

Reduced D-Kaup–Newell soliton hierarchies from $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{so}(3, \mathbb{R})$

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Two reduced D-Kaup–Newell spectral problems from $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{so}(3, \mathbb{R})$ are considered, and the corresponding soliton hierarchies are generated by using the zero curvature formulation. The resulting systems are shown to be bi-Hamiltonian and their hereditary recursion operators are explicitly computed.

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1. Introduction

There exist many interesting soliton hierarchies, which consist of commuting evolution equations. Typical examples of soliton hierarchies, which fit into the zero curvature formulation, include the Korteweg–de Vries hierarchy [1], the Ablowitz–Kaup–Newell–Segur hierarchy [2], the Kaup–Newell hierarchy [3, 4], the Wadati–Konno–Ichikawa hierarchy [5], the Boiti–Pempinelli–Tu hierarchy [6], the Dirac hierarchy [7], and the coupled AKNS–Kaup–Newell hierarchy [8]. Those hierarchies only contain dependent variables less than or equal to three, and soliton hierarchies with more dependent variables are highly complicated, which require considerable efforts in computations [9–11]. Integrable couplings associated with nonsemisimple loop algebras present such examples of soliton hierarchies, which can possess many dependent variables [12–15]. Spectral problems formulated using matrix Lie algebras are a starting point in generating soliton hierarchies (see, e.g. [16–22] for more details).

Very recently, the three-dimensional special orthogonal Lie algebra $\mathfrak{so}(3, \mathbb{R})$ has been used in constructing soliton hierarchies. The first few examples of soliton hierarchies associated with $\mathfrak{so}(3, \mathbb{R})$ are the Ablowitz–Kaup–Newell–Segur type soliton hierarchy [23], the Kaup–Newell type soliton hierarchy [24], the Wadati–Konno–Ichikawa type soliton hierarchy [25], and the Heisenberg type soliton hierarchy [26]. The Lie algebra $\mathfrak{so}(3, \mathbb{R})$ is the counterpart of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, and they are only the two three-dimensional real Lie algebras with a three-dimensional derived algebra.

In this paper, we would like to consider two reduced D-Kaup–Newell spectral problems possessing two dependent variables associated with $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{so}(3, \mathbb{R})$. The two D-Kaup–Newell spectral problems associated with $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{so}(3, \mathbb{R})$ are presented respectively in [19, 27], and both of them possess three dependent variables.

A standard procedure for generating soliton hierarchies [29, 28] is stated as follows. Let $\tilde{\mathfrak{g}}$ be the matrix loop algebra associated with a given matrix Lie algebra $\mathfrak{g}$. We first introduce a spatial spectral problem

$$\phi_x = U\phi, \quad U = U(u, \lambda) \in \tilde{\mathfrak{g}}, \quad (1.1)$$

where u denotes a column vector of dependent variables and λ is a spectral parameter. Then, we take a solution of the form

$$W = W(u, \lambda) = \sum_{i \geq 0} W_{0,i} \lambda^{-i}, \quad W_{0,i} \in \mathfrak{g}, \quad i \geq 0 \quad (1.2)$$

to the stationary zero curvature equation

$$W_x = [U, W]. \quad (1.3)$$

Next, we try to determine the Lax matrices

$$V^{[m]} = V^{[m]}(u, \lambda) = (\lambda^{f(m)} W)_+ + \Delta_m \in \tilde{\mathfrak{g}}, \quad m \geq 0, \quad (1.4)$$

