

A supertrace identity and its applications to superintegrable systems

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A supertrace identity on Lie superalgebras is established. It provides a tool for constructing super-Hamiltonian structures of zero curvature equations associated with Lie superalgebras. Applications in the case of the Lie superalgebra $B(0,1)$ present super-Hamiltonian structures of a super-AKNS soliton hierarchy and a super-Dirac soliton hierarchy. © 2008 American Institute of Physics.
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I. INTRODUCTION

The trace identity provides a powerful tool for constructing Hamiltonian structures of soliton equations.^{1,2} It is based on the Killing form on a semisimple Lie algebra. On nonsemisimple Lie algebras, the Killing form is always degenerate, and thus, the trace identity does not help in this case. Recently, a variational identity—a generalized trace identity—was presented in the theory of integrable couplings,³ which can be used to produce Hamiltonian structures in the case of nonsemisimple Lie algebras. In this paper, we would like to discuss and show a supertrace identity and its applications in the theory of superintegrable systems.^{4–8}

Let $\mathcal{A}$ be a commutative superalgebra over $\mathbb{R}$ or $\mathbb{C}$, and G a matrix loop superalgebra over $\mathcal{A}$ with the nondegenerate Killing form. We take a pair of matrix spectral problems:

$$\begin{aligned}\phi_x &= U\phi = U(u, \lambda)\phi, \\ \phi_t &= V\phi = V\left(u, u_x, \dots, \frac{\partial^{m_0} u}{\partial x^{m_0}}; \lambda\right)\phi,\end{aligned}\tag{1.1}$$

where ϕ_x and ϕ_t denote the partial derivatives with respect to x and t , $U, V \in G$ are a Lax pair, $u = (u_1, \dots, u_q)^T \in \mathcal{A}^q$ is a potential consisting of commuting and anticommuting variables, λ is a spectral parameter, and m_0 is a natural number indicating the differential order.

Assume that (1.1) determines^{2,9} a superevolution equation:

$$u_t = K(u) = K\left(u, u_x, \dots, \frac{\partial^{m_0} u}{\partial x^{m_0}}\right),\tag{1.2}$$

through their isospectral (i.e., $\lambda_t = 0$) compatibility condition (i.e., zero curvature equation):

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$$U_t - V_x + [U, V] = 0, \quad (1.3)$$

where $[U, V] = UV - VU$. The zero curvature equation (1.3) means that the triple (U, V, K) satisfies

$$U'(u)[K] - V_x + [U, V] = 0, \quad (1.4)$$

where $U'(u)[K] = \partial/\partial \varepsilon|_{\varepsilon=0} U(u + \varepsilon K)$. There exist Lie algebraic structures behind this equation for such triples.^{10,11}

For a functional $\mathcal{H}$, its variational derivative $\delta\mathcal{H}/\delta u = (\delta\mathcal{H}/\delta u_1, \dots, \delta\mathcal{H}/\delta u_q)^T \in \mathcal{A}^q$ is defined by

$$\sum_{i=1}^q \frac{\delta\mathcal{H}}{\delta u_i} v_i = \frac{\partial}{\partial \varepsilon} \mathcal{H}(u + \varepsilon v)|_{\varepsilon=0}, \quad (1.5)$$

where $v = (v_1, \dots, v_q)^T \in \mathcal{A}^q$ is a vector like u and $\varepsilon \in \mathcal{A}$ is an even parameter (see, for example, Refs. 12 and 13 for more information). For an operator $J = (J_{ij})_{q \times q}$ from $\mathcal{A}^q$ to $\mathcal{A}^q$, define its corresponding bracket:

$$\{\mathcal{H}_1, \mathcal{H}_2\} = \int \sum_{i,j=1}^q (-1)^{p(i)p(j)p(\mathcal{H}_2)} \left(J_{ij} \frac{\delta\mathcal{H}_2}{\delta u_j} \right) \frac{\delta\mathcal{H}_1}{\delta u_i} dx, \quad (1.6)$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two functionals, and $p(i) = p(u_i)$ and $p(\mathcal{H}_2)$ are the degrees of u_i and $\mathcal{H}_2$ (either 0 or 1). Such an operator J is called a super-Hamiltonian operator (or an even super-Hamiltonian operator) if its corresponding bracket $\{\cdot, \cdot\}$ defined above is a super-Lie bracket, i.e., it is superskewsymmetric:

$$\{\mathcal{H}_1, \mathcal{H}_2\} = -(-1)^{p(\mathcal{H}_1)p(\mathcal{H}_2)} \{\mathcal{H}_2, \mathcal{H}_1\}, \quad (1.7)$$

and satisfies the super-Jacobi identity:

$$(-1)^{p(\mathcal{H}_1)p(\mathcal{H}_3)} \{\mathcal{H}_1, \{\mathcal{H}_2, \mathcal{H}_3\}\} + \text{cycle}(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3) = 0, \quad (1.8)$$

where $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ are pure in the $\mathbb{Z}_2$ grading. A superevolution equation (1.2) is called a super-Hamiltonian equation if there is a super-Hamiltonian operator and a functional $\mathcal{H}$ such that

$$u_t = K(u) = J \frac{\delta\mathcal{H}}{\delta u}. \quad (1.9)$$

If so, we say that the evolution equation (1.2) has a super-Hamiltonian structure. A super-bi-Hamiltonian equation can also be defined, similar to the classical case.¹⁴

One interesting question for us is how to generate super-Hamiltonian structures for the evolution equation (1.2) based on the zero curvature equation (1.3). In what follows, we would like to establish a supertrace identity on matrix loop superalgebras to construct super-Hamiltonian structures of associated superintegrable equations. We will use the supertrace on a given matrix loop superalgebra G , which satisfies the symmetric property:

$$\text{str}(\text{ad}_a \text{ad}_b) = \text{str}(\text{ad}_b \text{ad}_a), \quad a, b \in G, \quad (1.10)$$

and the invariance property under the Lie product:

$$\text{str}(\text{ad}_a \text{ad}_{[b,c]}) = \text{str}(\text{ad}_{[a,b]} \text{ad}_c), \quad a, b, c \in G. \quad (1.11)$$

Here, ad_a denotes the adjoint action of $a \in G$ on G , which is defined by

$$\text{ad}_a b = [a, b], \quad b \in G, \quad (1.12)$$

and the bracket $[\cdot, \cdot]$ is the Lie superbracket of G .

