



Solving the (3 + 1)-dimensional generalized KP and BKP equations by the multiple exp-function algorithm

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ABSTRACT

The multiple exp-function algorithm, as a generalization of Hirota's perturbation scheme, is used to construct multiple wave solutions to the (3 + 1)-dimensional generalized KP and BKP equations. The resulting solutions involve generic phase shifts and wave frequencies containing many existing choices. It is also pointed out that the presented phase shifts for the two considered equations are all not of Hirota type.

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1. Introduction

Soliton equations possess nice mathematical features, e.g., elastic interactions of solutions. Such equations contain the KdV equation, the Boussinesq equation, the KP equation and the BKP equation, and they all have multi-soliton solutions [1,2]. The Hirota bilinear form plays a key role in generating soliton solutions [2,3].

Oriented towards construction of multiple wave solutions by using computer algebra systems, the multiple exp-function method [4] has been recently proposed as a generalization of Hirota's perturbation scheme [1]. The approach provides a direct and efficient way to construct generic multi-exponential wave solutions to nonlinear equations [5]. The resulting multiple wave solutions contain interesting exponential wave solutions, for example, linear combination solutions of exponential waves [6,7] and resonant solitons [8], the former of which presents an idea to construct linear subspaces of solutions for nonlinear equations.

Many existing approaches, such as the tanh-function method [9,10], the sub-equation method [11–13], the tri-function method [14] and the $\frac{c}{c}$ -expansion method [15,16], engender only traveling wave solutions. Aiming at multi-soliton solutions, Wronskian, Grammian and Pfaffian formulations of determinant type solutions articulate particular multiple wave solutions (see, e.g., [17–26]).

In this paper, we would like to use the multiple exp-function method to shed light on diversity of exact solutions to nonlinear equations, and two examples will be made in the (3 + 1)-dimensional case. One of the highlights is to deal with the nonlinearity of the resulting algebraic systems on the involved parameters and to show that there exist more generic phase shifts and wave frequencies in multiple wave solutions than the ones usually used in soliton solutions.

We apply the multiple exp-function algorithm to the (3 + 1)-dimensional generalized KP and BKP equations [6]:

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0$$

and

$$u_{ty} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xz} = 0,$$

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to construct multiple wave solutions. The multiple exp-function algorithm allows us to find general phase shifts and wave frequencies, under the help of Maple. Special reductions of the parameters present concrete examples of phase shifts and wave frequencies including many known ones. Moreover, it will be pointed out that the obtained expressions of phase shifts are not of Hirota type.

2. Multiple exp-function algorithm

We first describe the procedure for constructing multiple wave solutions to nonlinear equations [4], by considering the $(1+1)$ -dimensional evolution equation:

$$u_t = K(x, t, u_x, u_{xx}, \dots). \quad (2.1)$$

Step 1 – Defining solvable differential equations:

We introduce new variables $\eta_i = \eta_i(x, t)$, $1 \leq i \leq n$, by solvable partial differential equations, for example, the linear ones:

$$\eta_{i,x} = k_i \eta_i, \quad \eta_{i,t} = -\omega_i \eta_i, \quad 1 \leq i \leq n, \quad (2.2)$$

where k_i , $1 \leq i \leq n$, are the angular wave numbers and ω_i , $1 \leq i \leq n$, are the wave frequencies. Solving such linear equations yields the exponential function solutions:

$$\eta_i = c_i e^{\xi_i}, \quad \xi_i = k_i x - \omega_i t, \quad 1 \leq i \leq n, \quad (2.3)$$

where c_i , $1 \leq i \leq n$, are any constants, positive or negative. This explains why we called the approach the multiple exp-function method.

The arbitrariness of the constants c_i , $1 \leq i \leq n$, brings more choices for solutions than we used to [27,28]. Each of the functions η_i , $1 \leq i \leq n$, describes a single wave, and a multiple wave solution is a combination using all those single waves. We remark that the linear differential relations in (2.2) are extremely helpful while transforming differential equations to algebraic equations and carrying out necessary computations by computer algebra systems.

