

Semi-direct sums of Lie algebras and continuous integrable couplings

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Abstract

A relation between semi-direct sums of Lie algebras and integrable couplings of continuous soliton equations is presented, and correspondingly, a feasible way to construct integrable couplings is furnished. A direct application to the AKNS spectral problem leads to a novel hierarchy of integrable couplings of the AKNS hierarchy of soliton equations. It is also indicated that the study of integrable couplings using semi-direct sums of Lie algebras is an important step towards complete classification of integrable systems.

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1. Introduction

τ -symmetry algebras of integrable systems are a realization of Virasoro algebras in soliton theory. Such τ -symmetry algebras and their corresponding Virasoro algebras are all semi-direct sums of Lie algebras, of which master symmetries play the non-ideal part in the semi-direct sums. While studying relations between Virasoro algebras and hereditary operators, the problem of integrable couplings was formulated and related research was initialized [1,2]. A few ways to construct integrable couplings are then presented by using perturbations [1–3], enlarging spectral problems [4,5], and creating new loop algebras [6,7].

The problem of integrable couplings can be expressed as follows [2]: *For a given integrable system, how can we construct a non-trivial system of differential equations which is still integrable and includes the original integrable system as a sub-system?* Note that a change of orders of dependent variables

does not lose integrability. Therefore, up to a permutation, for a given integrable system of evolution type $u_t = K(u)$, we actually need to construct a new bigger triangular integrable system as follows:

$$\begin{cases} u_t = K(u), \\ v_t = S(u, v). \end{cases}$$

The vector-valued function S should satisfy the non-triviality condition $\partial S / \partial [u] \neq 0$, where $[u]$ denotes a vector consisting of all derivatives of u with respect to the space variable. For instance, we have $[u] = (u, u_x, u_{xx}, \dots)$ in the case of $1+1$ dimensions, where the x is the space variable. The non-triviality condition in the above statement of the problem means that the other differential equations in the bigger system involve the dependent variables of the original system, and thus it guarantees that trivial diagonal systems with $S(u, v) = S(v)$ are not within our business.

The study of integrable couplings not only generalizes the symmetry problem, but also provides clues towards complete classification of integrable systems. The first point was clearly presented and analyzed carefully in [1,2,8]. It will help us to see the second point to observe classification of square matrix spectral problems through the Jordan blocks under similar

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transformations. Each triangular Jordan block corresponds to an undecomposable sub-system in a given integrable system. We will show that semi-direct sums of Lie algebras can result in candidates for integrable couplings. Recall that an arbitrary Lie algebra has a semi-direct sum structure of a solvable Lie algebra and a semi-simple Lie algebra [9]. Therefore, the study of integrable couplings through semi-direct sums of Lie algebras is also an inevitable step towards complete classification of integrable systems. On the other hand, the theory of integrable couplings brings other interesting results such as Lax pairs of block form and with several spectral parameters [2,10,11], integrable constrained flows with higher multiplicity [12], local bi-Hamiltonian structures in higher dimensions [8] and hereditary recursion operators of higher order [2,13].

In this Letter, we would like to present a relation between semi-direct sums of Lie algebras and integrable couplings of continuous soliton equations and a feasible way to generate integrable couplings through semi-direct sums of Lie algebras. The general theory will be used to construct a novel class of integrable couplings of the AKNS hierarchy. It will also be indicated that the study of integrable couplings using semi-direct sums of Lie algebras is an important step towards complete classification of integrable systems. Finally, in the last section, a few concluding remarks will be given, especially on selection of semi-direct sums of Lie algebras.

2. From semi-direct sums of Lie algebras to integrable couplings

Let G be a matrix Lie algebra. We assume that a continuous integrable equation (or system) of evolution type

$$u_t = K(u) \quad (2.1)$$

is associated with G . More precisely, there is a pair of Lax matrices U and V in G so that the matrix spectral problem

$$\phi_x = U\phi = U(u, \lambda)\phi \quad (2.2)$$

and the associated matrix spectral problem

$$\phi_t = V\phi = V\left(u, u_x, \dots, \frac{\partial^{m_0} u}{\partial x^{m_0}}; \lambda\right)\phi, \quad (2.3)$$

where λ is a spectral parameter, generate [14,15] the integrable equation (2.1) through their isospectral (i.e., $\lambda_t = 0$) compatibility condition:

$$U_t - V_x + [U, V] = 0. \quad (2.4)$$

In one word, we have

$$U'[K] - V_x + [U, V] = 0, \quad (2.5)$$

where $U'[K] = \frac{\partial}{\partial \varepsilon}|_{\varepsilon=0} U(u + \varepsilon K)$.