In this paper, we would like to show that there exists a supertrace identity on Lie superalgebras with nondegenerate Killing forms and it can be used to construct super-Hamiltonian structures of soliton equations associated with such Lie superalgebras. Applications in the case of the Lie superalgebra $B(0,1)$ present super-Hamiltonian structures of a super-AKNS soliton hierarchy and a super-Dirac soliton hierarchy. A few concluding remarks are given in the last section.

II. SUPERTRACE IDENTITY ON LIE SUPERALGEBRAS

Let a spectral matrix U be defined by

$$U = U(u, \lambda) = E_0(\lambda) + u_1 E_1(\lambda) + \cdots + u_q E_q(\lambda), \quad u_i \in \mathcal{A}, \quad E_i \in G, \quad 1 \leq i \leq q, \quad (2.1)$$

where $\mathcal{A}$ is a commutative superalgebra over $\mathbb{R}$ or $\mathbb{C}$, G is a matrix loop superalgebra over $\mathcal{A}$ with the nondegenerate Killing form, and $E_i \in G$, $1 \leq i \leq q$, are $\mathcal{A}$ linearly independent. We need to select the appropriate ranks, $\text{rank}(\lambda)$ and $\text{rank}(u)$, such that U has the same rank, i.e., it is homogeneous in rank. This way, we can define

$$\text{rank}(U) = \text{rank}\left(\frac{\partial}{\partial x}\right) = \text{const}. \quad (2.2)$$

Assume that if two solutions $V_1, V_2 \in G$ of the stationary zero curvature equation

$$V_x = [U, V] \quad (2.3)$$

possess the same rank, then they are $\mathcal{A}$ linearly dependent of each other:

$$V_1 = \gamma V_2, \quad \gamma = \text{const}. \quad (2.4)$$

This is a condition on the selection of the spectral matrix U , which normally implies that there exists a unique recursion operator associated with the spectral problem (1.1). The condition has also been used in deriving the trace identity,¹⁵ the so-called quadratic form identity,¹⁶ and the variational identity under the general bilinear forms.³

A. Supertrace identity

Let us introduce a functional:

$$\mathcal{W} = \int [\text{str}(\text{ad}_V \text{ad}_{U_\lambda}) + \text{str}(\text{ad}_\Lambda \text{ad}_{V_x - [U, V]})] dx, \quad (2.5)$$

where $U_\lambda = \partial U / \partial \lambda$, and $V, \Lambda \in G$ are two matrices to be determined.

The directional derivative $\nabla_a \mathcal{R} \in G$ of a functional $\mathcal{R}$ with respect to $a \in G$ is defined by

$$\int \text{str}(\text{ad}_{\nabla_a \mathcal{R}} \text{ad}_b) dx = \left. \frac{\partial}{\partial \varepsilon} \mathcal{R}(a + \varepsilon b) \right|_{\varepsilon=0}, \quad b \in G. \quad (2.6)$$

It is obvious that based on the nondegenerate property of the Killing form, we can have

$$\nabla_b \int \text{str}(\text{ad}_a \text{ad}_b) dx = a, \quad \nabla_b \int \text{str}(\text{ad}_a \text{ad}_{b_x}) dx = -a_x.$$

Therefore, it follows from (1.10) and (1.11) that

$$\nabla_V \mathcal{W} = U_\lambda - \Lambda_x + [U, \Lambda], \quad \nabla_\Lambda \mathcal{W} = V_x - [U, V]. \quad (2.7)$$

In what follows, we would like to prove that the supertrace identity on Lie superalgebras holds, similar to the trace identity.²

Theorem 2.1 (The supertrace identity): *Let $U = U(u, \lambda) \in G$ be homogeneous in rank. Assume that the stationary zero curvature equation (2.3) has a unique solution $V \in G$ of a fixed rank*

up to a constant multiplier. Then, there is a constant γ such that

$$\frac{\delta}{\delta u} \int \text{str}(\text{ad}_V \text{ad}_{\partial U / \partial \lambda}) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \text{str}(\text{ad}_V \text{ad}_{\partial U / \partial u}) \quad (2.8)$$

holds for any solution $V \in G$ of (2.3), being homogeneous in rank.

Proof: Let us begin with the functional $\mathcal{W}$ defined by (2.5). In order to compute the variational derivative of $\mathcal{W}$ with respect to the potential u , we require the following constrained conditions on $\mathcal{W}$:

$$\nabla_V \mathcal{W} = U_\lambda - \Lambda_x + [U, \Lambda] = 0, \quad (2.9)$$

$$\nabla_\Lambda \mathcal{W} = V_x - [U, V] = 0, \quad (2.10)$$

which determine V and Λ . The two conditions also imply that V and Λ are related to U , and thus, they are related to the potential u .

Based on (2.10), we have

$$\frac{\delta}{\delta u} \int \text{str}(\text{ad}_V \text{ad}_{U_\lambda}) dx = \frac{\delta \mathcal{W}}{\delta u}, \quad (2.11)$$

where $\delta / \delta u$ is the variational derivative with respect to the potential u . Noting both the above constrained conditions [(2.9) and (2.10)] and the property that if $\nabla_a \mathcal{R}(a) = 0$, then $(\delta / \delta u) \mathcal{R}(a(u)) = 0$, we know that only the dependence of u in U (but not in V and Λ) needs to be considered in computing $\delta \mathcal{W} / \delta u$. Thus, based on the invariance property (1.11), we have

$$\frac{\delta}{\delta u} \int \text{str}(\text{ad}_V \text{ad}_{U_\lambda}) dx = \frac{\delta \mathcal{W}}{\delta u} = \text{str}(\text{ad}_V \text{ad}_{\partial U_\lambda / \partial u}) + \text{str}(\text{ad}_{[\Lambda, V]} \text{ad}_{\partial U / \partial u}). \quad (2.12)$$