Step 2 – Transforming nonlinear PDEs:

The second step is to consider rational solutions in η_i , $1 \leq i \leq n$:

$$u(x, t) = \frac{p(\eta_1, \eta_2, \dots, \eta_n)}{q(\eta_1, \eta_2, \dots, \eta_n)}, \quad p = \sum_{i,j=1}^n \sum_{s=0}^M p_{ij,rs} \eta_i^r \eta_j^s, \quad q = \sum_{i,j=1}^n \sum_{s=0}^N q_{ij,rs} \eta_i^r \eta_j^s, \quad (2.4)$$

where $p_{ij,rs}$ and $q_{ij,rs}$ are all constants to be determined from the original Eq. (2.1).

Based on the differential relations in (2.2), it is direct to express all partial derivatives of u with respect to x and t in terms of η_i , $1 \leq i \leq n$. All partial derivatives such as u_x and u_t will still be rational functions in η_i , $1 \leq i \leq n$. Substituting those new expressions of partial derivatives into the original Eq. (2.1) transforms (2.1) into a rational function equation in η_i , $1 \leq i \leq n$:

$$Q(x, t, \eta_1, \eta_2, \dots, \eta_n) = 0. \quad (2.5)$$

This is called the transformed equation of the original Eq. (2.1). This step makes it possible to construct solutions to differential equations directly by computer algebra systems.

Step 3 – Solving algebraic systems:

The third step is to set the numerator of the resulting rational function Q in (2.5) to be zero. This yields a system of algebraic equations on the parameters

$$k_i, \omega_i, p_{ij,rs}, q_{ij,rs}$$

and then, we solve this resulting algebraic system to determine the polynomials, p and q , and the wave exponents ξ_i , $1 \leq i \leq n$. It is always helpful to try to choose less variables to solve for. Finally, the multiple wave solution u is computed and given by

$$u(x, t) = \frac{p(c_1 e^{k_1 x - \omega_1 t}, \dots, c_n e^{k_n x - \omega_n t})}{q(c_1 e^{k_1 x - \omega_1 t}, \dots, c_n e^{k_n x - \omega_n t})}, \quad (2.6)$$

which completes the algorithm.

It is clear that the multiple exp-function method in the case of $n = 1$ becomes the so-called exp-function method proposed by He and Wu in [29], which deals only with traveling wave solutions (see, e.g., [30,31]).

Hirota's perturbation scheme for constructing multi-soliton solutions [1] begins with a special pair of polynomials p and q . The multiple exp-function method slightly generalizes Hirota's perturbation scheme; and can produce generic multiple wave solutions, which include special multiple wave solutions generated by Darboux transformations [32,33] and the simplified Hirota's method [34,35].

We will analyze three cases of polynomials p and q for the two $(3+1)$ -dimensional generalized KP and BKP equations to construct their multiple wave solutions in the next section.

3. Multiple wave solutions

3.1. (3 + 1)-dimensional generalized KP equation

We consider the (3 + 1)-dimensional generalized KP equation [6]:

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0, \quad (3.1)$$

which reduces to the KP equation if $y = x$. This equation does not belong to a class of generalized KP and Boussinesq equations discussed in [36]. There are many other similar or variable-coefficient generalizations for the KP equation (see, e.g., [37–39]).

Under the dependent variable transformation $u = 2(\ln f)_x$, the Eq. (3.1) is transformed into the Hirota bilinear form

$$(D_x^3 D_y + D_t D_x + D_t D_y - D_z^2) f \cdot f = 0, \quad (3.2)$$

where D_t , D_x , D_y and D_z are Hirota's bilinear operators [2,40]. It is shown [6] that the solution set of this bilinear equation has linear subspaces. We remark that the Bell polynomial theory tells when a nonlinear partial differential equation can be transformed into a Hirota bilinear equation [41], and that the invariant subspace method can also yield linear subspaces of solutions with generalized separated variables [42].

In what follows, we will construct some one-wave, two-wave and three-wave soliton type solutions.