$(P)_+$ denoting the polynomial part of P in λ and f being an appropriate function from $\mathbb{N}$ to $\mathbb{N}$, to formulate the temporal spectral problems

$$\phi_{t_m} = V^{[m]} \phi, \quad m \geq 0. \quad (1.5)$$

The modification terms $\Delta_m \in \tilde{\mathfrak{g}}$ should be selected, such that the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0, \quad (1.6)$$

will produce a hierarchy of soliton equations:

$$u_{t_m} = K_m(u), \quad m \geq 0. \quad (1.7)$$

A soliton hierarchy usually possesses Hamiltonian structures

$$u_{t_m} = K_m(u) = J \frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0, \quad (1.8)$$

where $\frac{\delta}{\delta u}$ is the variational derivative, J is a Hamiltonian operator, and $\mathcal{H}_m$, $m \geq 0$, are common conserved functionals. Such Hamiltonian structures can often be

generated by applying the trace identity, where $\mathfrak{g}$ is semisimple [28, 29]:

$$\frac{\delta}{\delta u} \int \text{tr} \left(\frac{\partial U}{\partial \lambda} W \right) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \text{tr} \left(\frac{\partial U}{\partial u} W \right), \quad \gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} \ln |\text{tr}(W^2)| \quad (1.9)$$

or the variational identity, where $\mathfrak{g}$ is nonsemisimple [30, 31]:

$$\frac{\delta}{\delta u} \int \left\langle \frac{\partial U}{\partial \lambda}, W \right\rangle dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \left\langle \frac{\partial U}{\partial u}, W \right\rangle, \quad \gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} \ln |\langle W, W \rangle|, \quad (1.10)$$

where $\langle \cdot, \cdot \rangle$ is a nondegenerate, symmetric and ad-invariant bilinear form on the underlying matrix loop algebra $\tilde{\mathfrak{g}}$.

The three-dimensional real special linear Lie algebra $sl(2, \mathbb{R})$ has the basis

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad (1.11)$$

whose commutators are

$$[e_1, e_2] = 2e_2, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = 2e_3, \quad (1.12)$$

whereas the special orthogonal Lie algebra $so(3, \mathbb{R})$ has the basis

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (1.13)$$

whose commutators are

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2. \quad (1.14)$$

We will adopt the following two matrix loop algebras associated with $sl(2, \mathbb{R})$ and $so(3, \mathbb{R})$:

$$\tilde{sl}(2, \mathbb{R}) = \left\{ \sum_{i \geq 0} M_i \lambda^{n-i} \mid M_i \in sl(2, \mathbb{R}), \ i \geq 0 \text{ and } n \in \mathbb{Z} \right\}, \quad (1.15)$$

and

$$\tilde{so}(3, \mathbb{R}) = \left\{ \sum_{i \geq 0} M_i \lambda^{n-i} \mid M_i \in so(3, \mathbb{R}), \ i \geq 0 \text{ and } n \in \mathbb{Z} \right\}. \quad (1.16)$$

Those are spaces of all Laurent series in λ with coefficients in $sl(2, \mathbb{R})$ or $so(3, \mathbb{R})$ and with a finite regular part.

The rest of the paper is structured as follows. In Secs. 2 and 3, we will introduce two reduced D-Kaup–Newell spectral problems associated with $sl(2, \mathbb{R})$ and $so(3, \mathbb{R})$, and then generate the corresponding soliton hierarchies of bi-Hamiltonian equations, respectively. In Sec. 4, we will give a conclusion and a few remarks.

2. A Reduced D-Kaup–Newell Soliton Hierarchy from $\mathfrak{sl}(2, \mathbb{R})$

In this section, we will derive a soliton hierarchy from matrix loop algebra $\widetilde{\mathfrak{sl}}(2, \mathbb{R})$. We begin with a spectral problem and let e_1, e_2, e_3 be defined by (1.11).

2.1. Spectral problem

Let α be an arbitrary real constant. Let us introduce a spectral matrix

$$U = U(u, \lambda) = (\lambda^2 + \alpha)e_1 + \lambda p e_2 + \lambda q e_3 = \begin{bmatrix} \lambda^2 + \alpha & \lambda p \\ \lambda q & -\lambda^2 - \alpha \end{bmatrix} \quad (2.1)$$

and consider the following isospectral problem

$$\phi_x = U\phi = \begin{bmatrix} \lambda^2 + \alpha & \lambda p \\ \lambda q & -\lambda^2 - \alpha \end{bmatrix} \phi, \quad u = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \quad (2.2)$$

associated with $\mathfrak{sl}(2, \mathbb{R})$.

