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Pfaffianized systems for a generalized Kadomtsev–Petviashvili equation

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Abstract
A one-parameter class of Pfaffian extensions is made for a $(3+1)$-dimensional generalized Kadomtsev–Petviashvili (KP) equation by the Pfaffianization procedure. Wronski-type Pfaffian and Gramm-type Pfaffian solutions of the resulting Pfaffianized bilinear systems of the generalized KP equation are constructed. Under a dimensional reduction, our results yield a class of Pfaffianized bilinear systems for the KP equation, which contains the standard Pfaffianized bilinear KP system as a particular example. Two kinds of Pfaffian identities provide the basis for our analysis.

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1. Introduction

The Korteweg–de Vries (KdV) equation, the Boussinesq equation, the Toda lattice and the Kadomtsev–Petviashvili (KP) equation possess multi-soliton solutions and Hirota's direct method plays a crucial role in constructing soliton solutions [1–3]. Oriented toward the ease of use and capability of computer algebra systems, a multiple exp-function method was presented [4] to construct generic multi-exponential wave solutions, which works for more general nonlinear equations than bilinear equations [4, 5]. Interestingly, some nonlinear equations even possess linear subspaces of their solutions [6]. Moreover, a sufficient and necessary condition was presented, which tests whether a given Hirota bilinear equation possesses linear combination solutions of exponential waves [7].

It was further found that solitons, positons and complexitons could be expressed as Wronskian determinants [8–10]. For higher-dimensional soliton equations, there exist Grammian determinant solutions (briefly, Grammian solutions) [1], and the Grammian solutions to the KP equation were presented by Nakamura [11]. Since Pfaffians generalize determinants, it is natural to ask whether there exist Pfaffian solutions complementing Wronskian and Grammian solutions. The answer is very positive. Pfaffian solutions to the B-type KP equation were constructed for the first time by Hirota [12]. However, Pfaffians may not solve given nonlinear equations, and so one often needs to generalize the given equations to some coupled systems of soliton equations. The procedure for doing this [13] is now called Pfaffianization [14–16], and two kinds of Pfaffian solutions—Wronski-type and Gramm-type Pfaffian solutions—can be often constructed while doing Pfaffianization [17–20].

In this paper, we would like to study a (3+1)-dimensional generalized KP equation [6]

$$u_{xxxy}+3(u_xu_y)_x+u_{tx}+u_{ty}-u_{zz}=0, \tag{1.1}$$

which does not belong to the generalized KP and Boussinesq equations discussed in [21]

$$(u_{x_1x_1x_1}-6uu_{x_1})_{x_1}+\sum_{i,j=1}^{M}a_{ij}u_{x_ix_j}=0, \tag{1.2}$$

where $a_{ij}=\text{constant}$ and $M\in\mathbb{N}$. We will carry out Pfaffianization to extend the above generalized KP equation to a one-parameter class of Pfaffianized generalized KP systems so that Wronski-type and Gramm-type Pfaffian solutions can exist. All the generating functions for Pfaffian entries satisfy a linear system of integrable partial differential equations (another primary idea to find exact solutions is to let generating functions satisfy integrable ordinary differential equations; see e.g. [22, 23]). The resulting Pfaffianized generalized KP systems contain one parameter, and this characteristic implies that Pfaffianization does not have the uniqueness property. The fundamental Pfaffian identities will provide the basis for presenting Pfaffian solutions, and examples of generating functions for Pfaffian entries will be computed. A dimensional reduction of our results gives a one-parameter class of Pfaffianized systems for the KP equation, which contains the standard Pfaffianized KP system as a particular example.

2. Basic notation and Pfaffian identities

Let us recall some basics used in Pfaffianization. The Pfaffian of even order denoted by $(\alpha_1, \alpha_2, \ldots, \alpha_{2N})$ is defined by [24]

$$(\alpha_1, \alpha_2, \ldots, \alpha_{2N}) = \sum_{\sigma} \text{sign}(\sigma)(\alpha_{i_1}, \alpha_{i_2})(\alpha_{i_3}, \alpha_{i_4}) \cdots \times (\alpha_{i_{2N-1}}, \alpha_{i_{2N}}), \tag{2.1}$$

where the summation is taken over all permutations

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & 2N \\ i_1 & i_2 & \cdots & i_{2N} \end{pmatrix}$$

with

$$i_1 < i_2,\ i_3 < i_4,\ \ldots,\ i_{2N-1} < i_{2N}, \quad i_1 < i_3 < \cdots < i_{2N-1}$$

and $\text{sign}(\sigma) = \pm 1$ denotes the parity of the permutations σ. The elements (α_i, α_j) are called the Pfaffian entries satisfying

$$(\alpha_i, \alpha_j) = -(\alpha_j, \alpha_i). \tag{2.2}$$

The interchange of labels α_i and α_j changes the parity of each permutation σ in the sum, and thus the Pfaffian has the skew-symmetric property

$$(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_j, \ldots, \alpha_{2N}) = -(\alpha_1, \ldots, \alpha_j, \ldots, \alpha_i, \ldots, \alpha_{2N}), \tag{2.3}$$

where $1 \leqslant i < j \leqslant 2N$.