One-wave solutions:

We begin with one-wave functions:

$$u = \frac{p}{q}, \quad p = a_1 + a_2 \varepsilon_1 k_1 e^{\theta_1}, \quad q = 1 + \varepsilon_1 e^{\theta_1}, \quad (3.3)$$

where a_1 , a_2 and ε_1 are constants and

$$\theta_1 = k_1 x + l_1 y + m_1 z - \omega_1 t, \quad (3.4)$$

with the dispersion relation being satisfied:

$$\omega_1 = \frac{k_1^3 l_1 - m_1^2}{k_1 + l_1}. \quad (3.5)$$

Then applying the multiple exp-function algorithm [4], we obtain by Maple:

$$a_1 = (a_2 - 1)k_1 \quad (3.6)$$

and so, the resulting one-wave solutions read

$$u = \frac{2(a_2 - 1)k_1 + 2a_2 \varepsilon_1 k_1 e^{k_1 x + l_1 y + m_1 z - \omega_1 t}}{1 + \varepsilon_1 e^{k_1 x + l_1 y + m_1 z - \omega_1 t}}, \quad (3.7)$$

where a_2 , ε_1 , k_1 , l_1 and m_1 are arbitrary but ω_1 is defined by (3.5).

Two-wave solutions:

We next consider two-wave functions:

$$u = \frac{p}{q} = \frac{2f_x}{f}, \quad (3.8)$$

with p and q being defined by

$$\begin{cases} p = 2f_x = 2[k_1 \varepsilon_1 e^{\theta_1} + k_2 \varepsilon_2 e^{\theta_2} + a_{12}(k_1 + k_2) \varepsilon_1 \varepsilon_2 e^{\theta_1} e^{\theta_2}], \\ q = f = 1 + \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2} + \varepsilon_1 \varepsilon_2 a_{12} e^{\theta_1 + \theta_2}, \end{cases} \quad (3.9)$$

where ε_1 and ε_2 are arbitrary and

$$\theta_i = k_i x + l_i y + m_i z - \omega_i t, \quad i = 1, 2. \quad (3.10)$$

Applying the multiple exp-function algorithm [4] by Maple leads to

$$\omega_i = \frac{k_i^3 l_i - m_i^2}{k_i + l_i}, \quad i = 1, 2 \quad \text{and} \quad a_{12} = \frac{b_{12}}{c_{12}}, \quad (3.11)$$

where

$$\begin{aligned} b_{12} = & (k_1 + l_1)^2 l_2^4 + (k_2 + l_2)^2 l_1^4 - 2(k_1 + l_1)(k_2 + l_2)l_1 l_2 (2l_1^2 - 3l_1 l_2 + 2l_2^2) \\ & + (k_1 + l_1)(k_2 + l_2)(k_1 - k_2 + l_1 - l_2)(k_1^2 l_2 + 2k_1 k_2 l_1 - 2k_1 k_2 l_2 - 2k_1 l_1 l_2 + k_1 l_2^2 - k_2^2 l_1 - k_2 l_1^2 + 2k_2 l_1 l_2 \\ & + 3l_1^2 l_2 - 3l_1 l_2^2) + [(k_1 + l_1)m_2 - (k_2 + l_2)m_1]^2, \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} c_{12} = & (k_1 + l_1)^2 l_2^4 + (k_2 + l_2)^2 l_1^4 - 2(k_1 + l_1)(k_2 + l_2)l_1 l_2 (2l_1^2 + 3l_1 l_2 + 2l_2^2) \\ & + (k_1 + l_1)(k_2 + l_2)(k_1 + k_2 + l_1 + l_2)(k_1^2 l_2 + 2k_1 k_2 l_1 + 2k_1 k_2 l_2 - 2k_1 l_1 l_2 - k_1 l_2^2 + k_2^2 l_1 - k_2 l_1^2 - 2k_2 l_1 l_2 \\ & + 3l_1^2 l_2 + 3l_1 l_2^2) + [(k_1 + l_1)m_2 - (k_2 + l_2)m_1]^2. \end{aligned} \quad (3.13)$$

This presents a kind of phase shifts that we have never seen before. It yields the most complicated example of phase shifts in soliton theory, to the best of our knowledge; and we were just fortunate enough to get them by using Maple programs.