On the other hand, using (2.9) and (2.10), and the Jacobi identity, we have

$$\begin{aligned} [\Lambda, V]_x &= [\Lambda_x, V] + [\Lambda, V_x] = [U_\lambda + [U, \Lambda], V] + [\Lambda, [U, V]] = [U_\lambda, V] + [V, [\Lambda, U]] + [\Lambda, [U, V]] \\ &= [U_\lambda, V] + [U, [\Lambda, V]], \end{aligned} \quad (2.13)$$

and from (2.10), we have

$$V_{\lambda, x} = V_{x, \lambda} = [U_\lambda, V] + [U, V_\lambda]. \quad (2.14)$$

Therefore, $Z := [\Lambda, V] - V_\lambda$ satisfies $Z_x = [U, Z]$. Now, based on the uniqueness condition in (2.4) and $\text{rank}(Z) = \text{rank}(V_\lambda) = \text{rank}((1/\lambda)V)$, there exists a constant γ such that

$$[\Lambda, V] - V_\lambda = \frac{\gamma}{\lambda} V, \quad (2.15)$$

because $(1/\lambda)V$ is a solution to (2.3).

Finally, (2.12) can be further computed as follows:

$$\begin{aligned} \frac{\delta}{\delta u} \int \text{str}(\text{ad}_V \text{ad}_{U_\lambda}) dx &= \text{str}(\text{ad}_V \text{ad}_{\partial U_\lambda / \partial u}) + \text{str}(\text{ad}_{V_\lambda} \text{ad}_{\partial U / \partial u}) + \frac{\gamma}{\lambda} \text{str}(\text{ad}_V \text{ad}_{\partial U / \partial u}) \\ &= \frac{\partial}{\partial \lambda} \text{str}(\text{ad}_V \text{ad}_{\partial U / \partial u}) + \left(\lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \right) \text{str}(\text{ad}_V \text{ad}_{\partial U / \partial u}) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \text{str}(\text{ad}_V \text{ad}_{\partial U / \partial u}). \end{aligned} \quad (2.16)$$

This completes the proof of the theorem. $\square$

B. An expression for the constant γ

Let us now consider how to compute the constant γ through the solution V to the stationary zero curvature equation.

Theorem 2.2: *Let V be a solution to the stationary zero curvature equation (2.3). Then we have*

$$\frac{d}{dx} \text{str}(\text{ad}_{V^m} \text{ad}_{V^m}) = 0, \quad m \geq 1. \quad (2.17)$$

Proof: Noting that $V_x = [U, V]$ leads to

$$(V^m)_x = [U, V^m], \quad m \geq 1,$$

and thus,

$$\text{ad}_{(V^m)_x} = [\text{ad}_U, \text{ad}_{V^m}], \quad m \geq 1,$$

it then follows from the symmetric property (1.10) and the invariance property under Lie bracket, (1.11), that

$$\begin{aligned} \frac{d}{dx} \text{str}(\text{ad}_{V^m} \text{ad}_{V^m}) &= 2 \text{str} \left(\left(\frac{d}{dx} \text{ad}_{V^m} \right) \text{ad}_{V^m} \right) = 2 \text{str}(\text{ad}_{(V^m)_x} \text{ad}_{V^m}) = 2 \text{str}([\text{ad}_U, \text{ad}_{V^m}] \text{ad}_{V^m}) \\ &= 2 \text{str}(\text{ad}_{[U, V^m]} \text{ad}_{V^m}) = 2 \text{str}(\text{ad}_U \text{ad}_{[V^m, V]}) = 0, \end{aligned}$$

where $m \geq 1$. This proves the theorem. $\square$

By (2.17), $\text{str}(\text{ad}_V \text{ad}_V)$ is independent of the space variable x .

Theorem 2.3: *Let V be a solution to the stationary zero curvature equation (2.3). Then the constant γ in the supertrace identity (2.8) is given by*

$$\gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} \ln |\text{str}(\text{ad}_V \text{ad}_V)|, \quad (2.18)$$

if $\text{str}(\text{ad}_V \text{ad}_V) \neq 0$.

Proof: Differentiating $\text{str}(\text{ad}_V \text{ad}_V)$ with respect to λ and using (2.15) engenders

$$\begin{aligned} \frac{d}{d\lambda} \text{str}(\text{ad}_V \text{ad}_V) &= 2 \text{str}(\text{ad}_{V_\lambda} \text{ad}_V) = 2 \text{str}(\text{ad}_{[\Lambda, V] - (\gamma/\lambda)V} \text{ad}_V) = 2 \text{str}(\text{ad}_{[\Lambda, V]} \text{ad}_V) - \frac{2\gamma}{\lambda} \text{str}(\text{ad}_V \text{ad}_V) \\ &= 2 \text{str}(\text{ad}_\Lambda \text{ad}_{[V, V]}) - \frac{2\gamma}{\lambda} \text{str}(\text{ad}_V \text{ad}_V) = -\frac{2\gamma}{\lambda} \text{str}(\text{ad}_V \text{ad}_V). \end{aligned}$$

This implies that (2.18) holds. $\square$

III. APPLICATIONS

We take the Lie superalgebra $B(0, 1)$ given by

$$g = g_0 E_0 + g_1 E_1 + g_2 E_2 + g_3 E_3 + g_4 E_4 = \begin{bmatrix} g_0 & g_1 & g_3 \\ g_2 & -g_0 & g_4 \\ g_4 & -g_3 & 0 \end{bmatrix}, \quad g_i \in \mathcal{A}, \quad 0 \leq i \leq 4, \quad (3.1)$$

where $\mathcal{A}$ is a commutative superalgebra over $\mathbb{R}$ or $\mathbb{C}$, g_0, g_1, g_2 are even, and g_3, g_4 are odd.^{17,18} The generators of the Lie superalgebra $B(0, 1)$, E_i , $0 \leq i \leq 4$, satisfy the following commutation relations:

$$[E_0, E_1] = 2E_1, \quad [E_0, E_2] = -2E_2, \quad [E_1, E_2] = E_0,$$

$$[E_0, E_3] = E_3, \quad [E_0, E_4] = -E_4, \quad [E_1, E_3] = 0,$$

$$[E_1, E_4] = E_3, \quad [E_2, E_3] = E_4, \quad [E_2, E_4] = 0,$$

$$[E_3, E_4]_+ = E_0, \quad [E_3, E_3]_+ = -2E_1, \quad [E_4, E_4]_+ = 2E_2,$$

where E_0, E_1, E_2 are even and E_3, E_4 are odd, and $[\cdot, \cdot]$ and $[\cdot, \cdot]_+$ denote the commutator and the anticommutator, respectively. The loop superalgebra

$$G = \tilde{B}(0, 1) = B(0, 1) \otimes \mathbb{C}[\lambda, \lambda^{-1}] \quad \text{or} \quad B(0, 1) \otimes \mathbb{R}[\lambda, \lambda^{-1}]$$

will be our starting Lie superalgebra.