The Pfaffian also satisfies

$$(\alpha_1, \alpha_2, \ldots, \alpha_{2N})^2 = \det(a_{ij})_{1 \leqslant i,j \leqslant 2N}, \quad a_{ij} = (\alpha_i, \alpha_j) \tag{2.4}$$

and so, it is denoted conventionally by

$$(\alpha_1, \alpha_2, \ldots, \alpha_{2N}) = \left| \begin{matrix} a_{12} & a_{13} & \cdots & a_{1,2N} \\ & a_{23} & \cdots & a_{2,2N} \\ & & \ddots & \vdots \\ & & & a_{2N-1,2N} \end{matrix} \right| . \tag{2.5}$$

When $N = 1, 2$, the Pfaffians read

$$(\alpha_1, \alpha_2) = a_{12}, \quad (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Moreover, the Pfaffian obeys an expansion rule about a row:

$$(\alpha_1, \alpha_2, \ldots, \alpha_{2N}) = \sum_{j=1}^{2N} (\alpha_i, \alpha_j)\Gamma(i, j), \quad 1 \leqslant i \leqslant 2N \tag{2.6}$$

with the cofactor $\Gamma(i, j)$ being defined by

$$\Gamma(i, j) = (-1)^{i+j-1}(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_{2N}), \quad i < j,$$
$$\Gamma(i, j) = -\Gamma(i, j), \quad i > j, \quad \Gamma(i, i) = 0,$$

where $\hat{\alpha}_k$ means that the label α_k is omitted.

Our construction for Wronski-type and Gramm-type Pfaffian solutions is totally based on the following two fundamental Pfaffian identities:

$$\begin{aligned} &(a_1, a_2, a_3, a_4, 1, 2, \ldots, 2N)(1, 2, \ldots, 2N) \\ &= (a_1, a_2, 1, 2, \ldots, 2N)(a_3, a_4, 1, 2, \ldots, 2N) \\ &\quad - (a_1, a_3, 1, 2, \ldots, 2N)(a_2, a_4, 1, 2, \ldots, 2N) \\ &\quad + (a_1, a_4, 1, 2, \ldots, 2N)(a_2, a_3, 1, 2, \ldots, 2N) \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} &(a_1, 1, 2, \ldots, 2N-1)(a_2, a_3, a_4, 1, 2, \ldots, 2N-1) \\ &- (a_2, 1, 2, \ldots, 2N-1)(a_1, a_3, a_4, 1, 2, \ldots, 2N-1) \\ &+ (a_3, 1, 2, \ldots, 2N-1)(a_1, a_2, a_4, 1, 2, \ldots, 2N-1) \\ &- (a_4, 1, 2, \ldots, 2N-1)(a_1, a_2, a_3, 1, 2, \ldots, 2N-1) = 0, \end{aligned} \tag{2.8}$$

whose direct proofs can be found in [1, 13].

3. Pfaffianization and Wronski-type Pfaffian solutions

Let us consider the (3+1)-dimensional generalized KP equation [6]

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0. \tag{3.1}$$

This equation has resonant soliton solutions [6], Wronskian and Grammian solutions [25], interesting three-wave soliton-type solutions [26] and a bilinear Bäcklund transform [27]. There are other generalized higher-dimensional soliton equations formulated from decompositions of (2+1)-dimensional equations into (1+1)-dimensional equations and various determinant Darboux transformations and Wronskian solutions exist [28].

Under the dependent variable transformation

$$u = 2(\ln f)_x, \tag{3.2}$$

the above (3+1)-dimensional generalized KP equation (3.1) is mapped into a Hirota bilinear equation

$$(D_x^3 D_y + D_t D_x + D_t D_y - D_z^2) f \cdot f = 0, \tag{3.3}$$

where D_x, D_y, D_z and D_t are Hirota bilinear differential operators [1, 29]. This exactly gives

$$\begin{aligned} &(f_{xxxy} + f_{tx} + f_{ty} - f_{zz})f - 3f_{xxy}f_x + 3f_{xy}f_{xx} \\ &- f_y f_{xxx} - f_t f_x - f_t f_y + (f_z)^2 = 0 \end{aligned} \tag{3.4}$$

and it is similar to, but different from, a (3+1)-dimensional Hirota bilinear equation studied in [20]:

$$(3D_x D_z - 2D_y D_t - D_y D_x^3) f \cdot f = 0.$$

We would like to carry out Pfaffianization for the above (3+1)-dimensional generalized KP equation (3.1). More specifically, we want to use the two fundamental Pfaffian identities, listed in section 2, to establish Pfaffian solution structures for the (3+1)-dimensional Hirota bilinear generalized KP equation (3.3).

Let us start to consider Wronski-type Pfaffian solutions

$$f = f_N = (1, 2, \ldots, 2N), \tag{3.5}$$

whose Pfaffian entries satisfy

$$\begin{cases} \dfrac{\partial}{\partial x}(i, j) = (i+1, j) + (i, j+1), \\ \dfrac{\partial}{\partial y}(i, j) = \gamma_1[(i+1, j) + (i, j+1)], \\ \dfrac{\partial}{\partial z}(i, j) = \gamma_2[(i+2, j) + (i, j+2)], \\ \dfrac{\partial}{\partial t}(i, j) = \gamma_3[(i+3, j) + (i, j+3)] \end{cases} \tag{3.6}$$

with

$$\gamma_1 = -\frac{a^2}{3}, \quad \gamma_2 = a, \quad \gamma_3 = \frac{4a^2}{3-a^2}, \tag{3.7}$$

where $a \neq 0, \pm\sqrt{3}$, but could be any other real number.