Three-wave solutions:

We now consider three-wave functions:

$$u = \frac{p}{q} = \frac{2f_x}{f}, \quad (3.14)$$

with f being defined by

$$\begin{aligned} f = & 1 + \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2} + \varepsilon_3 e^{\theta_3} + \varepsilon_1 \varepsilon_2 a_{12} e^{\theta_1 + \theta_2} + \varepsilon_1 \varepsilon_3 a_{13} e^{\theta_1 + \theta_3} + \varepsilon_2 \varepsilon_3 a_{23} e^{\theta_2 + \theta_3} + \varepsilon_1 \varepsilon_2 \varepsilon_3 a_{123} e^{\theta_1 + \theta_2 + \theta_3}, \\ a_{123} = & a_{12} a_{13} a_{23}, \end{aligned} \quad (3.15)$$

where ε_i , $1 \leq i \leq 3$, are arbitrary and

$$\theta_i = k_i x + l_i y + m_i z - \omega_i t, \quad 1 \leq i \leq 3. \quad (3.16)$$

We would like to search for three-wave soliton type solutions with the selection of

$$\omega_i = \frac{k_i^3 l_i - m_i^2}{k_i + l_i}, \quad 1 \leq i \leq 3 \quad \text{and} \quad a_{ij} = \frac{b_{ij}}{c_{ij}}, \quad 1 \leq i < j \leq 3, \quad (3.17)$$

where

$$\begin{aligned} b_{ij} = & (k_i + l_i)^2 l_j^4 + (k_j + l_j)^2 l_i^4 - 2(k_i + l_i)(k_j + l_j)l_i l_j (2l_i^2 - 3l_i l_j + 2l_j^2) + (k_i + l_i)(k_j + l_j)(k_i - k_j + l_i - l_j)(k_i^2 l_j \\ & + 2k_i k_j l_i - 2k_i k_j l_j - 2k_i l_i l_j + k_i l_j^2 - k_j^2 l_i - k_j l_i^2 + 2k_j l_i l_j + 3l_i^2 l_j - 3l_i l_j^2) + [(k_i + l_i)m_j - (k_j + l_j)m_i]^2 \end{aligned} \quad (3.18)$$

and

$$\begin{aligned} c_{ij} = & (k_i + l_i)^2 l_j^4 + (k_j + l_j)^2 l_i^4 - 2(k_i + l_i)(k_j + l_j)l_i l_j (2l_i^2 + 3l_i l_j + 2l_j^2) + (k_i + l_i)(k_j + l_j)(k_i + k_j + l_i + l_j)(k_i^2 l_j \\ & + 2k_i k_j l_i + 2k_i k_j l_j - 2k_i l_i l_j - k_i l_j^2 + k_j^2 l_i - k_j l_i^2 - 2k_j l_i l_j + 3l_i^2 l_j + 3l_i l_j^2) + [(k_i + l_i)m_j - (k_j + l_j)m_i]^2. \end{aligned} \quad (3.19)$$

It is worth pointing out that such phase shifts are not of Hirota type, i.e., they are not presented by Hirota's perturbation scheme as follows:

$$a_{ij} = -\frac{P_{KP}(\mathbf{P}_i - \mathbf{P}_j)}{P_{KP}(\mathbf{P}_i + \mathbf{P}_j)},$$

where

$$P_{KP}(x, y, z, t) = x^3 y + tx + ty - z^2, \quad \mathbf{P}_i = (k_i, l_i, m_i, -\omega_i), \quad 1 \leq i \leq 3. \quad (3.20)$$

Since the generalized KP Eq. (3.1) is partially integrable, we need to determine conditions on the wave numbers to obtain three-wave soliton type solutions.