It is known [19] that, the D-Kaup–Newell spectral problem associated with $\mathfrak{sl}(2, \mathbb{R})$ reads

$$\phi_x = U\phi = \begin{bmatrix} \lambda^2 + r & \lambda p \\ \lambda q & -\lambda^2 - r \end{bmatrix} \phi, \quad u = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix},$$

which possess three potentials: p, q and r . The new spectral problem (2.2) is just a reduced case of the above D-Kaup–Newell spectral problem under $r = \alpha$. It is, actually, also a generalization of the standard Kaup–Newell spectral problem [3], which corresponds to the case of $\alpha = 0$. There is another interesting reduction $r = \alpha pq$ of the D-Kaup–Newell spectral problem associated with $\mathfrak{sl}(2, \mathbb{R})$, which generates an integrable hierarchy (see [19, 32] for details).

2.2. Soliton hierarchy

Define

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \widetilde{\mathfrak{sl}}(2, \mathbb{R}) \quad (2.3)$$

and then, the stationary zero curvature equation $W_x = [U, W]$ becomes

$$\begin{cases} a_x = \lambda(pc - qb), \\ b_x = 2(\lambda^2 + \alpha)b - 2\lambda pa, \\ c_x = 2\lambda qa - 2(\lambda^2 + \alpha)c. \end{cases} \quad (2.4)$$

We further, assume that

$$a = \sum_{i \geq 0} a_i \lambda^{-2i}, \quad b = \sum_{i \geq 0} b_i \lambda^{-2i-1}, \quad c = \sum_{i \geq 0} c_i \lambda^{-2i-1} \quad (2.5)$$

and take the initial values

$$a_0 = 1, \quad b_0 = p, \quad c_0 = q \quad (2.6)$$

which are required by

$$a_{0,x} = pc_0 - qb_0, \quad b_0 - pa_0 = 0, \quad qa_0 - c_0 = 0. \quad (2.7)$$

Now based on (2.4), we have

$$\begin{cases} a_{i,x} = pc_i - qb_i, \\ b_{i,x} = 2\alpha b_i + 2b_{i+1} - 2pa_{i+1}, \\ c_{i,x} = 2qa_{i+1} - 2\alpha c_i - 2c_{i+1}. \end{cases} \quad i \geq 0. \quad (2.8)$$

From this, we can derive the recursion relations

$$\begin{cases} a_{i+1,x} = \alpha qb_i - \alpha pc_i - \frac{q}{2}b_{i,x} - \frac{p}{2}c_{i,x}, \\ b_{i+1} = \frac{1}{2}b_{i,x} - \alpha b_i + pa_{i+1}, \\ c_{i+1} = qa_{i+1} - \frac{1}{2}c_{i,x} - \alpha c_i, \end{cases} \quad i \geq 0, \quad (2.9)$$

since (2.8) tells

$$\begin{aligned} a_{i+1,x} &= pc_{i+1} - qb_{i+1} \\ &= p \left(qa_{i+1} - \frac{1}{2}c_{i,x} - \alpha c_i \right) - q \left(\frac{1}{2}b_{i,x} - \alpha b_i + pa_{i+1} \right) \\ &= \alpha qb_i - \alpha pc_i - \frac{q}{2}b_{i,x} - \frac{p}{2}c_{i,x}, \quad i \geq 0. \end{aligned}$$