Under the condition (3.6), we have the differential rules for the Wronski-type Pfaffian

$$\begin{cases} \dfrac{\partial}{\partial x}(i_1, i_2, \ldots, i_{2N}) = \displaystyle\sum_{k=1}^{2N}(i_1, i_2, \ldots, i_k+1, \ldots, i_{2N}), \\ \dfrac{\partial}{\partial y}(i_1, i_2, \ldots, i_{2N}) = \displaystyle\sum_{k=1}^{2N}\gamma_1(i_1, i_2, \ldots, i_k+1, \ldots, i_{2N}), \\ \dfrac{\partial}{\partial z}(i_1, i_2, \ldots, i_{2N}) = \displaystyle\sum_{k=1}^{2N}\gamma_2(i_1, i_2, \ldots, i_k+2, \ldots, i_{2N}), \\ \dfrac{\partial}{\partial t}(i_1, i_2, \ldots, i_{2N}) = \displaystyle\sum_{k=1}^{2N}\gamma_3(i_1, i_2, \ldots, i_k+3, \ldots, i_{2N}), \end{cases} \tag{3.8}$$

which are completely similar to the differential rules for determinants and can be deduced by using (2.6). Based on these rules, we can compute various derivatives of the Pfaffian $f_N = (1, 2, \ldots, 2N)$ with respect to the variables x, y, z, t:

$$\begin{aligned}
f_{N,x} &= (1, 2, \ldots, 2N-1, 2N+1), \\
f_{N,xx} &= (1, 2, \ldots, 2N-1, 2N+2) \\
&\quad + (1, 2, \ldots, 2N-2, 2N, 2N+1), \\
f_{N,xxx} &= (1, 2, \ldots, 2N-1, 2N+3) \\
&\quad + 2(1, 2, \ldots, 2N-2, 2N, 2N+2) \\
&\quad + (1, 2, \ldots, 2N-3, 2N-1, 2N, 2N+1), \\
f_{N,y} &= \gamma_1(1, 2 \ldots, 2N-1, 2N+1), \\
f_{N,xy} &= \gamma_1[(1, 2, \ldots, 2N-1, 2N+2) \\
&\quad + (1, 2, \ldots, 2N-2, 2N, 2N+1)], \\
f_{N,xxy} &= \gamma_1[(1, 2, \ldots, 2N-1, 2N+3) \\
&\quad + 2(1, 2, \ldots, 2N-2, 2N, 2N+2) \\
&\quad + (1, 2, \ldots, 2N-3, 2N-1, 2N, 2N+1)], \\
f_{N,xxxy} &= \gamma_1[(1, 2, \ldots, 2N-1, 2N+4) \\
&\quad + 3(1, 2, \ldots, 2N-2, 2N, 2N+3) \\
&\quad + 3(1, 2, \ldots, 2N-3, 2N-1, 2N, 2N+2) \\
&\quad + 2(1, 2, \ldots, 2N-2, 2N+1, 2N+2) \\
&\quad + (1, 2, \ldots, 2N-4, 2N-2, 2N-1, 2N, 2N+1)], \\
f_{N,z} &= \gamma_2[(1, 2, \ldots, 2N-1, 2N+2) \\
&\quad - (1, 2, \ldots, 2N-2, 2N, 2N+1)], \\
f_{N,zz} &= \gamma_2^2[(1, 2, \ldots, 2N-1, 2N+4) \\
&\quad + 2(1, 2, \ldots, 2N-2, 2N+1, 2N+2) \\
&\quad - (1, 2, \ldots, 2N-3, 2N-1, 2N, 2N+2) \\
&\quad - (1, 2, \ldots, 2N-2, 2N, 2N+3) \\
&\quad + (1, 2, \ldots, 2N-4, 2N-2, 2N-1, 2N, 2N+1)], \\
f_{N,t} &= \gamma_3[(1, 2, \ldots, 2N-1, 2N+3) \\
&\quad - (1, 2, \ldots, 2N-2, 2N, 2N+2) \\
&\quad + (1, 2, \ldots, 2N-3, 2N-1, 2N, 2N+1)], \\
f_{N,tx} &= \gamma_3[(1, 2, \ldots, 2N-1, 2N+4) \\
&\quad - (1, 2, \ldots, 2N-2, 2N+1, 2N+2), \\
&\quad + (1, 2, \ldots, 2N-4, 2N-2, 2N-1, 2N, 2N+1)], \\
f_{N,ty} &= \gamma_1\gamma_3[(1, 2, \ldots, 2N-1, 2N+4) \\
&\quad - (1, 2, \ldots, 2N-2, 2N+1, 2N+2), \\
&\quad + (1, 2, \ldots, 2N-4, 2N-2, 2N-1, 2N, 2N+1)].
\end{aligned}$$