Firstly, if we take the choices with different values of δ_i :

$$l_i = \alpha \delta_i k_i, \quad |\delta_i| = 1, \quad \alpha = \text{const.}, \quad 1 \leq i \leq 3, \quad (3.21)$$

then direct Maple computations show that the three-wave function (3.14) presents two distinct classes of three-wave soliton type solutions to (3.1). The first class of solutions is associated with the choice

$$l_i = \alpha k_i, \quad \alpha = \text{const.}, \quad 1 \leq i \leq 3 \quad (3.22)$$

discussed in [6]. This leads to the phase shifts

$$a_{ij} = \frac{3\alpha k_i^2 k_j^2 (k_i - k_j)^2 + (k_i m_j - m_i k_j)^2}{3\alpha k_i^2 k_j^2 (k_i + k_j)^2 + (k_i m_j - m_i k_j)^2}, \quad 1 \leq i < j \leq 3, \quad (3.23)$$

recently presented by Wazwaz in [43]. Further if taking

$$m_i = \beta k_i, \quad \beta = \text{const.}, \quad 1 \leq i \leq 3,$$

then we have

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \leq i < j \leq 3,$$

which presents a kind of standard phase shifts associated with soliton equations. The second class of solutions is associated with the choice

$$l_1 = \alpha k_1, \quad l_2 = \alpha k_2, \quad l_3 = -\alpha k_3, \quad \alpha = \text{const.} \quad (3.24)$$

This yields the phase shifts

$$a_{12} = \frac{3\alpha k_1^2 k_2^2 (k_1 - k_2)^2 + (k_1 m_2 - m_1 k_2)^2}{3\alpha k_1^2 k_2^2 (k_1 + k_2)^2 + (k_1 m_2 - m_1 k_2)^2}, \quad a_{13} = 1, \quad a_{23} = 1. \quad (3.25)$$

Obviously, the other choices lead to phase shifts falling into the above two equivalence classes of phase shifts.

Secondly, let us take the choice

$$m_i = \beta k_i, \quad \beta = \text{const.}, \quad 1 \leq i \leq 3. \quad (3.26)$$

If we introduce

$$k_1 = \lambda, \quad k_2 = 3\lambda, \quad k_3 = 2\lambda, \quad l_1 = \mu, \quad l_2 = 2\mu, \quad l_3 = 3\mu \quad (3.27)$$

and assign one of the following four values to β :

$$\begin{cases} \beta = \pm \frac{\sqrt{\lambda\mu(30\lambda^2 + 65\lambda\mu + 36\mu^2)}}{\mu}, \\ \beta = \pm \frac{\sqrt{\lambda\mu(6\lambda + 19\mu)(252\lambda^3 + 1188\lambda^2\mu + 1679\lambda\mu^2 + 756\mu^3)}}{\mu(6\lambda + 19\mu)}, \end{cases} \quad (3.28)$$

where λ and μ are constants, then a direct Maple computation ensures that the three-wave function (3.14) presents a class of three-wave soliton type solutions to (3.1).

Thirdly, let us take the choice

$$m_i = \gamma l_i, \quad \gamma = \text{const.}, \quad 1 \leq i \leq 3. \quad (3.29)$$

Similarly, if we introduce

$$k_1 = \lambda, \quad k_2 = 2\lambda, \quad k_3 = 5\lambda, \quad l_1 = \mu, \quad l_2 = 5\mu, \quad l_3 = 2\mu \quad (3.30)$$

and assign one of the following four values to β :

$$\begin{cases} \beta = \pm \frac{\sqrt{\lambda\mu(10\lambda^2 + 29\lambda\mu + 20\mu^2)}}{\mu}, \\ \beta = \pm \frac{\sqrt{\lambda\mu(10\lambda + 39\mu)(340\lambda^3 + 1616\lambda^2\mu + 2267\lambda\mu^2 + 1020\mu^3)}}{\mu(10\lambda + 39\mu)}, \end{cases} \quad (3.31)$$

where λ and μ are constants, then a direct Maple computation ensures that the three-wave function (3.14) presents a class of three-wave soliton type solutions to (3.1).

Fourthly, if we go with the choice (3.26) or the choice (3.29), and if we introduce

$$k_1 = \lambda, \quad k_2 = 2\lambda, \quad k_3 = 2\lambda, \quad l_1 = \mu, \quad l_2 = 3\mu, \quad l_3 = 2\mu, \quad (3.32)$$

where the constants λ and μ are required to satisfy

$$\lambda + 3\mu = 0, \quad (3.33)$$

then direct Maple computations verify that the three-wave function (3.14) gives two classes of three-wave soliton type solutions to (3.1), in which β is arbitrary.