It is direct to compute that

$$\text{ad}_a = \begin{bmatrix} 0 & -a_2 & a_1 & a_4 & a_3 \\ -2a_1 & 2a_0 & 0 & -2a_3 & 0 \\ 2a_2 & 0 & -2a_0 & 0 & 2a_4 \\ -a_3 & -a_4 & 0 & a_0 & a_1 \\ a_4 & 0 & -a_3 & a_2 & -a_0 \end{bmatrix}$$

for $a = a_0 E_0 + a_1 E_1 + a_2 E_2 + a_3 E_3 + a_4 E_4 \in \tilde{B}(0, 1)$, where ad_a is defined by (1.12). Therefore, if we define the compact supertrace

$$\text{str}(c) = c_{11} + c_{22} - c_{33}, \quad c = ab, \quad a, b \in \tilde{B}(0, 1), \quad (3.2)$$

where $c = (c_{ij})_{3 \times 3}$ and ab is the matrix product of a and b , then the Killing form on $\tilde{B}(0, 1)$ can be computed as follows:

$$\text{str}(\text{ad}_a \text{ad}_b) = 3 \text{str}(ab), \quad a, b \in \tilde{B}(0, 1), \quad (3.3)$$

where the supertrace is defined by

$$\text{str}(P) = p_{11} + p_{22} + p_{33} - p_{44} - p_{55} \quad \text{if } P = (p_{ij})_{5 \times 5}. \quad (3.4)$$

For other Lie superalgebras, an identity similar to (3.3) can be obtained (see, for example, Ref. 18).

A. The super-AKNS soliton hierarchy

1. The super-AKNS hierarchy

The super-AKNS spectral problem associated with $\tilde{B}(0, 1)$ is given by

$$\phi_x = U\phi, \quad U = \begin{bmatrix} \lambda & r & \alpha \\ s & -\lambda & \beta \\ \beta & -\alpha & 0 \end{bmatrix}, \quad u = \begin{bmatrix} r \\ s \\ \alpha \\ \beta \end{bmatrix}, \quad (3.5)$$

where λ is the spectral parameter, r and s are commuting variables, and α and β are anticommuting variables.¹⁹ When the anticommuting variables, α and β , are taken as zero, the above super-spectral problem reduces to the classical AKNS spectral problem.²⁰

As in Refs. 21 and 22, let us try to find the following solution to the stationary zero curvature equation (2.3):

$$V = \begin{bmatrix} A & B & \rho \\ C & -A & \sigma \\ \sigma & -\rho & 0 \end{bmatrix} = \sum_{i \geq 0} V_i \lambda^{-i} = \sum_{i \geq 0} \begin{bmatrix} A_i & B_i & \rho_i \\ C_i & -A_i & \sigma_i \\ \sigma_i & -\rho_i & 0 \end{bmatrix} \lambda^{-i}, \quad (3.6)$$

where A_i, B_i, C_i are commuting fields, and ρ_i, σ_i are anticommuting fields. Then, (2.3) is equivalent to

$$A_x = rC - sB + \alpha\sigma - \rho\beta,$$

$$B_x = 2\lambda B - 2rA - 2\alpha\rho,$$

$$C_x = -2\lambda C + 2sA + 2\beta\sigma,$$

$$\rho_x = \lambda\rho + r\sigma - \alpha A - \beta B,$$

$$\sigma_x = -\lambda\sigma + \beta A - \alpha C + s\rho.$$

Upon choosing the initial data:

$$A_0 = 1, \quad B_0 = C_0 = \rho_0 = \sigma_0 = 0,$$

we see that the above relations generate

$$B_{i+1} = \frac{1}{2}B_{i,x} + rA_i + \alpha\rho_i,$$

$$C_{i+1} = -\frac{1}{2}C_{i,x} + sA_i + \beta\sigma_i,$$

$$\rho_{i+1} = \rho_{i,x} - r\sigma_i + \alpha A_i + \beta B_i, \quad (3.7)$$

$$\sigma_{i+1} = -\sigma_{i,x} + \beta A_i - \alpha C_i + s\rho_i,$$

$$A_{i+1,x} = rC_{i+1} - sB_{i+1} + \alpha\sigma_{i+1} - \rho_{i+1}\beta,$$

where $i \geq 0$. Assume that the constants of integration are selected to be zero. Then, the recursion relations in (3.7) uniquely define a series of sets of differential polynomial functions in u with respect to x . The first three sets are as follows:

$$B_1 = r, \quad C_1 = s, \quad \rho_1 = \alpha, \quad \sigma_1 = \beta, \quad A_1 = 0;$$

$$B_2 = \frac{1}{2}r_x, \quad C_2 = -\frac{1}{2}s_x, \quad \rho_2 = \alpha_x, \quad \sigma_2 = -\beta_x, \quad A_2 = -\frac{1}{2}rs - \alpha\beta;$$

$$B_3 = \frac{1}{4}r_{xx} - \frac{1}{2}r^2s - r\alpha\beta + \alpha\alpha_x, \quad C_3 = \frac{1}{4}s_{xx} - \frac{1}{2}rs^2 - s\alpha\beta - \beta\beta_x;$$