Therefore, under the selection (3.7) of γ_1, γ_2 and γ_3, we have

$$\begin{aligned}
&f_{N,xxxy} + f_{N,tx} + f_{N,ty} - f_{N,zz} = -4a^2(1, 2, \ldots, 2N-2, \\
&\quad 2N+1, 2N+2) \\
&-3f_{N,xxy}f_{N,x} - f_{N,y}f_{N,xxx} - f_{N,t}f_{N,x} - f_{N,t}f_{N,y} \\
&\quad = 4a^2(1, 2, \ldots, 2N-1, 2N+1)(1, 2 \ldots, 2N-2, \\
&\quad 2N, 2N+2) \\
&3f_{N,xy}f_{N,xx} + (f_{N,z})^2 \\
&\quad = -4a^2(1, 2, \ldots, 2N-1, 2N+2)(1, 2, \ldots, 2N-2, \\
&\quad 2N, 2N+1)
\end{aligned}$$

and further obtain

$$\begin{aligned}
&(D_x^3D_y + D_tD_x + D_tD_y - D_z^2)f_N \cdot f_N \\
&= (f_{N,xxxy}+f_{N,tx}+f_{N,ty}-f_{N,zz})f_N - 3f_{N,xxy}f_{N,x} + 3f_{N,xy}f_{N,xx} \\
&\quad - f_{N,y}f_{N,xxx} - f_{N,t}f_{N,x} - f_{N,t}f_{N,y} + (f_{N,z})^2 \\
&= -4a^2[(1, 2, \ldots, 2N)(1, 2, \ldots, 2N-2, 2N+1, 2N+2) \\
&\quad -(1, 2, \ldots, 2N-1, 2N+1)(1, 2, \ldots, 2N-2, 2N, 2N+2) \\
&\quad +(1, 2, \ldots, 2N-1, 2N+2)(1, 2, \ldots, 2N-2, 2N, 2N+1)].
\end{aligned}$$

This is not equal to zero. By employing a Pfaffian identity of type (2.7), the right-hand side of the above equation is equal to

$$-4a^2(1,2,\ldots,2N+2)(1,2,\ldots,2N-2).$$

Therefore, if we introduce

$$g_N=(1,2,\ldots,2N-2),\ h_N=(1,2,\ldots,2N+2), \tag{3.9}$$

which have order of two less and two more, respectively, than that of $f_N=(1,2,\ldots,2N)$, then we have

$$(D_x^3D_y+D_tD_x+D_tD_y-D_z^2)f_N\cdot f_N=-4a^2g_Nh_N.$$

By using the Pfaffian identity of type (2.8):

$$\begin{aligned}&(\bullet,2N+1)(\bullet,2N-2,2N-1,2N)\\&-(\bullet,2N)(\bullet,2N-2,2N-1,2N+1)\\&+(\bullet,2N-1)(\bullet,2N-2,2N,2N+1)\\&-(\bullet,2N-2)(\bullet,2N-1,2N,2N+1)=0,\end{aligned}$$

where the abbreviated notation $\bullet$ stands for the list of indices $1,2,\ldots,2N-3$ common to each Pfaffian, and the Pfaffian identity of type (2.8):

$$\begin{aligned}&(\star,2N+3)(\star,2N,2N+1,2N+2)\\&-(\star,2N+2)(\star,2N,2N+1,2N+3)\\&+(\star,2N+1)(\star,2N,2N+2,2N+3)\\&-(\star,2N)(\star,2N+1,2N+2,2N+3)=0,\end{aligned}$$

where the abbreviated notation $\star$ stands for the list of indices $1,2,\ldots,2N-1$ common to each Pfaffian, we find g_N and h_N satisfy

$$\begin{aligned}&(D_x^3+\alpha D_t+\beta D_xD_z)g_N\cdot f_N=0,\\&(D_x^3+\alpha D_t-\beta D_xD_z)h_N\cdot f_N=0\end{aligned}$$

with the constants α and β being defined by

$$\alpha=\frac{3-a^2}{2a^2},\quad \beta=\frac{3}{a}. \tag{3.10}$$

Therefore, we have the following result on Wronski-type Pfaffian solutions.

Theorem 3.1. *Let a be any real number except $0,\pm\sqrt{3}$. If the Pfaffian entries (i,j) satisfy (3.6) where the constants γ_i are defined by (3.7), then*

$$\begin{cases}f=f_N=(1,2,\ldots,2N),\\ g=g_N=(1,2,\ldots,2N-2),\\ h=h_N=(1,2,\ldots,2N+2)\end{cases} \tag{3.11}$$

solves the following system of bilinear equations

$$\begin{cases}(D_x^3D_y+D_tD_x+D_tD_y-D_z^2)f\cdot f=-4a^2gh,\\ (D_x^3+\alpha D_t+\beta D_xD_z)g\cdot f=0,\\ (D_x^3+\alpha D_t-\beta D_xD_z)h\cdot f=0,\end{cases} \tag{3.12}$$

where α and β are defined by (3.10).

The system (3.12) is called a Pfaffianized bilinear system of the generalized KP equation (3.1). The differential condition (3.6) has various solutions and one of them is the following:

$$(i,j)=\sum_{k=1}^{M}(\phi_k^{(i)}\psi_k^{(j)}-\phi_k^{(j)}\psi_k^{(i)}) \tag{3.13}$$

with $M\in\mathbb{N}$ being arbitrary and ϕ_k and ψ_k satisfying

$$\chi_{k,y}=\gamma_1\chi_k^{(1)},\quad \chi_{k,z}=\gamma_2\chi_k^{(2)},\quad \chi_{k,t}=\gamma_3\chi_k^{(3)}, \tag{3.14}$$

where $\chi^{(i)}$ stands for the ith derivatives of χ with respect to x. It is easy to see that $(i,j)=-(j,i)$ and all (i,j) satisfy the conditions in (3.6). Examples of the functions ϕ_k and ψ_k can be the following:

$$\begin{cases}\phi_i=\sum_{j=1}^{p}d_{ij}e^{\eta_{ij}},\ \ \eta_{ij}=k_{ij}x+\gamma_1k_{ij}y+\gamma_2k_{ij}^2z+\gamma_3k_{ij}^3t,\\ \psi_i=\sum_{j=1}^{q}e_{ij}e^{\zeta_{ij}}\zeta_{ij}=l_{ij}x+\gamma_1l_{ij}y+\gamma_2l_{ij}^2z+\gamma_3l_{ij}^3t,\end{cases} \tag{3.15}$$

where d_{ij} and e_{ij} are the free parameters and p and q are the arbitrary natural numbers.