Finally, we point out that some other kinds of solutions such as Wronskian and Grammian solutions to the generalized KP Eq. (3.1) have been presented [44], and its Pfaffianization has been recently studied in [45].

3.2. (3 + 1)-dimensional generalized BKP equation

We consider the (3 + 1)-dimensional generalized BKP equation [6]:

$$u_{ty} - u_{xxx} - 3(u_x u_y)_x + 3u_{xz} = 0, \quad (3.34)$$

which reduces to the BKP equation if $z = x$. Under the dependent variable transformation $u = 2(\ln f)_x$, the above equation is transformed into the Hirota bilinear form

$$(D_t D_y - D_x^3 D_y + 3D_x D_z) f \cdot f = 0. \quad (3.35)$$

The solution set of this nonlinear equation also has linear subspaces [6]. In what follows, we'll construct some one-wave, two-wave and three-wave soliton type solutions.

One-wave solutions:

We begin with one-wave functions:

$$u = \frac{p}{q}, \quad p = a_1 + a_2 \varepsilon_1 k_1 e^{\theta_1}, \quad q = 1 + \varepsilon_1 e^{\theta_1}, \quad (3.36)$$

where a_1 , a_2 and ε_1 are constants and

$$\theta_1 = k_1 x + l_1 y + m_1 z - \omega_1 t, \quad (3.37)$$

with the dispersion relation being satisfied:

$$\omega_1 = -k_1^3 + \frac{3k_1 m_1}{l_1}. \quad (3.38)$$

Then applying the multiple exp-function algorithm [4], we obtain by Maple:

$$a_1 = (a_2 - 1)k_1 \quad (3.39)$$

and so, the resulting one-wave solutions read

$$u = \frac{2(a_2 - 1)k_1 + 2a_2 \varepsilon_1 k_1 e^{k_1 x + l_1 y + m_1 z - \omega_1 t}}{1 + \varepsilon_1 e^{k_1 x + l_1 y + m_1 z - \omega_1 t}}, \quad (3.40)$$

where a_2 , ε_1 , k_1 , l_1 and m_1 are arbitrary but ω_1 is defined by (3.38).

Two-wave solutions:

We next consider two-wave functions:

$$u = \frac{p}{q} = \frac{2f_x}{f}, \quad (3.41)$$

with p and q being defined by

$$\begin{cases} p = 2f_x = 2[k_1 \varepsilon_1 e^{\theta_1} + k_2 \varepsilon_2 e^{\theta_2} + a_{12}(k_1 + k_2) \varepsilon_1 \varepsilon_2 e^{\theta_1} e^{\theta_2}], \\ q = f = 1 + \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2} + \varepsilon_1 \varepsilon_2 a_{12} e^{\theta_1 + \theta_2}, \end{cases} \quad (3.42)$$

where ε_1 and ε_2 are arbitrary and

$$\theta_i = k_i x + l_i y + m_i z - \omega_i t, \quad i = 1, 2. \quad (3.43)$$

Applying the multiple exp-function algorithm [4] by Maple leads to

$$\omega_i = -k_i^3 + \frac{3k_i m_i}{l_i}, \quad i = 1, 2 \quad (3.44)$$

and

$$a_{12} = \frac{k_1 k_2 l_1 l_2 (k_1 - k_2)(l_1 - l_2) - (k_1 l_2 - k_2 l_1)(l_1 m_2 - l_2 m_1)}{k_1 k_2 l_1 l_2 (k_1 + k_2)(l_1 + l_2) - (k_1 l_2 - k_2 l_1)(l_1 m_2 - l_2 m_1)}. \quad (3.45)$$

We will explore some interesting reductions of this kind of phase shifts later when discussing three-wave solutions.