To construct an integrable coupling of Eq. (2.1), we enlarge the Lie algebra G by using semi-direct sums of Lie algebras and take a semi-direct sum of G with another matrix Lie algebra G_c :

$$\tilde{G} = G \in G_c. \quad (2.6)$$

The notion of semi-direct sums means that G and G_c satisfy

$$[G, G_c] \subseteq G_c, \quad (2.7)$$

where $[G, G_c] = \{[A, B] \mid A \in G, B \in G_c\}$. Therefore, G_c is an ideal Lie sub-algebra of $\tilde{G}$. The subscript c indicates a contribution to the construction of couplings. Take a pair of new Lax matrices in the semi-direct sum $\tilde{G}$ of G and G_c :

$$\bar{U} = U + U_c, \quad \bar{V} = V + V_c, \quad U_c, V_c \in G_c, \quad (2.8)$$

and make a pair of enlarged matrix spectral problems

$$\begin{cases} \bar{\phi}_x = \bar{U}\bar{\phi} = \bar{U}(\bar{u}, \lambda)\bar{\phi}, \\ \bar{\phi}_t = \bar{V}\bar{\phi} = \bar{V}(\bar{u}, \bar{u}_x, \dots, \frac{\partial^{m_0} \bar{u}}{\partial x^{m_0}}; \lambda)\bar{\phi}, \end{cases} \quad (2.9)$$

where the matrix U_c in $\bar{U}$ introduces additional dependent variables and $\bar{u}$ consists of both the original dependent variables and the additional dependent variables. Also, the matrix U_c could depend on the spectral parameter λ , and the matrix V_c in $\bar{V}$ does almost in all cases. Obviously, we have

$$[\bar{U}, \bar{V}] = [U, V] + ([U, V_c] + [U_c, V] + [U_c, V_c]) \in G \in G_c.$$

Therefore, under $u_t = K(u)$, the corresponding enlarged zero curvature equation

$$\bar{U}_t - \bar{V}_x + [\bar{U}, \bar{V}] = 0 \quad (2.10)$$

is equivalent to

$$\begin{cases} U_t - V_x + [U, V] = 0, \\ U_{c,t} - V_{c,x} + [U, V_c] + [U_c, V] + [U_c, V_c] = 0. \end{cases} \quad (2.11)$$

The first equation here exactly presents Eq. (2.1), and thus, this provides a coupling system for Eq. (2.1). To summarize, semi-direct sums of G with new Lie algebras provide a great choice of candidates of integrable couplings for Eq. (2.1) associated with G .

Let us next shed light on the above general idea of constructing coupling systems by a particular class of semi-direct sums of Lie algebras. Consider the following semi-direct sum of Lie algebras:

$$G \in G_c, \quad G = \left\{ \text{diag}(A, \dots, A) \right\}, \quad (2.12)$$

$$G_c = \left\{ \begin{pmatrix} 0 & B_1 & \cdots & B_v \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & B_1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \right\},$$

where the matrices $A, B_i, 1 \leq i \leq v$, are arbitrary matrices of the same size as U . Notice that the spectral matrices, $\text{diag}(U, \dots, U)$ and $\text{diag}(V, \dots, V)$, generate the same equation as U and V . So, new enlarged spectral matrices $\bar{U}$ and $\bar{V}$ in $G \in G_c$ can be chosen as

$$\bar{U} = \begin{pmatrix} U & U_{a_1} & \cdots & U_{a_v} \\ 0 & U & \ddots & \vdots \\ \vdots & \ddots & \ddots & U_{a_1} \\ 0 & \cdots & 0 & U \end{pmatrix},$$

$$\bar{V} = \begin{pmatrix} V & V_{a_1} & \cdots & V_{a_v} \\ 0 & V & \ddots & \vdots \\ \vdots & \ddots & \ddots & V_{a_1} \\ 0 & \cdots & 0 & V \end{pmatrix}. \quad (2.13)$$

Then, the coupling system (2.11) becomes

$$\begin{cases} U_t - V_x + [U, V] = 0, \\ U_{a_i,t} - V_{a_i,x} + \sum_{k+l=i, k,l \geq 0} [U_{a_k}, V_{a_l}] = 0, \\ 1 \leq i \leq v, \end{cases} \quad (2.14)$$

where $U_{a_0} = U$ and $V_{a_0} = V$.

