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Partial differential equations possessing Frobenius integrable decompositions

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Abstract

Frobenius integrable decompositions are introduced for partial differential equations. A procedure is provided for determining a class of partial differential equations of polynomial type, which possess specified Frobenius integrable decompositions. Two concrete examples with logarithmic derivative Bäcklund transformations are given, and the presented partial differential equations are transformed into Frobenius integrable ordinary differential equations with cubic nonlinearity. The resulting solutions are illustrated to describe the solution phenomena shared with the KdV and potential KdV equations.

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1. Introduction

One often uses reductions to solve partial differential equations (PDEs), in particular, nonlinear PDEs such as soliton equations. It is very common to reduce given PDEs to simpler PDEs (normally linear) and/or integrable ordinary differential equations (ODEs). A new class of exact solutions—complexitons—of soliton equations was recently constructed this way, indeed [1,2]. More recently, integrable time–space decompositions of evolution equations were discussed, which transform the evolution equations into finite-dimensional integrable Hamiltonian systems [3]. In this Letter, we want to discuss a generalization of such integrable time–space decompositions for PDEs.

If a scalar PDE or a system of PDEs

$$K(u, u_t, u_x, \ldots) = 0 \tag{1.1}$$

has solutions of the form

$$u = f(\Phi) = f\big(\Phi(x,t)\big), \tag{1.2}$$

where f is a function of Φ and Φ satisfies two compatible scalar ODEs or systems of ODEs:

$$\Phi_x = \Gamma(\Phi), \qquad \Phi_t = \Omega(\Phi), \tag{1.3}$$

where Γ and Ω are functions of Φ, then we say that (1.1) possesses a Frobenius integrable decomposition (FID). In this definition, we do not require the Hamiltonian structures of the Frobenius integrable ODEs in (1.3), and thus, FIDs generalize integrable time–space decompositions requiring Hamiltonian structures [3], which aim to guarantee the Liouville integrability [4,5].

Symmetry constraints provide integrable time–space decompositions, and thus, FIDs of soliton equations, with Bäcklund transformations (1.2) being polynomial functions [6,7]. The main benefit of finding FIDs is that a PDE problem would be transformed into two ODE problems. Therefore, the existence of solutions can be guaranteed easily by using the theory of ODEs, and the plots of solutions can be made stably and simply as well.

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Naturally, one may ask what kind of PDEs can possess FIDs, and what kind of Frobenius integrable ODEs the PDEs can be transformed into. In this Letter, we will restrict these two "what" to polynomial type functions, but take rational functions for Bäcklund transformations from PDEs to ODEs. Two classes of examples of such PDEs possessing FIDs will be presented.

2. Polynomial type PDEs possessing FIDs

We consider a class of evolution equations of the form

$$u_t = P(u, u_x, u_{xx}, u_{xxx}), \tag{2.1}$$

where P is a differential polynomial in the dependent variable u. This class of equations includes many interesting wave equations such as the Burgers, KdV and potential KdV equations. Motivated by the KdV theory of symmetry constraints [7], we consider the case that

$$\Phi = (\phi, \psi)^T = \big(\phi(x,t), \psi(x,t)\big)^T \tag{2.2}$$

satisfies

$$\phi_x = \psi, \qquad \psi_x = \lambda\phi, \tag{2.3}$$

and

$$\phi_t = \Theta_1(\phi, \psi), \qquad \psi_t = \Theta_2(\psi, \psi), \tag{2.4}$$

where λ is a real parameter, and Θ_1 and Θ_2 are polynomials in ϕ and ψ to be determined. Obviously, the system (2.3) is a second-order ODE, which can be generated from the Schrödinger spectral problem with zero potential.

The two systems of ODEs, (2.3) and (2.4), need to be compatible, in order to generate solutions to the evolution equation (2.1). Evidently, we have

$$\begin{cases} \phi_{xt} = \psi_t = \Theta_2(\phi, \psi), \\ \phi_{tx} = \Theta_{1,\phi}\psi + \lambda\Theta_{1,\psi}\phi, \end{cases}$$

and

$$\begin{cases} \psi_{xt} = \lambda\phi_t = \lambda\Theta_1(\phi, \psi), \\ \psi_{tx} = \Theta_{2,\phi}\psi + \lambda\Theta_{2,\psi}\phi. \end{cases}$$