$$\rho_3 = \alpha_{xx} + r\beta_x + \frac{1}{2}r_x\beta - \frac{1}{2}rs\alpha, \quad \sigma_3 = \beta_{xx} + s\alpha_x + \frac{1}{2}s_x\alpha - \frac{1}{2}rs\beta;$$

$$A_3 = \frac{1}{4}(rs_x - r_x s) + (\alpha\beta_x - \alpha_x\beta).$$

The compatibility conditions of the matrix spectral problems

$$\phi_x = U\phi, \quad \phi_{t_m} = V^{[m]}\phi, \quad V^{[m]} = (\lambda^m V)_+, \quad m \geq 0, \quad (3.8)$$

determine the super-AKNS soliton hierarchy:

$$u_{t_m} = K_m = (-2B_{m+1}, 2C_{m+1}, -\rho_{m+1}, \sigma_{m+1})^T, \quad m \geq 0, \quad (3.9)$$

the first nonlinear supersystem of which is as follows:

$$\begin{aligned} r_{t_2} &= -\frac{1}{2}r_{xx} + r^2s + 2r\alpha\beta - 2\alpha\alpha_x, \\ s_{t_2} &= \frac{1}{2}s_{xx} - rs^2 - 2s\alpha\beta - 2\beta\beta_x, \\ \alpha_{t_2} &= -\alpha_{xx} - r\beta_x - \frac{1}{2}r_x\beta + \frac{1}{2}rs\alpha, \\ \beta_{t_2} &= \beta_{xx} + s\alpha_x + \frac{1}{2}s_x\alpha - \frac{1}{2}rs\beta. \end{aligned} \quad (3.10)$$

From the recursion relations in (3.7), we know that the hereditary recursion operator Φ of the super-AKNS hierarchy reads

$$\Phi = \begin{bmatrix} \frac{1}{2}\partial - r\partial^{-1}s & -r\partial^{-1}r & 2r\partial^{-1}\beta + 2\alpha & -2r\partial^{-1}\alpha \\ s\partial^{-1}s & -\frac{1}{2}\partial + s\partial^{-1}r & -2s\partial^{-1}\beta & 2s\partial^{-1}\alpha + 2\beta \\ -\frac{1}{2}\alpha\partial^{-1}s + \frac{1}{2}\beta & -\frac{1}{2}\alpha\partial^{-1}r & \partial + \alpha\partial^{-1}\beta & r - \alpha\partial^{-1}\alpha \\ \frac{1}{2}\beta\partial^{-1}s & \frac{1}{2}\beta\partial^{-1}r - \frac{1}{2}\alpha & -\beta\partial^{-1}\beta - s & -\partial + \beta\partial^{-1}\alpha \end{bmatrix}. \quad (3.11)$$

2. Super-Hamiltonian structures of the super-AKNS hierarchy

Let us now apply the supertrace identity (2.8) to construct super-Hamiltonian structures for the super-AKNS soliton hierarchy (3.9).

Noting $\text{ad}_{U_z} = \partial_z \text{ad}_U$ and using (3.3), we have

$$\text{str}(\text{ad}_V \text{ad}_{U_\lambda}) = 6A,$$

$$\text{str}(\text{ad}_V \text{ad}_{\partial U/\partial r}) = 3C, \quad \text{str}(\text{ad}_V \text{ad}_{\partial U/\partial s}) = 3B,$$

$$\text{str}(\text{ad}_V \text{ad}_{\partial U/\partial \alpha}) = -6\sigma, \quad \text{str}(\text{ad}_V \text{ad}_{\partial U/\partial \beta}) = 6\rho.$$

For example, we can compute that

$$\text{str}(\text{ad}_V \text{ad}_{U_\lambda}) = 3 \text{str}(VU_\lambda) = 3 \text{str} \left(\begin{bmatrix} A & B & \rho \\ C & -A & \sigma \\ \sigma & -\rho & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = 3 \text{str} \begin{bmatrix} A & -B & 0 \\ C & A & 0 \\ \sigma & \rho & 0 \end{bmatrix} = 3(2A) = 6A.$$

Then, an application of the supertrace identity (2.8) gives rise to

$$\frac{\delta}{\delta u} \int 2A dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma (C, B, -2\sigma, 2\rho)^T.$$

Comparing the coefficients of λ^{-m-1} , this identity equivalently yields

$$\frac{\delta}{\delta u} \int 2A_{m+1} dx = (-m + \gamma)(C_m, B_m, -2\sigma_m, 2\rho_m)^T, \quad m \geq 0.$$

Since $\text{str}(\text{ad}_V \text{ad}_V) = 6$, we know the constant $\gamma = 0$ by Theorem 2.3. Therefore, we obtain

$$\frac{\delta}{\delta u} \int \frac{2A_{m+1}}{m} dx = (-C_m, -B_m, 2\sigma_m, -2\rho_m)^T, \quad m \geq 1.$$

It then follows that the super-AKNS soliton hierarchy (3.9) possesses the following super-Hamiltonian structure:

$$u_{t_m} = K_m = J \frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0, \quad (3.12)$$

where the super-Hamiltonian operator J and the Hamiltonian functional $\mathcal{H}_m$ are given by

$$J = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}, \quad \mathcal{H}_m = \int \frac{2A_{m+2}}{m+1} dx, \quad m \geq 0. \quad (3.13)$$

Now obviously, this super-AKNS soliton hierarchy has the following super-bi-Hamiltonian structure:

$$u_{t_m} = K_m = J \frac{\delta \mathcal{H}_m}{\delta u} = M \frac{\delta \mathcal{H}_{m-1}}{\delta u}, \quad m \geq 1, \quad (3.14)$$

where the second compatible super-Hamiltonian operator reads

$$M = \Phi J = \begin{bmatrix} 2r\partial^{-1}r & \partial - 2r\partial^{-1}s & -r\partial^{-1}\alpha & r\partial^{-1}\beta + \alpha \\ \partial - 2s\partial^{-1}r & 2s\partial^{-1}s & s\partial^{-1}\alpha + \beta & -s\partial^{-1}\beta \\ \alpha\partial^{-1}r & -\alpha\partial^{-1}s + \beta & \frac{1}{2}r - \frac{1}{2}\alpha\partial^{-1}\alpha & \frac{1}{2}\partial + \frac{1}{2}\alpha\partial^{-1}\beta \\ -\beta\partial^{-1}r + \alpha & \beta\partial^{-1}s & -\frac{1}{2}\partial + \frac{1}{2}\beta\partial^{-1}\alpha & -\frac{1}{2}\beta\partial^{-1}\beta - \frac{1}{2}s \end{bmatrix}. \quad (3.15)$$

This super-bi-Hamiltonian structure also implies that each super-AKNS system in the hierarchy (3.9) possesses infinitely many commuting conserved quantities $\{\mathcal{H}_n\}_{n=0}^{\infty}$ and infinitely many commuting symmetries $\{K_n\}_{n=0}^{\infty}$.