Let us further assume that the spectral matrix U depends linearly on the spectral parameter λ (see, for example, [15–17]):

$$U = U(u, \lambda) = \lambda U_0 + U_1, \quad \frac{\partial U_0}{\partial \lambda} = \frac{\partial U_1}{\partial \lambda} = 0. \quad (2.15)$$

Suppose that

$$W = \sum_{j \geq 0} W_j \lambda^{-j}, \quad W_{a_i} = \sum_{j \geq 0} W_{a_i,j} \lambda^{-j}, \quad 1 \leq i \leq v,$$

$$\frac{\partial W_j}{\partial \lambda} = 0, \quad \frac{\partial W_{a_i,j}}{\partial \lambda} = 0, \quad 1 \leq i \leq v, \quad j \geq 0,$$

define a solution

$$\bar{W} = \begin{pmatrix} W & W_{a_1} & \cdots & W_{a_v} \\ 0 & W & \ddots & \vdots \\ \vdots & \ddots & \ddots & W_{a_1} \\ 0 & \cdots & 0 & W \end{pmatrix}$$

to the enlarged stationary zero curvature equation $\bar{W}_x = [\bar{U}, \bar{W}]$. Then for each $m \geq 0$, choose

$$\bar{V}^{[m]} = \begin{pmatrix} V^{[m]} & V_{a_1}^{[m]} & \cdots & V_{a_v}^{[m]} \\ 0 & V^{[m]} & \ddots & \vdots \\ \vdots & \ddots & \ddots & V_{a_1}^{[m]} \\ 0 & \cdots & 0 & V^{[m]} \end{pmatrix} = (\lambda^m \bar{W})_+ + \bar{\Delta}_m,$$

$$\bar{\Delta}_m = \begin{pmatrix} \Delta_m & \Delta_{1,m} & \cdots & \Delta_{v,m} \\ 0 & \Delta_m & \ddots & \vdots \\ \vdots & \ddots & \ddots & \Delta_{1,m} \\ 0 & \cdots & 0 & \Delta_m \end{pmatrix},$$

where the subscript ‘+’ denotes to select the polynomial part in λ , and Δ_m and $\Delta_{i,m}$ do not depend on λ and satisfy

$$[U_0, \Delta_m] = 0, \quad [U_0, \Delta_{i,m}] = 0, \quad 1 \leq i \leq v.$$

Based on (2.14) and $\bar{W}_x = [\bar{U}, \bar{W}]$, we can directly show that the enlarged zero curvature equation

$$\bar{U}_{t_m} - \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0$$

gives

$$\begin{cases} U_{t_m} = V_x^{[m]} - [U, V^{[m]}], \\ U_{a_i,t_m} = (\Delta_{i,m})_x + [U_0, W_{a_i,m+1}] - [U_1, \Delta_{i,m}], \\ 1 \leq i \leq v. \end{cases} \quad (2.16)$$

We will see that this enlarged hierarchy of Lax integrable coupling systems can keep other integrable properties in special

cases, e.g., existence of infinitely many higher symmetries [18]. Therefore, each system in the hierarchy (2.16) is a good candidate for integrable couplings of its first sub-system. In the next section, we will discuss in more detail an example in the AKNS case.