Therefore, the compatibility conditions, $\phi_{xt} = \phi_{tx}$ and $\psi_{xt} = \psi_{tx}$, lead to

$$\Theta_2(\phi, \psi) = \Theta_{1,\phi}\psi + \lambda\Theta_{1,\psi}\phi, \tag{2.5}$$

and further

$$\lambda\Theta_1(\phi, \psi) = \Theta_{1,\phi\phi}\psi^2 + 2\lambda\Theta_{1,\phi\psi}\phi\psi + \lambda^2\Theta_{1,\psi\psi}\phi^2 + \lambda(\Theta_{1,\phi}\phi + \Theta_{1,\psi}\psi). \tag{2.6}$$

Eq. (2.6) is the only condition on Θ_1, and (2.5) defines Θ_2 through Θ_1.

To determine PDEs of the form (2.1) possessing the specified FIDs, we first search for polynomial solutions Θ_1 to Eq. (2.6). Then fix a rational transformation $u = f(\phi, \psi)$, compute u_t, u_x, u_{xx} and u_{xxx} using two Frobenius integrable systems (2.3) and (2.4), and finally search for the polynomial type P such that the evolution equation (2.1) holds for the selected type of functions $u = f(\phi, \psi)$.

Procedure for finding PDEs and FIDS that the PDEs possess:

Step 1 Search for Θ_1 which satisfies (2.6), to obtain Frobenius integrable ODEs.

Step 2 Decide a Bäcklund transformation $u = f(\Phi) = f(\phi, \psi)$, and compute the partial derivatives u_t, u_x, u_{xx} and u_{xxx} in terms of the resulting Frobenius integrable ODEs.

Step 3 Solve $u_t = P(u, u_x, u_{xx}, u_{xxx})$ for P and Θ_1, to determine the PDEs and the associated Frobenius integrable ODEs.

Let us begin with the following class of functions for Θ_1:

$$\Theta_1(\phi, \psi) = \sum_{i=0}^{3}\sum_{j=0}^{3} c_{i,j}\phi^i\psi^j, \tag{2.7}$$

where c_{ij}'s are arbitrary constants. It is easy to know that among the functions defined by (2.7), we have the following solution to (2.6):

$$\Theta_1(\phi, \psi) = -\lambda c_{1,2}\phi^3 - \lambda c_{0,3}\phi^2\psi + c_{1,2}\phi\psi^2 + c_{0,3}\psi^3 + c_{1,0}\phi + c_{0,1}\psi. \tag{2.8}$$

Moreover, we will determine P among the following class of functions:

$$\begin{aligned} &P(u, u_x, u_{xx}, u_{xxx}) \\ &\quad = d_0u + d_1u_x + d_2u_{xx} + d_3u_{xxx} + d_{0,1}uu_x + d_{0,2}uu_{xx} \\ &\qquad + d_{0,3}uu_{xxx} + d_{1,1}u_xu_x + d_{1,2}u_xu_{xx} + d_{1,3}u_xu_{xxx}, \end{aligned} \tag{2.9}$$

where d_i's and $d_{i,j}$'s are also arbitrary constants. The following theorem gives the only two classes of examples so produced.

Theorem 2.1. (a) *If we take a Bäcklund transformation*

$$u = \frac{\psi}{\phi} = (\ln\phi)_x \tag{2.10}$$

from the Frobenius integrable systems (2.3) *and* (2.4) *to the evolution equation* (1.1), *then Θ_1 and P defined by* (2.8) *and* (2.9) *must be of the form*

$$\Theta_1(\phi, \psi) = -\lambda c_{1,2}\phi^3 + c_{1,2}\phi\psi^2 + c_{1,0}\phi + (d_1 - 2d_{0,2}\lambda + 4d_3\lambda)\psi, \tag{2.11}$$

$$\begin{aligned} &P(u, u_x, u_{xx}, u_{xxx}) \\ &\quad = d_1u_x + d_2u_{xx} + d_3u_{xxx} + 2(d_2 - 2\lambda d_{0,3})uu_x + d_{0,2}uu_{xx} \\ &\qquad + d_{0,3}uu_{xxx} + 2(3d_3 - d_{0,2})u_x^2 - 3d_{0,3}u_xu_{xx}, \end{aligned} \tag{2.12}$$

where $c_{1,2}$, $c_{1,0}$, d_1, d_2, d_3, $d_{0,2}$ and $d_{0,3}$ are arbitrary constants.