Three-wave solutions:

We now consider three-wave functions:

$$u = \frac{p}{q} = \frac{2f_x}{f}, \quad (3.46)$$

with f being defined by

$$\begin{aligned} f &= 1 + \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2} + \varepsilon_3 e^{\theta_3} + \varepsilon_1 \varepsilon_2 a_{12} e^{\theta_1 + \theta_2} + \varepsilon_1 \varepsilon_3 a_{13} e^{\theta_1 + \theta_3} + \varepsilon_2 \varepsilon_3 a_{23} e^{\theta_2 + \theta_3} + \varepsilon_1 \varepsilon_2 \varepsilon_3 a_{123} e^{\theta_1 + \theta_2 + \theta_3}, \\ a_{123} &= a_{12} a_{13} a_{23}, \end{aligned} \quad (3.47)$$

where ε_i , $1 \leq i \leq 3$, are arbitrary and

$$\theta_i = k_i x + l_i y + m_i z - \omega_i t, \quad 1 \leq i \leq 3. \quad (3.48)$$

We would like to search for three-wave soliton type solutions with the selection of

$$\omega_i = -k_i^3 + \frac{3k_i m_i}{l_i}, \quad 1 \leq i \leq 3 \quad (3.49)$$

and

$$a_{ij} = \frac{k_i k_j l_i l_j (k_i - k_j)(l_i - l_j) - (k_i l_j - k_j l_i)(l_i m_j - l_j m_i)}{k_i k_j l_i l_j (k_i + k_j)(l_i + l_j) - (k_i l_j - k_j l_i)(l_i m_j - l_j m_i)}, \quad 1 \leq i < j \leq 3. \quad (3.50)$$

It is worth noting that such phase shifts are not of Hirota type, i.e., they are not presented by Hirota's perturbation scheme as follows:

$$a_{ij} = -\frac{P_{BKP}(\mathbf{P}_i - \mathbf{P}_j)}{P_{BKP}(\mathbf{P}_i + \mathbf{P}_j)},$$

where

$$P_{BKP}(x, y, z, t) = ty - x^3 y + 3xz, \quad \mathbf{P}_i = (k_i, l_i, m_i, -\omega_i), \quad 1 \leq i \leq 3. \quad (3.51)$$

Similarly, because the generalized BKP Eq. (3.34) is not completely integrable, we need to determine conditions on the wave numbers to generate three-wave soliton type solutions.

Firstly, if we take the choices with different values of δ_i :

$$l_i = \alpha \delta_i k_i, \quad |\delta_i| = 1, \quad \alpha = \text{const.}, \quad 1 \leq i \leq 3, \quad (3.52)$$

then we have two distinct classes of three-wave soliton type solutions defined by (3.46). The first class of solutions is associated with

$$l_i = \alpha k_i, \quad \alpha = \text{const.}, \quad 1 \leq i \leq 3, \quad (3.53)$$

which leads to the wave frequencies

$$\omega_i = -k_i^3 + \frac{3m_i}{\alpha}, \quad 1 \leq i \leq 3 \quad (3.54)$$

and the phase shifts

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \leq i < j \leq 3. \quad (3.55)$$

The second class of solutions is associated with

$$l_1 = \alpha k_1, \quad l_2 = -\alpha k_2, \quad l_3 = \alpha k_3, \quad \alpha = \text{const.}, \quad (3.56)$$

which leads to the wave frequencies

$$\omega_1 = -k_1^3 + \frac{3m_1}{\alpha}, \quad \omega_2 = -k_2^3 - \frac{3m_2}{\alpha}, \quad \omega_3 = -k_3^3 + \frac{3m_3}{\alpha} \quad (3.57)$$

and the phase shifts

$$a_{12} = 1, \quad a_{13} = \frac{(k_1 - k_3)^2}{(k_1 + k_3)^2}, \quad a_{23} = 1. \quad (3.58)$$

Obviously, the other choices in (3.52) yield wave frequencies and phase shifts falling into the above two equivalence classes of wave frequencies and phase shifts. In the case of (3.53), a special reduction

$$m_i = \beta k_i^3, \quad \beta = \text{const.}, \quad 1 \leq i \leq 3, \quad (3.59)$$

presents the choice for the wave numbers discussed in [46].