3. Application to the AKNS hierarchy

3.1. The AKNS hierarchy

Let us here recall the AKNS hierarchy [16]. The AKNS spectral problem is given by

$$\begin{aligned} \phi_x &= U\phi, \\ U &= U(u, \lambda) = \begin{pmatrix} -\lambda & p \\ q & \lambda \end{pmatrix}, \quad u = \begin{pmatrix} p \\ q \end{pmatrix}, \end{aligned} \quad (3.1)$$

where p and q are two dependent variables. Upon setting

$$W = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \sum_{i \geq 0} W_i \lambda^{-i} = \sum_{i \geq 0} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-i}, \quad (3.2)$$

and choosing the initial data

$$a_0 = -1, \quad b_0 = c_0 = 0, \quad (3.3)$$

we see that the stationary zero curvature equation $W_x = [U, W]$ generates

$$\begin{cases} b_{i+1} = -\frac{1}{2}b_{i,x} + pa_i, \\ c_{i+1} = \frac{1}{2}c_{i,x} - qa_i, \\ a_{i+1,x} = pc_{i+1} - qb_{i+1}, \quad i \geq 0. \end{cases} \quad (3.4)$$

Assume $a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0$, $i \geq 1$, or equivalently select constants of integration to be zero. Then, the recursion relation (3.4) uniquely defines a series of sets of differential polynomial functions in u with respect to x . The first three sets are as follows:

$$\begin{cases} b_1 = p, & c_1 = q, & a_1 = 0; \\ b_2 = -\frac{1}{2}p_x, & c_2 = \frac{1}{2}q_x, & a_2 = \frac{1}{2}pq; \\ b_3 = \frac{1}{4}p_{xx} - \frac{1}{2}p^2q, & c_3 = \frac{1}{4}q_{xx} - \frac{1}{2}pq^2, \\ a_3 = \frac{1}{4}(pq_x - p_xq). \end{cases}$$

The compatibility conditions of the matrix spectral problems

$$\begin{aligned} \phi_x &= U\phi, & \phi_t &= V^{[m]}\phi, \\ V^{[m]} &= (\lambda^m W)_+, & m &\geq 0, \end{aligned} \quad (3.5)$$

determine the AKNS hierarchy of soliton equations

$$u_{t_m} = \begin{pmatrix} p \\ q \end{pmatrix}_{t_m} = K_m = \begin{pmatrix} -2b_{m+1} \\ 2c_{m+1} \end{pmatrix} = \Phi^m \begin{pmatrix} -2p \\ 2q \end{pmatrix}, \quad m \geq 0, \quad (3.6)$$

where the hereditary recursion operator Φ is defined by

$$\Phi = \begin{pmatrix} -\frac{1}{2}\partial + p\partial^{-1}q & p\partial^{-1}p \\ -q\partial^{-1}q & \frac{1}{2}\partial - q\partial^{-1}p \end{pmatrix}, \quad \partial = \frac{\partial}{\partial x}. \quad (3.7)$$

It is also known [19] that for each $m \geq 0$, we have

$$W_{t_m} = [V^{[m]}, W],$$

when $u_{t_m} = K_m$, i.e., $U_{t_m} - (V^{[m]})_x + [U, V^{[m]}] = 0$. It then follows that

$$[(V^{[k]})_{t_l} - (V^{[l]})_{t_k} - [V^{[k]}, V^{[l]}], W] = 0, \quad k, l \geq 0.$$

Actually, we can show the initial value equality

$$((V^{[k]})_{t_l} - (V^{[l]})_{t_k} - [V^{[k]}, V^{[l]}])|_{u=0} = 0, \quad k, l \geq 0,$$

or equivalently,

$$((V^{[k]})'[K_l] - (V^{[l]})'[K_k] - [V^{[k]}, V^{[l]}])|_{u=0} = 0, \quad k, l \geq 0;$$

and the uniqueness property of the spectral problem (3.1) in the loop algebra $G = \text{span}\{\lambda^n A \mid n \geq 0, A \in \text{sl}(2)\}$: if a matrix solution $W \in G$ to the equation $U'[K] - W_x + [U, W] = 0$ with some vector-valued function K satisfies $W|_{u=0} = 0$, then $W = 0$. Therefore, it directly follows (see [20,21] for more information) that

$$(V^{[k]})_{t_l} - (V^{[l]})_{t_k} - [V^{[k]}, V^{[l]}] = 0, \quad k, l \geq 0.$$

This implies the commutativity of the flows of the AKNS hierarchy (3.6) and thus all AKNS equations in (3.6) are symmetries to each other.