We impose the conditions for integration:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1 \quad (2.10)$$

to determine the sequence of $\{a_i, b_i, c_i | i \geq 1\}$ uniquely. We list the first two sets as follows:

$$\begin{aligned} a_1 &= -\frac{1}{2}pq, \\ b_1 &= \frac{1}{2}p_x - \alpha p - \frac{1}{2}p^2q, \\ c_1 &= -\frac{1}{2}q_x - \alpha q - \frac{1}{2}pq^2; \\ a_2 &= \alpha pq + \frac{1}{4}(pq_x - p_xq) + \frac{3}{8}p^2q^2, \\ b_2 &= \alpha^2p - \alpha p_x + \frac{1}{4}p_{xx} - \frac{3}{4}pp_xq + \frac{3}{2}\alpha p^2q + \frac{3}{8}p^3q^2, \\ c_2 &= \alpha^2q + \alpha q_x + \frac{1}{4}q_{xx} + \frac{3}{4}pqq_x + \frac{3}{2}\alpha pq^2 + \frac{3}{8}p^2q^3. \end{aligned}$$

Based on the recursion relations (2.9), we can have

$$\begin{bmatrix} c_{i+1} \\ b_{i+1} \end{bmatrix} = \Psi \begin{bmatrix} c_i \\ b_i \end{bmatrix}, \quad i \geq 0, \quad (2.11)$$

where

$$\Psi = \begin{bmatrix} -\alpha - \frac{1}{2}\partial - \alpha q \partial^{-1} p - \frac{1}{2} q \partial^{-1} p \partial & \alpha q \partial^{-1} q - \frac{1}{2} q \partial^{-1} q \partial \\ -\alpha p \partial^{-1} p - \frac{1}{2} p \partial^{-1} p \partial & -\alpha + \frac{1}{2}\partial + \alpha p \partial^{-1} q - \frac{1}{2} p \partial^{-1} q \partial \end{bmatrix} \quad (2.12)$$

in which $\partial = \frac{\partial}{\partial x}$. We will see that all vectors $(c_i, b_i)^T$, $i \geq 0$, are gradient, and will generate conserved functionals.

Now for each $m \geq 0$, we introduce

$$\begin{aligned} V^{[m]} &= \lambda(\lambda^{2m+1}W)_+ \\ &= (\lambda^{2m+2}W)_+ - a_{m+1}e_1 \\ &= \sum_{i=0}^m [a_i \lambda^{2(m-i)+2} e_1 + b_i \lambda^{2(m-i)+1} e_2 + c_i \lambda^{2(m-i)+1} e_3] \end{aligned} \quad (2.13)$$

and the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0 \quad (2.14)$$

engender a hierarchy of solution equations

$$u_{t_m} = K_m = \begin{bmatrix} b_{m,x} - 2\alpha b_m \\ c_{m,x} + 2\alpha c_m \end{bmatrix} = J \begin{bmatrix} c_m \\ b_m \end{bmatrix}, \quad m \geq 0, \quad (2.15)$$

where

$$J = \begin{bmatrix} 0 & \partial - 2\alpha \\ \partial + 2\alpha & 0 \end{bmatrix}. \quad (2.16)$$

It is obvious that J is a Hamiltonian operator, since it is skew-adjoint and does not depend on the potentials [33].

2.3. Bi-Hamiltonian structures

It is easy to compute that

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 2\lambda & p \\ q & -2\lambda \end{bmatrix}, \quad \frac{\partial U}{\partial p} = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial q} = \begin{bmatrix} 0 & 0 \\ \lambda & 0 \end{bmatrix}$$

and so, we have

$$\text{tr} \left(W \frac{\partial U}{\partial \lambda} \right) = 4\lambda a + qb + pc, \quad \text{tr} \left(W \frac{\partial U}{\partial p} \right) = \lambda c, \quad \text{tr} \left(W \frac{\partial U}{\partial q} \right) = \lambda b.$$

By the trace identity (1.9), we get

$$\frac{\delta}{\delta u} \int (4\lambda a + qb + pc) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \begin{bmatrix} \lambda c \\ \lambda b \end{bmatrix}.$$