Secondly, the choice

$$m_i = \beta k_i, \quad \beta = \text{const.}, \quad 1 \leq i \leq 3, \quad (3.60)$$

presents a class of three-wave soliton type solutions defined by (3.46) with

$$\omega_i = -k_i^3 + \frac{3\beta k_i^2}{l_i}, \quad 1 \leq i \leq 3 \quad (3.61)$$

and

$$a_{ij} = \frac{k_i k_j l_i l_j (k_i - k_j)(l_i - l_j) + \beta (k_i l_j - k_j l_i)^2}{k_i k_j l_i l_j (k_i + k_j)(l_i + l_j) + \beta (k_i l_j - k_j l_i)^2}, \quad 1 \leq i < j \leq 3. \quad (3.62)$$

Thirdly, the choice

$$m_i = \gamma l_i, \quad \gamma = \text{const.}, \quad 1 \leq i \leq 3, \quad (3.63)$$

generates a class of three-wave soliton type solutions defined by (3.46) with

$$\omega_i = -k_i^3 + 3\gamma k_i, \quad 1 \leq i \leq 3 \quad (3.64)$$

and

$$a_{ij} = \frac{(k_i - k_j)(l_i - l_j)}{(k_i + k_j)(l_i + l_j)}, \quad 1 \leq i < j \leq 3. \quad (3.65)$$

Finally, we point out that some other kinds of solutions to the generalized BKP Eq. (3.34), including traveling wave solutions and Grammian solutions, have been discussed in [47,48].

4. Concluding remarks

The multiple exp-function algorithm has been applied to the $(3+1)$ -dimensional generalized KP and BKP equations. Maple was used to compute multiple wave solutions with generic phase shifts and wave frequencies. The particular cases reduce to the discussed choices for the parameters available in the literature. We were just fortunate enough to work out, under the help of Maple, the two very complicated classes of phase shifts: one defined by (3.17) with (3.18) and (3.19), and the other, by (3.50).

We remark that there is no resonant phenomenon in the presented solitons. Let us assume

$$f = 1 + \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2},$$

where ε_1 and ε_2 are arbitrary. It is clear that f solves the bilinear generalized KP Eq. (3.2) or the bilinear generalized BKP Eq. (3.35) iff ω_1 and ω_2 are defined by (3.11) and

$$P_{KP}(\mathbf{P}_1 - \mathbf{P}_2) = (k_1 - k_2)^3(l_1 - l_2) - (\omega_1 - \omega_2)(k_1 - k_2 + l_1 - l_2) - (m_1 - m_2)^2 = 0$$

or ω_1 and ω_2 are defined by (3.44) and

$$P_{BKP}(\mathbf{P}_1 - \mathbf{P}_2) = -(\omega_1 - \omega_2)(m_1 - m_2) - (k_1 - k_2)^3(l_1 - l_2) + 3(k_1 - k_2)(m_1 - m_2) = 0.$$

By comparing coefficients of l_1 or l_2 , we see that none of these two classes of conditions imply the corresponding phase shifts $a_{12} = 0$ or $a_{12} = \infty$. Therefore, neither the phase shifts a_{12} defined by (3.11) with (3.12) and (3.13) nor the phase shifts a_{12} defined by (3.45) show any resonant phenomena.

It is also known that there are other interesting solutions such as Wronskian, Grammian and Pfaffian determinant type solutions to soliton equations (see, e.g., [17–26]), and even soliton equations with self-consistent sources [49]. Applications of the multiple exp-function algorithm can tell what kind of linear differential rules that matrix entries in those determinant type solutions should obey. The involved linear conditions generate exponential functions, and so, the resulting exact solutions belong to the class of solutions which the multiple exp-function algorithm produces. Moreover, we remark that the multiple exp-function algorithm can also be applied to more generalized bilinear equations than Hirota ones, introduced in [50].

However, the nonlinearity of the resulting algebraic systems on the parameters involved in multiple wave solutions brings huge difficulty in finding appropriate phase shifts and wave frequencies. While using symbolic and algebraic manipulation programs, we often need to make reductions of the choices for the parameters so that the multiple exp-function algorithm works smoothly and efficiently on the computer. The guess and check strategy is always helpful, and one way to proceed is to take special values of the involved parameters and even the spatial and temporal variables.

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