3.2. Integrable couplings from a specific semi-direct sum

Let us now introduce two Lie algebras of 4×4 matrices:

$$G = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \middle| A \in \mathbb{R}[\lambda] \otimes \text{sl}(2) \right\},$$

$$G_c = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \middle| B \in \mathbb{R}[\lambda] \otimes \text{sl}(2) \right\}, \quad (3.8)$$

where $\mathbb{R}[\lambda] \otimes \text{sl}(2)$ is the loop algebra defined by $\text{span}\{\lambda^n A \mid n \geq 0, A \in \text{sl}(2)\}$, and form a semi-direct sum $\tilde{G} = G \ltimes G_c$ of these two Lie algebras G and G_c . In this case, G_c is an Abelian ideal of $\tilde{G}$. For the AKNS spectral problem (3.1), we write

$$U = U_0 \lambda + U_1,$$

$$U_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U_1 = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \quad (3.9)$$

and define the corresponding enlarged spectral matrix as follows:

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = \begin{pmatrix} U & U_a \\ 0 & U \end{pmatrix} \in G \ltimes G_c,$$

$$U_a = U_a(v) = \begin{pmatrix} -v_1 & v_2 \\ v_3 & v_1 \end{pmatrix}, \quad (3.10)$$

where v_i , $1 \leq i \leq 3$, are new dependent variables and

$$v = (v_1, v_2, v_3)^T,$$

$$\bar{u} = (u^T, v^T)^T = (p, q, v_1, v_2, v_3)^T. \quad (3.11)$$

To solve the corresponding enlarged stationary zero curvature equation $\bar{W}_x = [\bar{U}, \bar{W}]$, we set

$$\bar{W} = \begin{pmatrix} W & W_a \\ 0 & W \end{pmatrix}, \quad W_a = W_a(\bar{u}, \lambda) = \begin{pmatrix} e & f \\ g & -e \end{pmatrix}, \quad (3.12)$$

where W is a solution to $W_x = [U, W]$, defined by (3.2). Then, the enlarged stationary zero curvature equation becomes

$$W_{a,x} = [U, W_a] + [U_a, W]. \quad (3.13)$$

Obviously, we have

$$[U, W_a] = \begin{pmatrix} pg - qf & -2\lambda f - 2pe \\ 2qe + 2\lambda g & qf - pg \end{pmatrix},$$

$$[U_a, W] = \begin{pmatrix} v_2c - v_3b & -2v_1b - 2v_2a \\ 2v_3a + 2v_1c & v_3b - v_2c \end{pmatrix},$$

and thus, Eq. (3.13) is equivalent to

$$\begin{cases} e_x = pg - qf + v_2c - v_3b, \\ f_x = -2\lambda f - 2pe - 2v_1b - 2v_2a, \\ g_x = 2qe + 2\lambda g + 2v_3a + 2v_1c. \end{cases}$$

Trying a solution

$$e = \sum_{i \geq 0} e_i \lambda^{-i}, \quad f = \sum_{i \geq 0} f_i \lambda^{-i}, \quad g = \sum_{i \geq 0} g_i \lambda^{-i},$$

we obtain

$$\begin{cases} e_{i+1,x} = pg_{i+1} - qf_{i+1} + v_2c_{i+1} - v_3b_{i+1}, \\ f_{i+1} = -\frac{1}{2}f_{i,x} - pe_i - v_1b_i - v_2a_i, \\ g_{i+1} = \frac{1}{2}g_{i,x} - qe_i - v_3a_i - v_1c_i, \quad i \geq 0, \end{cases} \quad (3.14)$$

upon setting

$$e_0 = -1, \quad f_0 = g_0 = 0. \quad (3.15)$$

Assuming $e_i|_{u=0} = f_i|_{u=0} = g_i|_{u=0} = 0$, $i \geq 1$, we see that all sets of functions e_i , f_i and g_i are uniquely determined. In particular, the first few sets are:

$$\begin{cases} f_1 = p + v_2, & g_1 = q + v_3, & e_1 = 0; \\ f_2 = -\frac{1}{2}(p + v_2)_x - v_1p, \\ g_2 = \frac{1}{2}(q + v_3)_x - v_1q, & e_2 = \frac{1}{2}pq + \frac{1}{2}v_3p + \frac{1}{2}v_2q; \\ f_3 = \frac{1}{4}(p + v_2)_{xx} + \frac{1}{2}(v_1p)_x - \frac{1}{2}p(pq + v_3p + v_2q) \\ \quad + \frac{1}{2}v_1p_x - \frac{1}{2}v_2pq, \\ g_3 = \frac{1}{4}(q + v_3)_{xx} - \frac{1}{2}(v_1q)_x - \frac{1}{2}(pq + v_3p + v_2q)q \\ \quad - \frac{1}{2}v_3pq - \frac{1}{2}v_1q_x, \\ e_3 = \frac{1}{4}(pq_x - p_xq) + \frac{1}{4}(v_{3,x}p - p_xv_3) + \frac{1}{4}(v_2q_x - v_{2,x}q) \\ \quad - v_1pq. \end{cases}$$