(b) *If we take a Bäcklund transformation*

$$u = \frac{\lambda\phi^2 - \psi^2}{\phi^2} = (\ln\phi)_{xx} \tag{2.13}$$

from the Frobenius integrable systems (2.3) *and* (2.4) *to the evolution equation* (1.1), *then Θ_1 and P defined by* (2.8) *and* (2.9)

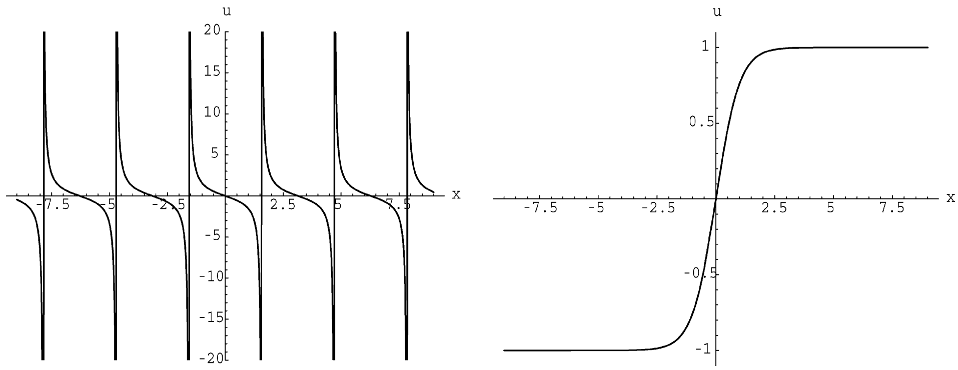

Fig. 1. Periodic-like solution with $\lambda > 0$ (left); 1-kink-like solution with $\lambda < 0$ (right).

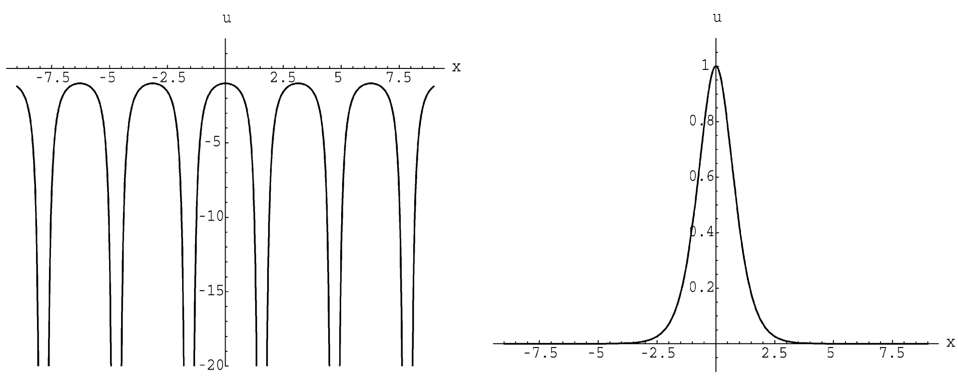

Fig. 2. 1-positon-like solution with $\lambda < 0$ (left); 1-soliton-like solution with $\lambda > 0$ (right).

must be of the form

$$\Theta_1(\phi, \psi) = -\lambda c_{1,2}\phi^3 + c_{1,2}\phi\psi^2 + c_{1,0}\phi + (d_1 + 4d_3\lambda)\psi, \tag{2.14}$$

$$\begin{aligned} &P(u, u_x, u_{xx}, u_{xxx}) \\ &\quad = \frac{4}{3}\lambda^2 d_{0,2}u + d_1u_x - \frac{1}{3}\lambda d_{0,2}u_{xx} + d_3u_{xxx} \\ &\qquad + 4(3d_3 + \lambda d_{0,3})uu_x + d_{0,2}uu_{xx} \\ &\qquad + d_{0,3}uu_{xxx} - \frac{3}{2}d_{0,2}u_x^2 - 2d_{0,3}u_xu_{xx}, \end{aligned} \tag{2.15}$$

where $c_{1,2}$, $c_{1,0}$, d_1, d_3 *and* $d_{0,3}$ *are arbitrary constants.*

A direct computation can verify all of the above results. The theorem tells us that under the Bäcklund transformations (2.10) and (2.13), two classes of PDEs defined by (1.1) with (2.12) and (2.15) can be transformed into Frobenius integrable ODEs (2.3) and (2.4), with Θ_1 being given by (2.11) and (2.14), and Θ_2 by (2.5). Those obtained PDEs possessing FIDS could be further classified by normalizing the involved arbitrary constants such as the constants $c_{1,2}$ and $c_{1,0}$ as in [8] and [9], and by a change of independent variables, the linear term u_x can also be incorporated into the linear term u_t.