Balancing coefficients of all power of λ in above equality tells

$$\frac{\delta}{\delta u} \int (4a_{m+1} + qb_m + pc_m) dx = (\gamma - 2m) \begin{bmatrix} c_m \\ b_m \end{bmatrix}, \quad m \geq 0.$$

Taking $m = 1$, we obtain $\gamma = 0$, and further, we arrive at

$$\frac{\delta}{\delta u} \int \left(-\frac{4a_{m+1} + qb_m + pc_m}{2m} \right) dx = \begin{bmatrix} c_m \\ b_m \end{bmatrix}, \quad m \geq 1. \quad (2.17)$$

Therefore, we get Hamiltonian structures for the reduced D-Kaup–Newell soliton hierarchy (2.15) associated with $sl(2, \mathbb{R})$:

$$u_{t_m} = K_m = J \begin{bmatrix} c_m \\ b_m \end{bmatrix} = J \frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0, \quad (2.18)$$

where the Hamiltonian functionals are given by

$$\begin{aligned} \mathcal{H}_0 &= \int pq dx, \\ \mathcal{H}_m &= \int \left(-\frac{4a_{m+1} + qb_m + pc_m}{2m} \right) dx, \quad m \geq 1. \end{aligned} \quad (2.19)$$

Now, we introduce a second Hamiltonian operator

$$M = J\Psi = \Psi^\dagger J = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad (2.20)$$

where all elements can be explicitly worked out:

$$\left\{ \begin{aligned} M_{11} &= 2\alpha^2 p \partial^{-1} p + \alpha p \partial^{-1} p \partial - \alpha \partial p \partial^{-1} p - \frac{1}{2} \partial p \partial^{-1} p \partial, \\ M_{12} &= \frac{1}{2} \partial^2 - 2\alpha \partial + 2\alpha^2 - 2\alpha^2 p \partial^{-1} q + \alpha \partial p \partial^{-1} q + \alpha p \partial^{-1} q \partial \\ &\quad - \frac{1}{2} \partial p \partial^{-1} q \partial, \\ M_{21} &= -\frac{1}{2} \partial^2 - 2\alpha \partial - 2\alpha^2 - 2\alpha^2 q \partial^{-1} p - \alpha \partial q \partial^{-1} p - \alpha q \partial^{-1} p \partial \\ &\quad - \frac{1}{2} \partial q \partial^{-1} p \partial, \\ M_{22} &= 2\alpha^2 q \partial^{-1} q + \alpha \partial q \partial^{-1} q - \alpha q \partial^{-1} q \partial - \frac{1}{2} \partial q \partial^{-1} q \partial. \end{aligned} \right. \quad (2.21)$$

It is easy to check that M is skew-adjoint. By a long and tedious computation with Maple, we can verify that J and M constitute a Hamiltonian pair [34, 35], i.e. any linear combination N of J and M is skew-symmetry and satisfies the Jacobi identity:

$$\int K^T N'(u) [NS] T dx + \text{cycle}(K, S, T) = 0$$

for all vector fields K, S and T . Thus, the operator

$$\Phi = \Psi^\dagger = \begin{bmatrix} -\alpha + \frac{1}{2}\partial + \alpha p\partial^{-1}q - \frac{1}{2}\partial p\partial^{-1}q & \alpha p\partial^{-1}p - \frac{1}{2}\partial p\partial^{-1}p \\ -\alpha q\partial^{-1}q - \frac{1}{2}\partial q\partial^{-1}q & -\alpha - \frac{1}{2}\partial - \alpha q\partial^{-1}p - \frac{1}{2}\partial q\partial^{-1}p \end{bmatrix} \quad (2.22)$$

is hereditary, that is, it satisfies that

$$\Phi'(u)[\Phi K]S - \Phi\Phi'(u)[K]S = \Phi'(u)[\Phi S]K - \Phi\Phi'(u)[S]K$$

for all vector fields K and S (see [36] for definition of hereditary operators). Here, Φ' denotes the Gateaux derivative of Φ as usual.