Expand W_a as

$$W_a = \sum_{i \geq 0} W_{a,i} \lambda^{-i}, \quad (3.16)$$

and then it follows from (3.13) that

$$(W_{a,i})_x = [U_0, W_{a,i+1}] + [U_1, W_{a,i}] + [U_a, W_i], \quad i \geq 0. \quad (3.17)$$

Now, we define

$$\bar{V}^{[m]} = \begin{pmatrix} V^{[m]} & V_a^{[m]} \\ 0 & V^{[m]} \end{pmatrix} \in \tilde{G},$$

$$V_a^{[m]} = (\lambda^m W_a)_+ + \Delta_{m,a}, \quad m \geq 0, \quad (3.18)$$

where $V^{[m]}$ is defined as in (3.5), and choose $\Delta_{m,a}$ as

$$\Delta_{m,a} = \begin{pmatrix} e_{m+\delta(m)} & 0 \\ 0 & -e_{m+\delta(m)} \end{pmatrix}, \quad m \geq 0,$$

where the integer $\delta(m)$ satisfies $m + \delta(m) \geq 0$. Then, the m th enlarged zero curvature equation

$$\bar{U}_{t_m} - (\bar{V}^{[m]})_x + [\bar{U}, \bar{V}^{[m]}] = 0$$

leads to

$$U_{a,t_m} - (V_a^{[m]})_x + [U, V_a^{[m]}] + [U_a, V^{[m]}] = 0,$$

together with the m th AKNS equation in (3.6). Based on (3.17), this can be simplified to

$$U_{a,t_m} - (\Delta_{m,a})_x - [U_0, W_{a,m+1}] + [U_1, \Delta_{m,a}] = 0,$$

which gives rise to

$$v_{t_m} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{t_m} = S_m(u, v) = \begin{pmatrix} -e_{m+\delta(m),x} \\ -2f_{m+1} + 2pe_{m+\delta(m)} \\ 2g_{m+1} - 2qe_{m+\delta(m)} \end{pmatrix}, \quad m \geq 0, \quad (3.19)$$

where $v = (v_1, v_2, v_3)^T$ defined as in (3.11). Therefore, we obtain a hierarchy of coupling systems:

$$\bar{u}_{t_m} = \begin{pmatrix} u \\ v \end{pmatrix}_{t_m} = \bar{K}_m(u) = \begin{pmatrix} K_m(u) \\ S_m(u, v) \end{pmatrix}, \quad m \geq 0 \quad (3.20)$$

for the AKNS hierarchy (3.6).

The hierarchy of evolution equations in (3.20) is Lax integrable and keep the original infinitely many conservation laws of the AKNS hierarchy (3.6). The first three systems in (3.20) with $\delta(m) = 1$ are:

$$\begin{cases} p_{t_0} = -2p, & q_{t_0} = 2q, \\ v_{1,t_0} = 0, & v_{2,t_0} = -2(p + v_2), & v_{3,t_0} = 2(q + v_3), \end{cases}$$

$$\begin{cases} p_{t_1} = p_x, & q_{t_1} = q_x, \\ v_{1,t_1} = -\frac{1}{2}(pq + v_3p + v_2q)_x, \\ v_{2,t_1} = (p + v_2)_x + 2v_1p + p(pq + v_3p + v_2q), \\ v_{3,t_1} = (q + v_3)_x - 2v_1q - (pq + v_3p + v_2q)q, \end{cases}$$