If we introduce the dependent variable transformation,

$$u=2(\ln f)_x,\quad v=\frac{g}{f},\quad w=\frac{h}{f}, \tag{3.16}$$

then the coupled (3+1)-dimensional system of bilinear equations, (3.12), is mapped into

$$\begin{cases}u_{xxxy}+3(u_xu_y)_x+u_{tx}+u_{ty}-u_{zz}+4a^2(vw)_x=0,\\ \alpha v_t+3u_xv_x+v_{xxx}+\beta(v_{xz}+vu_z)=0,\\ \alpha w_t+3u_xw_x+w_{xxx}-\beta(w_{xz}+wu_z)=0,\end{cases} \tag{3.17}$$

which is called a Pfaffianized system of the (3+1)-dimensional generalized KP equation (3.1). Here, a can take any real number except $0,\pm\sqrt{3}$, and thus this shows that there may exist infinitely many Pfaffianized systems for a given soliton equation.

We also point out that the dependent variable transformation adopted above is different from the one used for the KP equation and each in the above resulting class of Pfaffianized systems is local unlike the standard Pfaffianized KP system [1].

If we take a dimensional reduction $y=\gamma_1x$ as we did while deducing Pfaffian solutions, then the (3+1)-dimensional generalized KP equation (3.1) becomes the (2+1)-dimensional KP equation, and the above results give a one-parameter class of Pfaffianized bilinear systems for the KP equation, together with their Pfaffian solutions. If we further take $a=3$, then under the transformation

$$t\mapsto 2t,\quad g\mapsto\frac{\sqrt{2}}{2}g,\quad h\mapsto\frac{\sqrt{2}}{2}h$$

the corresponding Pfaffianized bilinear system (3.12) becomes the standard Pfaffianized bilinear KP system [1, 13]:

$$\begin{cases}(D_x^4-4D_tD_x+3D_z^2)f\cdot f=24gh,\\ (D_x^3+2D_t+3D_xD_z)g\cdot f=0,\\ (D_x^3+2D_t-3D_xD_z)h\cdot f=0.\end{cases} \tag{3.18}$$

However, for other values of a, there is no scaling transformation of dependent and independent variables which puts the resulting Pfaffianized bilinear system (3.12) into the above standard Pfaffianized one (3.18).

4. Gramm-type Pfaffian solutions

In this section, we would like to establish a class of Gramm-type Pfaffian solutions to the (3+1)-dimensional Pfaffianized system (3.17). Let us introduce the following Gramm-type Pfaffians:

$$\begin{cases} f = f_N = (1, 2, \ldots, 2N), \\ g = g_N = (c_1, c_0, 1, 2, \ldots, 2N), \\ h = h_N = (d_0, d_1, 1, 2, \ldots, 2N), \end{cases} \tag{4.1}$$

where the different types of Pfaffian entries are defined by

$$\begin{cases} (i,j) = c_{ij} + \displaystyle\int^x (\phi_i\psi_j - \phi_j\psi_i)\mathrm{d}x, \ c_{ij} = -c_{ji}, \ c_{ij} = \text{const.}, \\ (d_n, i) = \dfrac{\partial^n}{\partial x^n}\phi_i, \ (c_n, i) = \dfrac{\partial^n}{\partial x^n}\psi_i, \\ (d_m, d_n) = (c_m, c_n) = (c_m, d_n) = 0. \end{cases} \tag{4.2}$$

In the above definition of (i, j), the lower limit of integration is chosen so that the functions ϕ_i, ψ_j and their derivatives are zero at the lower limit.

Theorem 4.1. *Let ϕ_i and ψ_j satisfy*

$$\begin{cases} \phi_{i,y} = \gamma_1\phi_{i,x}, & \psi_{i,y} = \gamma_1\psi_{i,x}, \\ \phi_{i,z} = \gamma_2\phi_{i,xx}, & \psi_{i,z} = -\gamma_2\psi_{i,xx}, \\ \phi_{i,t} = \gamma_3\phi_{i,xxx}, & \psi_{i,t} = \gamma_3\psi_{i,xxx}, \end{cases} \tag{4.3}$$

where the constants γ_i are defined by (3.7). Then f_N, g_N and h_N defined by (4.1) and (4.2) solve the Pfaffianized bilinear system (3.12).