In the above theorem, what we have considered is the first and second logarithmic derivatives of ϕ for the Bäcklund transformation on u, and the Hamiltonian structure on Frobenius integrable systems of ODEs are not required. The techniques used here is similar to but different from symmetry constraints in soliton theory [6,7], which lead to Hamiltonian structures. The resulting solutions by the specified Bäcklund transformations could be more diverse. We call the integral curves (or surfaces) determined by such solutions Frobenius integral curves (or surfaces), and some examples are plotted in the above figures.

Now, we discuss the exact solutions generated from the Bäcklund transformations (2.10) and (2.13), assuming that $d_1 = 0$ and $d_3 \neq 0$. Note that the above first class of nonlinear PDEs includes the Burgers equation $u_t = u_{xx} + 2uu_x$, the potential KdV equation $u_t = u_{xxx} + 6u_x^2$ and the following equation

$$u_t = u_{xxx} + \frac{3}{2}uu_{xx}, \tag{2.16}$$

while three particular members of the second class are the KdV equation $u_t = u_{xxx} + 12uu_x$ and the following two equations

$$u_t = u_{xxx} - \frac{3}{\lambda}uu_{xxx} + \frac{6}{\lambda}u_xu_{xx}, \tag{2.17}$$

$$u_t = \frac{4}{3}\lambda^2u - \frac{1}{3}\lambda u_{xx} + uu_{xx} - \frac{3}{2}u_x^2. \tag{2.18}$$

The solutions given by (2.10) contains the cotangent function type solutions when $\lambda < 0$, and 1-kink type solutions when $\lambda > 0$ (see Fig. 1). The waves represented by such solutions move in the negative x-axis direction when either $\lambda < 0$ and $d_{0,2} < 2d_3$, or $\lambda > 0$ and $d_{0,2} > 2d_3$; they move in the positive x-axis direction when either $\lambda < 0$ and $d_{0,2} > 2d_3$, or $\lambda > 0$ and $d_{0,2} < 2d_3$; and they are steady when $d_{0,2} = 2d_3$.

The solutions given by (2.13) contains 1-positon type solutions when $\lambda < 0$, and 1-soliton type solutions when $\lambda > 0$ (see Fig. 2). The waves corresponding to such solutions move in the negative x-axis direction when $\lambda < 0$; and they move in the positive x-axis direction when $\lambda > 0$.

Therefore, we have presented one family of nonlinear PDEs having the cotangent-like and 1-kink-like solutions like the potential KdV equation and the other family of nonlinear PDEs possessing the 1-positon-like and 1-soliton-like solutions like the KdV equation.

3. Concluding remarks

We have presented two classes of evolution equations possessing two classes of specified FIDs under two rational Bäcklund transformations of dependent variables. These rational transformations were generalized from logarithmic derivatives in soliton theory. The primary advantage of the resulting FIDs is to transform the nonlinear PDEs into the systems of Frobenius integrable ODEs with the third-order nonlinearity.

It will be interesting to analyze other integrable properties of new members from the two families of PDEs defined through (2.12) and (2.15). In particular, do Lax pairs and Hamiltonian structures exit for Eqs. (2.16), (2.17) and (2.18)?

We can generalize the dependent variable transformation to $u = f(\Phi, v)$, where v can be a solution to either the considered PDE or another PDE, and then the resulting transformations are auto-Bäcklund transformations or Miura type transformations. It is possible to classify such Bäcklund or Miura type transformations based on Frobenius integrable systems of ODEs (see, for example, [8,9]).

A higher-dimensional PDEs can have similar FIDs with more than two Frobenius integrable ODEs. For example, for a three-dimensional PDE, there will be three systems of Frobenius integrable ODEs. Moreover, we can set more equations in Frobenius integrable systems of ODEs. We plan to analyze these situations and present examples in a future publication.

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