$$\begin{cases} p_{t_2} = -\frac{1}{2}p_{xx} + p^2q, & q_{t_2} = \frac{1}{2}q_{xx} - pq^2, \\ v_{1,t_2} = -\frac{1}{4}pq_{xx} + \frac{1}{4}p_{xx}q - \frac{1}{4}v_{3,xx}p + \frac{1}{4}v_3p_{xx} \\ \quad - \frac{1}{4}v_2q_{xx} + \frac{1}{4}v_{2,xx}q + (v_1pq)_x, \\ v_{2,t_2} = -\frac{1}{2}(p + v_2)_{xx} - v_{1,x}p - 2v_1p_x \\ \quad + p(pq + v_3p + 2v_2q) + \frac{1}{2}p(pq_x - p_xq) \\ \quad + \frac{1}{2}p(v_{3,x}p - v_3p_x) + \frac{1}{2}p(v_2q_x - v_{2,x}q) - 2v_1p^2q, \\ v_{3,t_2} = \frac{1}{2}(p + v_3)_{xx} - v_{1,x}q - 2v_1q_x \\ \quad - (pq + 2v_3p + v_2q)q - \frac{1}{2}(pq_x - p_xq)q \\ \quad - \frac{1}{2}(v_{3,x}p - v_3p_x)q - \frac{1}{2}(v_2q_x - v_{2,x}q)q + 2v_1pq^2. \end{cases}$$

Other integrable properties of the above coupling systems defining t_1 - and t_2 -flows need further investigation. For example, do they possess infinitely many higher symmetries? Are they integrable by the inverse scattering transform?

Let us now set $v_1 = \alpha = \text{const}$. Then the corresponding enlarged spectral problem $\bar{\phi}_x = \bar{U}\bar{\phi}$ has the uniqueness property in the underlying loop algebra $\bar{G} = G \in G_c$ (see (3.8) for the definition of G and G_c). When $\alpha = 0$, the resulting enlarged spectral problem is a perturbation of the spectral problem (3.1). Moreover, we choose $\Delta_{m,a} = 0$, and then from (3.19), we have

$$v_{t_m} = \begin{pmatrix} v_2 \\ v_3 \end{pmatrix}_{t_m} = T_m(u, v) = \begin{pmatrix} -2f_{m+1} \\ 2g_{m+1} \end{pmatrix}, \quad m \geq 0, \quad (3.21)$$

where $v = (v_2, v_3)^T$. Further, from (3.20), we obtain a reduced hierarchy of coupling systems for the AKNS hierarchy (3.6):

$$\bar{u}_{t_m} = \begin{pmatrix} u \\ v \end{pmatrix}_{t_m} = \bar{K}_m(u) = \begin{pmatrix} K_m(u) \\ T_m(u, v) \end{pmatrix}, \quad m \geq 0, \quad (3.22)$$

where $\bar{u} = (u^T, v^T)^T = (p, q, v_2, v_3)^T$. The hierarchy (3.22) is commutative, since we can easily prove that

$$((\bar{V}^{[k]})'[\bar{K}_l] - (\bar{V}^{[l]})'[\bar{K}_k] - [\bar{V}^{[k]}, \bar{V}^{[l]}])|_{\bar{u}=0} = 0, \quad k, l \geq 0. \quad (3.23)$$

This implies that every system $\bar{u}_{t_m} = \bar{K}_m$ in the hierarchy (3.22) has infinitely many higher symmetries as well as infinitely many conservation laws, and thus, it provides an integrable coupling of $u_{t_m} = K_m$ in the AKNS hierarchy (3.6). Here we mean by integrability the existence of Lax pairs and infinitely many higher symmetries. We remark that any choice for $\bar{V}^{[m]}$, which could be different from (3.18), will generate integrable couplings possessing infinitely many higher symmetries, if the uniqueness property in the underlying loop algebra and the initial value condition (3.23) are satisfied. The first non-linear system in (3.22) is

$$\begin{cases} p_{t_2} = -\frac{1}{2}p_{xx} + p^2q, & q_{t_2} = \frac{1}{2}q_{xx} - pq^2, \\ v_{2,t_2} = -\frac{1}{2}(p + v_2)_{xx} - 2\alpha p_x \\ \quad + p(pq + v_3p + v_2q) + v_2pq, \\ v_{3,t_2} = \frac{1}{2}(p + v_3)_{xx} - 2\alpha q_x \\ \quad - (pq + v_3p + v_2q)q - v_3pq. \end{cases} \quad (3.24)$$