B. The super-Dirac soliton hierarchy

1. The super-Dirac hierarchy

Let us take a super-Dirac spectral problem associated with $\tilde{B}(0, 1)$:

$$\phi_x = U\phi, \quad U = \begin{bmatrix} r & \lambda + s & \alpha \\ -\lambda + s & -r & \beta \\ \beta & -\alpha & 0 \end{bmatrix}, \quad u = \begin{bmatrix} r \\ s \\ \alpha \\ \beta \end{bmatrix}, \quad (3.16)$$

where λ is the spectral parameter, r and s are commuting variables, and α and β are anticommuting variables. Similarly, when the anticommuting variables, α and β , are chosen as zero, the above superspectral problem reduces to the classical Dirac spectral problem (see, for example, Refs. 23 and 24).

Let us start to find the following solution to the stationary zero curvature equation (2.3):

$$V = \begin{bmatrix} C & A+B & \rho \\ A-B & -C & \sigma \\ \sigma & -\rho & 0 \end{bmatrix} = \sum_{i \geq 0} V_i \lambda^{-i} = \sum_{i \geq 0} \begin{bmatrix} C_i & A_i+B_i & \rho_i \\ A_i-B_i & -C_i & \sigma_i \\ \sigma_i & -\rho_i & 0 \end{bmatrix} \lambda^{-i}, \quad (3.17)$$

where A_i, B_i, C_i are commuting fields, and ρ_i, σ_i are anticommuting fields. Then, (2.3) is equivalent to

$$A_x = -2\lambda C + 2rB - \alpha\rho + \beta\sigma,$$

$$B_x = 2rA - 2sC - \alpha\rho - \beta\sigma,$$

$$C_x = 2\lambda A - 2sB + \alpha\sigma + \beta\rho,$$

$$\rho_x = -\beta(A+B) - \alpha C + (\lambda+s)\sigma + r\rho,$$

$$\sigma_x = (-\lambda+s)\rho - r\sigma - \alpha(A-B) + \beta C.$$

Upon choosing the initial data:

$$B_0 = 1, \quad A_0 = C_0 = \rho_0 = \sigma_0 = 0,$$

we see that the above relations yield

$$A_{i+1} = \frac{1}{2}C_{i,x} + sB_i - \frac{1}{2}\alpha\sigma_i - \frac{1}{2}\beta\rho_i,$$

$$C_{i+1} = -\frac{1}{2}A_{i,x} + rB_i - \frac{1}{2}\alpha\rho_i + \frac{1}{2}\beta\sigma_i,$$

$$\rho_{i+1} = -\sigma_{i,x} - r\sigma_i + s\rho_i - \alpha(A_i - B_i) + \beta C_i, \quad (3.18)$$

$$\sigma_{i+1} = \rho_{i,x} - r\rho_i - s\sigma_i + \beta(A_i + B_i) + \alpha C_i,$$

$$B_{i+1,x} = 2rA_{i+1} - 2sC_{i+1} - \alpha\rho_{i+1} - \beta\sigma_{i+1},$$

where $i \geq 0$. Assume that the constants of integration are selected to be zero. Then, the recursion relations in (3.18) uniquely determine a series of sets of differential polynomial functions in u with respect to x . The first three sets read

$$A_1 = s, \quad C_1 = r, \quad \rho_1 = \alpha, \quad \sigma_1 = \beta, \quad B_1 = 0;$$

$$A_2 = \frac{1}{2}r_x, \quad C_2 = -\frac{1}{2}s_x, \quad \rho_2 = -\beta_x, \quad \sigma_2 = \alpha_x, \quad B_2 = \frac{1}{2}(r^2 + s^2) + \alpha\beta;$$

$$A_3 = -\frac{1}{4}s_{xx} + \frac{1}{2}(r^2 + s^2)s + s\alpha\beta - \frac{1}{2}\alpha\alpha_x + \frac{1}{2}\beta\beta_x;$$

$$C_3 = -\frac{1}{4}r_{xx} + \frac{1}{2}r(r^2 + s^2) + r\alpha\beta + \frac{1}{2}\alpha\beta_x - \frac{1}{2}\alpha_x\beta;$$

$$\rho_3 = -\alpha_{xx} - r\alpha_x - s\beta_x - \frac{1}{2}r_x\alpha - \frac{1}{2}s_x\beta + \frac{1}{2}(r^2 + s^2)\alpha;$$

$$\sigma_3 = -\beta_{xx} + r\beta_x - s\alpha_x + \frac{1}{2}r_x\beta - \frac{1}{2}s_x\alpha + \frac{1}{2}(r^2 + s^2)\beta;$$

$$B_3 = -\frac{1}{2}(rs_x - r_x s) + \alpha\alpha_x + \beta\beta_x.$$

Evidently, the compatibility conditions of the matrix spectral problems:

$$\phi_x = U\phi, \quad \phi_{t_m} = V^{[m]}\phi, \quad V^{[m]} = (\lambda^m V)_+, \quad m \geq 0, \quad (3.19)$$

determine the super-Dirac soliton hierarchy:

$$u_{t_m} = K_m = (-2A_{m+1}, 2C_{m+1}, -\sigma_{m+1}, \rho_{m+1})^T, \quad m \geq 0, \quad (3.20)$$

the first nonlinear supersystem of which is given by

$$\begin{aligned} r_{t_2} &= \frac{1}{2}s_{xx} - (r^2 + s^2)s - 2s\alpha\beta + \alpha\alpha_x - \beta\beta_x, \\ s_{t_2} &= -\frac{1}{2}r_{xx} + r(r^2 + s^2) + 2r\alpha\beta + \alpha\beta_x - \alpha_x\beta, \\ \alpha_{t_2} &= \beta_{xx} - r\beta_x + s\alpha_x - \frac{1}{2}r_x\beta + \frac{1}{2}s_x\alpha - \frac{1}{2}(r^2 + s^2)\beta, \\ \beta_{t_2} &= -\alpha_{xx} - r\alpha_x - s\beta_x - \frac{1}{2}q_x\alpha - \frac{1}{2}s_x\beta + \frac{1}{2}(r^2 + s^2)\alpha. \end{aligned} \quad (3.21)$$