Proof. Using the condition (4.3), we first have the derivatives of the Pfaffian entries

$$\begin{aligned}
\frac{\partial}{\partial x}(i,j) &= \phi_i\psi_j - \phi_j\psi_i = (c_0, d_0, i, j), \\
\frac{\partial}{\partial y}(i,j) &= \int^x (\phi_{i,y}\psi_j + \phi_i\psi_{j,y} - \phi_{j,y}\psi_i - \phi_j\psi_{i,y})\,\mathrm{d}x \\
&= \gamma_1\int^x (\phi_{i,x}\psi_j + \phi_i\psi_{j,x} - \phi_{j,x}\psi_i - \phi_j\psi_{i,x})\,\mathrm{d}x \\
&= \gamma_1(\phi_i\psi_j - \phi_j\psi_i) = \gamma_1(c_0, d_0, i, j), \\
\frac{\partial}{\partial z}(i,j) &= \int^x (\phi_{i,z}\psi_j + \phi_i\psi_{j,z} - \phi_{j,z}\psi_i - \phi_j\psi_{i,z})\,\mathrm{d}x \\
&= \gamma_2\int^x (\phi_{i,xx}\psi_j - \phi_i\psi_{j,xx} - \phi_{j,xx}\psi_i + \phi_j\psi_{i,xx})\,\mathrm{d}x \\
&= \gamma_2(\phi_{i,x}\psi_j - \phi_i\psi_{j,x} - \phi_{j,x}\psi_i + \phi_j\psi_{i,x}) \\
&= \gamma_2[(c_0, d_1, i, j) - (c_1, d_0, i, j)], \\
\frac{\partial}{\partial t}(i,j) &= \int^x (\phi_{i,t}\psi_j + \phi_i\psi_{j,t} - \phi_{j,t}\psi_i - \phi_j\psi_{i,t})\,\mathrm{d}x \\
&= \gamma_3\int^x (\phi_{i,xxx}\psi_j + \phi_i\psi_{j,xxx} - \phi_{j,xxx}\psi_i - \phi_j\psi_{i,xxx})\,\mathrm{d}x \\
&= \gamma_3(\phi_{i,xx}\psi_j - \phi_{j,xx}\psi_i - \phi_{i,x}\psi_{j,x} + \phi_{j,x}\psi_{i,x} + \phi_i\psi_{j,xx} \\
&\quad - \phi_j\psi_{i,xx}) \\
&= \gamma_3[(c_0, d_2, i, j) - (c_1, d_1, i, j) + (c_2, d_0, i, j)].
\end{aligned}$$

Then, we can develop differential rules for Pfaffians, and compute various derivatives of the Gramm-type Pfaffians $f_N = (1, 2, \ldots, 2N)$ with respect to the variables x, y, z, t as follows:

$$\begin{aligned}
f_{N,x} &= (c_0, d_0, \bullet), \\
f_{N,xx} &= (c_0, d_1, \bullet) + (c_1, d_0, \bullet), \\
f_{N,xxx} &= (c_0, d_2, \bullet) + 2(c_1, d_1, \bullet) + (c_2, d_0, \bullet), \\
f_{N,y} &= \gamma_1(c_0, d_0, \bullet), \\
f_{N,xy} &= \gamma_1[(c_0, d_1, \bullet) + (c_1, d_0, \bullet)], \\
f_{N,xxy} &= \gamma_1[(c_0, d_2, \bullet) + 2(c_1, d_1, \bullet) + (c_2, d_0, \bullet)], \\
f_{N,xxxy} &= \gamma_1[(c_0, d_3, \bullet) + 3(c_1, d_2, \bullet) + 2(c_0, d_0, c_1, d_1, \bullet) \\
&\qquad + 3(c_2, d_1, \bullet) + (c_3, d_0, \bullet)], \\
f_{N,z} &= \gamma_2[(c_0, d_1, \bullet) - (c_1, d_0, \bullet)], \\
f_{N,zz} &= \gamma_2^2[(c_0, d_3, \bullet) - (c_1, d_2, \bullet) + 2(c_0, d_0, c_1, d_1, \bullet) \\
&\qquad - (c_2, d_1, \bullet) + (c_3, d_0, \bullet)], \\
f_{N,t} &= \gamma_3[(c_0, d_2, \bullet) - (c_1, d_1, \bullet) + (c_2, d_0, \bullet)], \\
f_{N,tx} &= \gamma_3[(c_0, d_3, \bullet) - (c_0, d_0, c_1, d_1, \bullet) + (c_3, d_0, \bullet)], \\
f_{N,ty} &= \gamma_1\gamma_3[(c_0, d_3, \bullet) - (c_0, d_0, c_1, d_1, \bullet) + (c_3, d_0, \bullet)],
\end{aligned}$$

where the abbreviated notation $\bullet$ denotes the list of indices $1, 2, \ldots, 2N$ common to each Pfaffian.

Noting the selection (3.7) of γ_1, γ_2 and γ_3, we can now compute

$$\begin{aligned}
&f_{N,xxxy} + f_{N,tx} + f_{N,ty} - f_{N,zz} = -4a^2(c_0, d_0, c_1, d_1, \bullet), \\
&\quad -3f_{N,xxy}f_{N,x} - f_{N,y}f_{N,xxx} - f_{N,t}f_{N,x} - f_{N,t}f_{N,y} \\
&\quad = 4a^2(c_0, d_0, \bullet)(c_1, d_1, \bullet), \\
&\quad 3f_{N,xy}f_{N,xx} + (f_{N,z})^2 = -4a^2(c_1, d_0, \bullet)(c_0, d_1, \bullet)
\end{aligned}$$

and further obtain

$$\begin{aligned}
&(D_x^3D_y + D_tD_x + D_tD_y - D_z^2)f_N \cdot f_N \\
&= (f_{N,xxxy} + f_{N,tx} + f_{N,ty} - f_{N,zz})f_N - 3f_{N,xxy}f_{N,x} \\
&\quad + 3f_{N,xy}f_{N,xx} - f_{N,y}f_{N,xxx} - f_{N,t}f_{N,x} - f_{N,t}f_{N,y} + (f_{N,z})^2 \\
&= -4a^2[(\bullet)(c_0, d_0, c_1, d_1, \bullet) - (c_0, d_0, \bullet)(c_1, d_1, \bullet) \\
&\quad + (c_1, d_0, \bullet)(c_0, d_1, \bullet)] \\
&= 4a^2(c_0, c_1, \bullet)(d_0, d_1, \bullet) = -4a^2 g_N h_N.
\end{aligned}$$

This last but one equality is nothing but the Pfaffian identity of type (2.7).