The above condition for the hereditary operators is equivalent to

$$L_{\Phi K}\Phi = \Phi L_K\Phi \quad (2.23)$$

for any vector field K . Here $L_K\Phi$ is the Lie derivative defined by

$$(L_K\Phi)S := \Phi[K, S] - [K, \Phi S],$$

where $[\cdot, \cdot]$ is the Lie bracket of vector fields. Note that, for any autonomous operator $\Psi = \Psi(u, u_x, \dots)$ is a recursion operator of a given evolution equation $u_t = K(u)$, if and only if Φ satisfies

$$L_K\Phi = \Phi'[K] - [K', \Phi] = 0,$$

S' denoting the Gateaux derivative operator of a vector field S (see [33] for more details).

For the reduced D-Kaup–Newell hierarchy (2.15), it is direct to show that

$$L_{K_0}\Phi = \Phi'[K_0] - [K'_0, \Phi] = 0. \quad (2.24)$$

Thus, it follows now that the hierarchy (2.15) is bi-Hamiltonian:

$$u_{t_m} = K_m = J \frac{\delta \mathcal{H}_m}{\delta u} = M \frac{\delta \mathcal{H}_{m-1}}{\delta u}, \quad m \geq 1 \quad (2.25)$$

and Φ is a common hereditary recursion operator for the whole hierarchy (2.15). All this implies that the reduced D-Kaup–Newell hierarchy (2.15) is Liouville integrable [26]. We point out that no bi-Hamiltonian structure was presented for the D-Kaup–Newell soliton hierarchy in [19], though the hierarchy was shown to have infinitely many symmetries.

When $m = 0$, we get a linear system:

$$u_{t_0} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_0} = K_0 = \begin{bmatrix} p_x - 2\alpha p \\ q_x + 2\alpha q \end{bmatrix} = J \frac{\delta \mathcal{H}_0}{\delta u}. \quad (2.26)$$

When $m = 1$, we have a nonlinear system of bi-Hamiltonian equations:

$$\begin{aligned}
 u_{t_1} &= \begin{bmatrix} p \\ q \end{bmatrix}_{t_1} = K_1 \\
 &= \begin{bmatrix} 2\alpha^2 p + \alpha p^2 q + \frac{1}{2} p_{xx} - 2\alpha p_x - \frac{1}{2} p^2 q_x - p q p_x \\ -2\alpha^2 q - \alpha p q^2 - 2\alpha q_x - \frac{1}{2} q_{xx} - p q q_x - \frac{1}{2} p_x q^2 \end{bmatrix} \\
 &= J \frac{\delta \mathcal{H}_1}{\delta u} = M \frac{\delta \mathcal{H}_0}{\delta u},
 \end{aligned} \tag{2.27}$$

where $\mathcal{H}_1$ can also be explicitly given by

$$\mathcal{H}_1 = \int \left[-\alpha p q - \frac{1}{4} p^2 q^2 - \frac{1}{4} (p q_x - p_x q) \right] dx. \tag{2.28}$$

3. A Reduced D-Kaup–Newell Soliton Hierarchy from $so(3, \mathbb{R})$

In this section, we will construct the second soliton hierarchy based on the three-dimensional orthogonal Lie algebra $so(3, \mathbb{R})$. The basis of $so(3, \mathbb{R})$ consisting of e_1, e_2, e_3 is defined by (1.13).

3.1. Spectral problem

Let us introduce the second spectral matrix with an arbitrary real constant α :

$$\begin{aligned}
 U &= U(u, \lambda) = (\lambda^2 + \alpha) e_1 + \lambda p e_2 + \lambda q e_3 \\
 &= \begin{bmatrix} 0 & -\lambda q & -\lambda^2 - \alpha \\ \lambda q & 0 & -\lambda p \\ \lambda^2 + \alpha & \lambda p & 0 \end{bmatrix}
 \end{aligned} \tag{3.1}$$

and consider the following isospectral problem

$$\begin{aligned}
 \phi_x &= U \phi = \begin{bmatrix} 0 & -\lambda q & -\lambda^2 - \alpha \\ \lambda q & 0 & -\lambda p \\ \lambda^2 + \alpha & \lambda p & 0 \end{bmatrix} \phi, \\
 u &= \begin{bmatrix} p \\ q \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix},
 \end{aligned} \tag{3.2}$$

associated with $so(3, \mathbb{R})$.

