principle does not apply to nonlinear differential equations. But, it can hold for some special kind of wave solutions to Hirota bilinear equations, for example, for exponential waves as we explored. This also tells, to some extent, why Hirota bilinear forms are so effective in presenting determinant solutions and pfaffian solutions [16–18], and of course, determinant and pfaffian solutions are more general than the considered N -wave solutions. Among further interesting questions is to establish the linear superposition principle that applies to constrained soliton

equations, soliton equations with self-consistent sources and extended soliton equations (see, e.g., [19], [20] and [21–23], respectively).

We would also like to repeat a sub-mathematical problem related to our opposite question: is it feasible to design an algorithm to construct multivariate polynomials $P(x_1, x_2, \dots, x_M)$ which satisfy the following property:

$$P(k_{1,1} - k_{1,2}, k_{2,1} - k_{2,2}, \dots, k_{M,1} - k_{M,2}) = 0,$$

when

$$P(k_{1,i}, k_{2,i}, \dots, k_{M,i}) = 0, \quad i = 1, 2?$$

This is an interesting mathematical problem in the study of polynomials whose zeros form a vector space, and it determines one class of Hirota bilinear equations possessing the linear superposition principle of exponential waves. In our discussion above, we only analyzed some specific such Hirota bilinear equations. We expect to see more examples and, of course, a systematical theory finally.

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