Similarly, noting the selection (3.10) of α and β, we can have

$$
\begin{aligned}
&(D_x^3+\alpha D_t+\beta D_xD_z)g_N\cdot f_N\\
&=(g_{N,xxx}+\alpha g_{N,t}+\beta g_{N,xz})f_N-(3g_{N,xx}+\beta g_{N,z})f_{N,x}\\
&\quad+g_{N,x}(3f_{N,xx}-\beta f_{N,z})-g_N(f_{N,xxx}+\alpha f_{N,t}-\beta f_{N,xz})\\
&=6[(\bullet)(c_2,c_1,c_0,d_0,\bullet)-(c_2,c_1,\bullet)(c_0,d_0,\bullet)\\
&\quad+(c_2,c_0,\bullet)(c_1,d_0,\bullet)-(c_1,c_0,\bullet)(c_2,d_0,\bullet)]=0.
\end{aligned}
$$

This last equality is nothing but the Pfaffian identity of type (2.7). The third equation in (3.12) is obtained by interchanging c and d in the above equation. Therefore, we have shown that f_N, g_N and h_N defined by (4.1) and (4.2) solve the Pfaffianized bilinear system (3.12), under the condition of (4.3). □

Since the system (4.3) is linear, examples of generating functions for the Pfaffian entries can be easily computed as follows:

$$
\begin{cases}
\phi_i=\displaystyle\sum_{j=1}^{p}d_{ij}\mathrm{e}^{\eta_{ij}},\ \ \eta_{ij}=k_{ij}x+\gamma_1k_{ij}y+\gamma_2k_{ij}^2z+\gamma_3k_{ij}^3t,\\
\psi_j=\displaystyle\sum_{i=1}^{q}e_{ji}\mathrm{e}^{\zeta_{ji}},\ \ \zeta_{ji}=l_{ji}x+\gamma_1l_{ji}y-\gamma_2l_{ji}^2z+\gamma_3l_{ji}^3t,
\end{cases}
\tag{4.4}
$$

where d_{ij}, e_{ji}, k_{ij} and l_{ji} are the free parameters and p and q are the two arbitrary natural numbers. In particular, the case of $p=q=1$ leads to the following specific solutions:

$$
\begin{cases}
\phi_i=\mathrm{e}^{\eta_i},\ \ \eta_i=k_ix+\gamma_1k_iy+\gamma_2k_i^2z+\gamma_3k_i^3t,\\
\psi_j=\mathrm{e}^{\zeta_j},\ \ \zeta_j=l_jx+\gamma_1l_jy-\gamma_2l_j^2z+\gamma_3l_j^3t,
\end{cases}
\tag{4.5}
$$

where k_i and l_j are the free parameters.

5. Conclusions and discussions

We have made a one-parameter class of Pfaffian extensions

$$
\begin{cases}
u_{xxxy}+3(u_xu_y)_x+u_{tx}+u_{ty}-u_{zz}+4a^2(vw)_x=0,\\
\alpha v_t+3u_xv_x+v_{xxx}+\beta(v_{xz}+vu_z)=0,\\
\alpha w_t+3u_xw_x+w_{xxx}-\beta(w_{xz}+wu_z)=0
\end{cases}
\tag{5.1}
$$

for the (3+1)-dimensional generalized KP equation

$$
u_{xxxy}+3(u_xu_y)_x+u_{tx}+u_{ty}-u_{zz}=0 \tag{5.2}
$$

by using the fundamental Pfaffian identities (2.7) and (2.8), where a can be any real number, but not 0 and $\pm\sqrt{3}$, and

$$
\alpha=\frac{3-a^2}{2a^2},\qquad \beta=\frac{3}{a}. \tag{5.3}
$$

Under the dependent variable transformation,

$$
u=2(\ln f)_x,\qquad v=\frac{g}{f},\qquad w=\frac{h}{f}, \tag{5.4}
$$

the above one-parameter class of Pfaffianized systems becomes

$$
\begin{cases}
(D_x^3D_y+D_tD_x+D_tD_y-D_z^2)f\cdot f=-4a^2gh,\\
(D_x^3+\alpha D_t+\beta D_xD_z)g\cdot f=0,\\
(D_x^3+\alpha D_t-\beta D_xD_z)h\cdot f=0.
\end{cases}
\tag{5.5}
$$

It is worth mentioning that the above dependent variable transformation is different from the one used for the KP equation [1]; the resulting one-parameter class of Pfaffianized systems is local, but the standard Pfaffianized KP system is non-local [1].