The D-Kaup–Newell spectral problem associated with $so(3, \mathbb{R})$ was studied recently in [27]:

$$\phi_x = U \phi = \begin{bmatrix} 0 & -\lambda q & -\lambda^2 - r \\ \lambda q & 0 & -\lambda p \\ \lambda^2 + r & \lambda p & 0 \end{bmatrix} \phi, \quad u = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix},$$

which possesses three potentials: p, q and r . The new spectral problem (3.2) is a reduced case of the above D-Kaup–Newell spectral problem under $r = \alpha$. In fact, it is also a generalization of the standard Kaup–Newell spectral problem in [24], which corresponds to the case of $\alpha = 0$. Another interesting reduction $r = \alpha(p^2 + q^2)$ of the D-Kaup–Newell type spectral problem associated with $\mathfrak{so}(3, \mathbb{R})$, generating an integrable hierarchy, has been proposed and studied in [37].

3.2. Soliton hierarchy

Define

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} 0 & -c & -a \\ c & 0 & -b \\ a & b & 0 \end{bmatrix} \in \widetilde{\mathfrak{so}}(3, \mathbb{R}) \quad (3.3)$$

and then, the stationary zero curvature equation $W_x = [U, W]$ becomes

$$\begin{cases} a_x = \lambda(pc - qb), \\ b_x = \lambda qa - (\lambda^2 + \alpha)c, \\ c_x = (\lambda^2 + \alpha)b - \lambda pa. \end{cases} \quad (3.4)$$

We further, assume that

$$a = \sum_{i \geq 0} a_i \lambda^{-2i}, \quad b = \sum_{i \geq 0} b_i \lambda^{-2i-1}, \quad c = \sum_{i \geq 0} c_i \lambda^{-2i-1} \quad (3.5)$$

and take the initial values

$$a_0 = 1, \quad b_0 = p, \quad c_0 = q \quad (3.6)$$

which are required by

$$a_{0,x} = pc_0 - qb_0, \quad -c_0 + qa_0 = 0, \quad -pa_0 + b_0 = 0. \quad (3.7)$$

Now based on (3.4), we have

$$\begin{cases} a_{i,x} = pc_i - qb_i, \\ b_{i,x} = qa_{i+1} - c_{i+1} - \alpha c_i, \\ c_{i,x} = b_{i+1} + \alpha b_i - pa_{i+1}. \end{cases} \quad i \geq 0. \quad (3.8)$$

From this, we can derive the recursion relations

$$\begin{cases} a_{i+1,x} = -pb_{i,x} + \alpha qb_i - qc_{i,x} - \alpha pc_i, \\ b_{i+1} = c_{i,x} - \alpha b_i + pa_{i+1}, \\ c_{i+1} = -b_{i,x} - \alpha c_i + qa_{i+1}, \end{cases} \quad i \geq 0, \quad (3.9)$$

since (3.8) tells

$$\begin{aligned} a_{i+1,x} &= pc_{i+1} - qb_{i+1} \\ &= p(-b_{i,x} - \alpha c_i + qa_{i+1}) - q(c_{i,x} - \alpha b_i + pa_{i+1}) \\ &= -pb_{i,x} + \alpha qb_i - qc_{i,x} - \alpha pc_i, \quad i \geq 0. \end{aligned}$$