From the recursion relations in (3.18), we know that the hereditary recursion operator Φ of the super-Dirac hierarchy reads

$$\Phi = \begin{bmatrix} 2s\partial^{-1}r & -\frac{1}{2}\partial + 2s\partial^{-1}s & -2s\partial^{-1}\beta - \alpha & 2s\partial^{-1}\alpha + \beta \\ \frac{1}{2}\partial - 2r\partial^{-1}r & -2r\partial^{-1}s & 2r\partial^{-1}\beta - \beta & -2r\partial^{-1}\alpha - \alpha \\ \beta\partial^{-1}r + \frac{1}{2}\beta & \beta\partial^{-1}s - \frac{1}{2}\alpha & -\beta\partial^{-1}\beta - s & -\partial + \beta\partial^{-1}\alpha + r \\ -\alpha\partial^{-1}r + \frac{1}{2}\alpha & -\alpha\partial^{-1}s + \frac{1}{2}\beta & \partial + \alpha\partial^{-1}\beta + r & -\alpha\partial^{-1}\alpha + s \end{bmatrix}. \quad (3.22)$$

2. Super-Hamiltonian structures of the super-Dirac hierarchy

Let us now apply the supertrace identity (2.8) to construct super-Hamiltonian structures for the super-Dirac soliton hierarchy (3.20).

Similarly, noting $\text{ad}_{U_z} = \partial_z \text{ad}_U$ and using (3.3), we have

$$\text{str}(\text{ad}_V \text{ad}_{U_\lambda}) = -6B,$$

$$\text{str}(\text{ad}_V \text{ad}_{\partial U/\partial r}) = 6C, \quad \text{str}(\text{ad}_V \text{ad}_{\partial U/\partial s}) = 6A,$$

$$\text{str}(\text{ad}_V \text{ad}_{\partial U/\partial \alpha}) = -6\sigma, \quad \text{str}(\text{ad}_V \text{ad}_{\partial U/\partial \beta}) = 6\rho.$$

For example, we can compute that

$$\begin{aligned} \text{str}(\text{ad}_V \text{ad}_{U_\lambda}) &= 3 \text{str}(VU_\lambda) = 3 \text{str} \left(\begin{bmatrix} C & A+B & \rho \\ A-B & -C & \sigma \\ \sigma & -\rho & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = 3 \text{str} \begin{bmatrix} -(A+B) & C & 0 \\ C & A-B & 0 \\ \rho & \sigma & 0 \end{bmatrix} \\ &= 3(-2B) = -6B. \end{aligned}$$

Then, an application of the supertrace identity (2.8) gives rise to

$$\frac{\delta}{\delta u} \int (-B) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma (C, A, -\sigma, \rho)^T.$$

Comparing the coefficients of λ^{-m-1} , this identity equivalently leads to

$$\frac{\delta}{\delta u} \int (-B_{m+1}) dx = (-m + \gamma)(C_m, A_m, -\sigma_m, \rho_m)^T, \quad m \geq 0.$$

Since $\text{str}(\text{ad}_V \text{ad}_V) = -6$, we know from Theorem 2.3 that the constant γ is zero. Therefore, we arrive at

$$\frac{\delta}{\delta u} \int \frac{B_{m+1}}{m} dx = (C_m, A_m, -\sigma_m, \rho_m)^T, \quad m \geq 1.$$

It then follows that the super-Dirac soliton hierarchy (3.20) possesses the following super-Hamiltonian structure:

$$u_{t_m} = K_m = J \frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0, \quad (3.23)$$

where the super-Hamiltonian operator J and the Hamiltonian functional $\mathcal{H}_m$ are defined by

$$J = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{H}_m = \int \frac{B_{m+2}}{m+1} dx, \quad m \geq 0. \quad (3.24)$$

Now similarly, this super-Dirac soliton hierarchy has the following super-bi-Hamiltonian structure:

$$u_{t_m} = K_m = J \frac{\delta \mathcal{H}_m}{\delta u} = M \frac{\delta \mathcal{H}_{m-1}}{\delta u}, \quad m \geq 1, \quad (3.25)$$

where the second compatible super-Hamiltonian operator reads

$$M = \Phi J = \begin{bmatrix} -\partial + 4s\partial^{-1}s & -4s\partial^{-1}r & -2s\partial^{-1}\beta - \alpha & 2s\partial^{-1}\alpha + \beta \\ -4r\partial^{-1}s & -\partial + 4r\partial^{-1}r & 2r\partial^{-1}\beta - \beta & -2r\partial^{-1}\alpha - \alpha \\ 2\beta\partial^{-1}s - \alpha & -2\beta\partial^{-1}r - \beta & -\beta\partial^{-1}\beta - s & -\partial + \beta\partial^{-1}\alpha + r \\ -2\alpha\partial^{-1}s + \beta & 2\alpha\partial^{-1}r - \alpha & \partial + \alpha\partial^{-1}\beta + r & -\alpha\partial^{-1}\alpha + s \end{bmatrix}. \quad (3.26)$$

It now follows from this super-bi-Hamiltonian structure that each super-Dirac system in the hierarchy (3.20) possesses infinitely many commuting conserved quantities $\{\mathcal{H}_n\}_{n=0}^\infty$ and infinitely many commuting symmetries $\{K_n\}_{n=0}^\infty$.