Theorems 3.1 and 4.1 present the main results on Wronski-type and Gramm-type Pfaffian solutions. The first result is that each Pfaffianized bilinear system in (5.5) has the following Wronski-type Pfaffian solutions

$$
\begin{cases}
f=f_N=(1,2,\ldots,2N),\\
g=g_N=(1,2,\ldots,2N-2),\\
h=h_N=(1,2,\ldots,2N+2),
\end{cases}
\tag{5.6}
$$

whose Pfaffian entries satisfy

$$
\begin{cases}
\dfrac{\partial}{\partial x}(i,j)=(i+1,j)+(i,j+1),\\
\dfrac{\partial}{\partial y}(i,j)=\gamma_1[(i+1,j)+(i,j+1)],\\
\dfrac{\partial}{\partial z}(i,j)=\gamma_2[(i+2,j)+(i,j+2)],\\
\dfrac{\partial}{\partial t}(i,j)=\gamma_3[(i+3,j)+(i,j+3)]
\end{cases}
$$

with

$$
\gamma_1=-\frac{a^2}{3},\qquad \gamma_2=a,\qquad \gamma_3=\frac{4a^2}{3-a^2}. \tag{5.7}
$$

The second result is that each Pfaffianized bilinear system in (5.5) has the following Gramm-type Pfaffian solutions:

$$
\begin{cases}
f=f_N=(1,2,\ldots,2N),\\
g=g_N=(c_1,c_0,1,2,\ldots,2N),\\
h=h_N=(d_0,d_1,1,2,\ldots,2N),
\end{cases}
\tag{5.8}
$$

where the different types of Pfaffian entries are defined by

$$
\begin{cases}
(i,j)=c_{ij}+\displaystyle\int^x(\phi_i\psi_j-\phi_j\psi_i\,\mathrm{d}x,\ \ c_{ij}=-c_{ji},\ \ c_{ij}=\mathrm{const.},\\
(d_n,i)=\dfrac{\partial^n}{\partial x^n}\phi_i,\ \ (c_n,i)=\dfrac{\partial^n}{\partial x^n}\psi_i,\\
(d_m,d_n)=(c_m,c_n)=(c_m,d_n)=0
\end{cases}
$$

with ϕ_i and ψ_j satisfying

$$
\begin{cases}
\phi_{i,y}=\gamma_1\phi_{i,x},\ \ \psi_{i,y}=\gamma_1\psi_{i,x},\\
\phi_{i,z}=\gamma_2\phi_{i,xx},\ \ \psi_{i,z}=-\gamma_2\psi_{i,xx},\\
\phi_{i,t}=\gamma_3\phi_{i,xxx},\ \ \psi_{i,t}=\gamma_3\psi_{i,xxx}.
\end{cases}
$$

Examples of Wronski-type and Gramm-type Pfaffian entries are given. The involvement of the parameter a also implies that the Pfaffianization procedure does possess the uniqueness property.

Our Pfaffian solutions are essentially (2+1)-dimensional due to the dimensional reduction $y = \gamma_1 x$ we imposed. Nevertheless, under the reduction $y = \gamma_1 x$, the first equation in each resulting Pfaffianized bilinear system in (5.5) becomes

$$[a^2 D_x^4 - (3-a^2) D_t D_x + 3D_z^2] f \cdot f = 12a^2 gh \qquad (5.9)$$

and under the condition of $a \neq 0$ and $a \neq \pm\sqrt{3}$, this equation with $g = 0$ or $h = 0$ can be transformed into the standard Hirota bilinear KP equation

$$(D_x^4 - 4D_t D_x + 3D_z^2) f \cdot f = 0.$$

Therefore, all the resulting Pfaffianized bilinear systems in the above one-parameter class (5.5) present Pfaffianized bilinear systems for the KP equation. As we pointed out, the case of $a = 3$ leads to the standard Pfaffianized bilinear KP system [1, 13]

$$\begin{cases} (D_x^4 - 4D_t D_x + 3D_z^2) f \cdot f = 24gh, \\ (D_x^3 + 2D_t + 3D_x D_z) g \cdot f = 0, \\ (D_x^3 + 2D_t - 3D_x D_z) h \cdot f = 0 \end{cases} \qquad (5.10)$$

through the transformation

$$t \mapsto 2t, \quad g \mapsto \frac{\sqrt{2}}{2} g, \quad h \mapsto \frac{\sqrt{2}}{2} h.$$

But for other values of a, a scaling transformation of dependent and independent variables cannot put any obtained Pfaffianized bilinear system of the KP equation into the standard Pfaffianized one (5.10).

On the other hand, in theorems 3.1 and 4.1, we only considered a kind of specific sufficient conditions: (3.14) and (4.3), albeit one parameter a in the conditions. There should exist more general conditions involving the combined equations for Wronskian and Grammian solutions. Such conditions were presented for Wronskian solutions of the KdV equation [30], the Boussinesq equation [31, 32], the Toda lattice [33] and the two-dimensional Toda lattice [34] and for Grammian solutions of the KP equation [35]. It should be interesting to search for such a generalized condition consisting of combined equations for the above (3+1)-dimensional generalized KP equation (5.2).

Another interesting question for us is whether there exist other kinds of Pfaffian solutions, besides Wronski-type Pfaffian and Gramm-type Pfaffian solutions. Again, we should start with the Pfaffian indentities to answer what other kinds of Pfaffians will lead nonlinear equations, for example, generalized bilinear equations [36], to the Pfaffian identities.

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