Based on (3.4) and (3.14), we find the following recursion operator of the system (3.24):

$$\bar{\Phi} = \bar{\Phi}(\bar{u}) = \begin{pmatrix} \Phi(u) & 0 \\ \Phi_c(\bar{u}) - \alpha I_2 & \Phi(u) \end{pmatrix}, \quad (3.25)$$

where $\Phi(u)$ is given by (3.7), I_2 is the identity matrix of order 2 and $\Phi_c(\bar{u})$ is defined by

$$\Phi_c(\bar{u}) = \begin{pmatrix} v_2\partial^{-1}q + p\partial^{-1}v_3 & v_2\partial^{-1}p + p\partial^{-1}v_2 \\ -v_3\partial^{-1}q - q\partial^{-1}v_3 & -v_3\partial^{-1}p - q\partial^{-1}v_2 \end{pmatrix}. \quad (3.26)$$

It can be shown that the recursion operator $\bar{\Phi}$ in (3.25) is hereditary, whatever value α takes.

4. Concluding remarks

A technically-feasible way to construct integrable couplings of continuous soliton equations has been presented by using

semi-direct sums of Lie algebras, and an application of the general theory to the AKNS hierarchy of soliton equations has been made. The key idea in our construction is to establish a relation between semi-direct sums of Lie algebras and integrable couplings of soliton equations. The underlying matrix spectral problems come from Lie algebras in the form of semi-direct sums, and the Lax spectral matrices associated with given soliton equations play the non-ideal part in semi-direct sums.

We point out that the enlarged Lax spectral matrix

$$\bar{U} = \begin{pmatrix} U & U_a \\ 0 & U \end{pmatrix}$$

from a semi-direct sum of Lie algebras was also previously mentioned in [4]. Another enlargement of matrix spectral problems discussed in [4],

$$\bar{U} = \begin{pmatrix} U & U_a \\ 0 & 0 \end{pmatrix},$$

corresponds to the semi-direct sum of Lie algebras:

$$G \in G_c, \quad G = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right\}, \quad G_c = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \right\},$$

where the partitions of matrices in G and G_c are the same as for $\bar{U}$. More general, the enlargement of Lax spectral matrices

$$\bar{U} = \begin{pmatrix} U & U_{a_1} \\ 0 & U_{a_2} \end{pmatrix}$$

could be taken from the following semi-direct sum of Lie algebras:

$$G \in G_c, \quad G = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right\}, \quad G_c = \left\{ \begin{pmatrix} 0 & B_1 \\ 0 & B_2 \end{pmatrix} \right\},$$

where A , B_1 and B_2 possess the same sizes as U , U_{a_1} and U_{a_2} , respectively. In this case, U_{a_1} and U_{a_2} are two matrices which introduce additional dependent variables, but U_{a_2} could be of different order from U and even not square. Furthermore, combining the above enlarging ideas together, we can make even more general cases such as

$$\bar{U} = \begin{pmatrix} U & U_{a_1} & U_{a_2} \\ 0 & U & U_{a_3} \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{U} = \begin{pmatrix} U & U_{a_1} & U_{a_2} \\ 0 & U & U_{a_3} \\ 0 & 0 & U_{a_4} \end{pmatrix}.$$

On the other hand, in the enlarged Lax spectral matrices, the sub-matrices, e.g., U_a and U_{a_1} , could depend on the spectral parameter λ . Then the resulting spectral problems will engender much more diverse integrable couplings, which possess multi-component potentials with higher multiplicity. For instance, the enlargements of the third kind above can generate integrable

couplings for a joint system of equations associated with U and U_{a_2} . Moreover, e_0 in (3.15) can take any non-zero value, and $\Delta_{m,a}$ in the case of $v_1 = \text{const}$ can be a diagonal matrix $\text{diag}(\beta, -\beta)$, where β is an arbitrary constant. Therefore, semi-direct sums of Lie algebras really provide a great source for constructing integrable couplings.

The general idea in our analysis could also be applied to other types of soliton equations such as discrete soliton equations and higher-dimensional soliton equations. All integrable couplings generated this way provide examples of multi-component integrable systems and show richness of integrable systems. We hope that any contributions related to applications of semi-direct sums of Lie algebras to integrable systems could help overcome difficulties in working towards complete classification of integrable systems.

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