IV. CONCLUDING REMARKS

The supertrace identity has been established for zero curvature equations associated with Lie superalgebras with nondegenerate Killing forms. It can be used to construct super-Hamiltonian structures of supersoliton equations. Two applications in the case of the Lie superalgebra $B(0,1)$ present the super-bi-Hamiltonian structures for the super-AKNS and Dirac soliton hierarchies.

The supertrace identity provides a tool for us to construct super-Hamiltonian structures, based on zero curvature equations. It shows us that there is always a bridge to go between superspectral problems and super-Hamiltonian structures, if the spectral matrix U is well selected. The supertrace identity was also mentioned without proof in Ref. 25 and applied in Refs. 21 and 25. We presented here a proof of the supertrace identity with an explicit expression for computing the constant γ appearing in the identity and illustrated by two examples how to generate super-Hamiltonian structures. Our construction of superintegrable systems can be also applied to other soliton hierarchies,²⁶⁻²⁸ and the supertrace identity can be used to generate their corresponding

super-Hamiltonian structures. A more general variational identity, which could be applied to supersymmetric integrable systems, e.g., fully supersymmetric KdV and AKNS hierarchies,^{29–32} should be more interesting.

There is a question to us related to the supertrace identity. In the expression for the constant γ , we required the condition $\text{str}(\text{ad}_V \text{ad}_V) \neq 0$. Is there any general expression for the constant γ that works for all cases even if $\text{str}(\text{ad}_V \text{ad}_V) = 0$? It seems that we should find another way to express the constant γ while $\text{str}(\text{ad}_V \text{ad}_V) = 0$.

There are other interesting questions. For example, how can one construct Darboux transformations for the super-AKNS and Dirac hierarchies? Do there exist any Darboux matrices similar to the ones for the nonsuper-AKNS and Dirac hierarchies?^{33,34} We also mention that like the classical case of binary nonlinearization, symmetry constraints³⁵ (or adjoint symmetry constraints³⁶) could link supersoliton equations with super finite-dimensional Hamiltonian systems. We hope to see some good supplements to existing theories on Darboux transformations and binary nonlinearization in the near future.

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- ¹G. Z. Tu, *J. Math. Phys.* **30**, 330 (1989).
- ²G. Z. Tu, *J. Phys. A* **22**, 2375 (1989).
- ³W. X. Ma and M. Chen, *J. Phys. A* **39**, 10787 (2006).
- ⁴B. A. Kupershmidt, *Lett. Math. Phys.* **9**, 323 (1985).
- ⁵D. A. Leites and M. A. Semenov-Tyan-Shanskii, *J. Sov. Math.* **28**, 525 (1985).
- ⁶B. A. Kupershmidt *Elements of Superintegrable Systems* (Kluwer, Dordrecht, 1987).
- ⁷P. Mathieu, *Lett. Math. Phys.* **16**, 199 (1988).
- ⁸Y. Watanabe, *Ann. Mat. Pura Appl.* **154**, 127 (1989).
- ⁹W. X. Ma, *Chin. Ann. Math., Ser. A* **13**, 115 (1992).
- ¹⁰W. X. Ma, *J. Phys. A* **26**, 2573 (1993).
- ¹¹W. X. Ma, *Phys. Lett. A* **179**, 179 (1993).
- ¹²Yu. I. Manin and A. O. Radul, *Commun. Math. Phys.* **98**, 65 (1985).
- ¹³J. S. He and Y. Cheng, *Phys. Lett. B* **449**, 194 (1999).
- ¹⁴F. Magri, *J. Math. Phys.* **19**, 1156 (1978).
- ¹⁵G. Z. Tu, *Sci. Sin., Ser. A* **29**, 138 (1986).
- ¹⁶F. K. Guo and Y. F. Zhang, *J. Phys. A* **38**, 8537 (2005).
- ¹⁷M. Scheunert *The Theory of Superalgebras* (Springer, New York, 1978).
- ¹⁸H. Z. Sun and Q. Z. Han *Lie Algebras and Lie Superalgebras and Their Applications in Physics* (Peking University Press, Beijing, 1999).
- ¹⁹M. Gürses and O. Oguz, *Phys. Lett. A* **108**, 437 (1985).
- ²⁰M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, *Stud. Appl. Math.* **53**, 249 (1974).
- ²¹A. R. Chowdhury and R. Swapna, *J. Math. Phys.* **27**, 2464 (1986).
- ²²Y. S. Li and L. N. Zhang, *J. Math. Phys.* **31**, 470 (1990).
- ²³W. X. Ma and K. S. Li, *Inverse Probl.* **12**, L25 (1996).
- ²⁴W. X. Ma, *Chin. Ann. Math., Ser. B* **18**, 79 (1997).
- ²⁵X. B. Hu, *J. Phys. A* **30**, 619 (1997).
- ²⁶W. X. Ma and Z. X. Zhou, *J. Math. Phys.* **42**, 4345 (2001).
- ²⁷W. X. Ma and R. G. Zhou, *Chin. Ann. Math., Ser. B* **23**, 373 (2002).
- ²⁸Z. Y. Qin, *Phys. Lett. A* **355**, 452 (2006).
- ²⁹P. Mathieu, *J. Math. Phys.* **29**, 2499 (1988).
- ³⁰C. Morosi and L. Pizzocchero, *Commun. Math. Phys.* **158**, 267 (1993).
- ³¹C. Morosi and L. Pizzocchero, *Commun. Math. Phys.* **176**, 353 (1996).
- ³²H. Aratyn, C. Rasinariu, and A. Das, *Mod. Phys. Lett. A* **12**, 2623 (1997).
- ³³C. H. Gu, H. S. Hu, and Z. X. Zhou, *Darboux Transformations in Integrable Systems* (Springer, Dordrecht, 2005).
- ³⁴E. G. Fan, *J. Phys. A* **38**, 1063 (2005).
- ³⁵W. X. Ma and W. Strampp, *Phys. Lett. A* **185**, 277 (1994).
- ³⁶W. X. Ma and R. G. Zhou, *J. Nonlinear Math. Phys.* **9**, Suppl. 1, 106 (2002).