We impose the conditions for integration:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1 \quad (3.10)$$

to determine the sequence of $\{a_i, b_i, c_i | i \geq 1\}$ uniquely. We list the first two sets as follows:

$$\begin{aligned} a_1 &= -\frac{1}{2}(p^2 + q^2), \\ b_1 &= q_x - \alpha p - \frac{1}{2}p(p^2 + q^2), \\ c_1 &= -p_x - \alpha q - \frac{1}{2}q(p^2 + q^2); \\ a_2 &= \alpha(p^2 + q^2) + p_x q - p q_x + \frac{3}{8}(p^2 + q^2)^2, \\ b_2 &= -p_{xx} - 2\alpha q_x + \alpha^2 p + \frac{3}{2}(\alpha p - q_x)(p^2 + q^2) + \frac{3}{8}p(p^2 + q^2)^2, \\ c_2 &= -q_{xx} + 2\alpha p_x + \alpha^2 q + \frac{3}{2}(\alpha q + p_x)(p^2 + q^2) + \frac{3}{8}q(p^2 + q^2)^2. \end{aligned}$$

It follows directly from the recursion relations (3.9), that, we have

$$\begin{bmatrix} b_{i+1} \\ c_{i+1} \end{bmatrix} = \Psi \begin{bmatrix} b_i \\ c_i \end{bmatrix}, \quad i \geq 0, \quad (3.11)$$

where

$$\Psi = \begin{bmatrix} -\alpha - p\partial^{-1}p\partial + \alpha p\partial^{-1}q & \partial - \alpha p\partial^{-1}p - p\partial^{-1}q\partial \\ -\partial - q\partial^{-1}p\partial + \alpha q\partial^{-1}q & -\alpha - \alpha q\partial^{-1}p - q\partial^{-1}q\partial \end{bmatrix}. \quad (3.12)$$

We will see that all vectors $(b_i, c_i)^T$, $i \geq 0$, above are gradient, and will generate conserved functionals.

Now for each $m \geq 0$, we introduce

$$\begin{aligned} V^{[m]} &= \lambda(\lambda^{2m+1}W)_+ \\ &= (\lambda^{2m+2}W)_+ - a_{m+1}e_1 \\ &= \sum_{i=0}^m [a_i \lambda^{2(m-i)+2} e_1 + b_i \lambda^{2(m-i)+1} e_2 + c_i \lambda^{2(m-i)+1} e_3] \end{aligned} \quad (3.13)$$

and the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0 \quad (3.14)$$

engender a hierarchy of solution equations

$$u_{t_m} = K_m = \begin{bmatrix} b_{m,x} + \alpha c_m \\ c_{m,x} - \alpha b_m \end{bmatrix} = J \begin{bmatrix} b_m \\ c_m \end{bmatrix}, \quad m \geq 0, \quad (3.15)$$

with J being defined by

$$J = \begin{bmatrix} \partial & \alpha \\ -\alpha & \partial \end{bmatrix}. \quad (3.16)$$

It is direct to check that J is Hamiltonian.

3.3. Bi-Hamiltonian structures

It is easy to see that

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 0 & -q & -2\lambda \\ q & 0 & -p \\ 2\lambda & p & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial p} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial q} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and so, we obtain

$$\text{tr} \left(W \frac{\partial U}{\partial \lambda} \right) = -4\lambda a - 2pb - 2qc, \quad \text{tr} \left(W \frac{\partial U}{\partial p} \right) = -2\lambda b, \quad \text{tr} \left(W \frac{\partial U}{\partial q} \right) = -2\lambda c.$$

By the trace identity (1.9), we get

$$\frac{\delta}{\delta u} \int (2\lambda a + pb + qc) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \begin{bmatrix} \lambda b \\ \lambda c \end{bmatrix}.$$

Balancing coefficients of all power of λ in above equality tells

$$\frac{\delta}{\delta u} \int (2a_{m+1} + pb_m + qc_m) dx = (\gamma - 2m) \begin{bmatrix} b_m \\ c_m \end{bmatrix}, \quad m \geq 0.$$

Taking $m = 1$, we obtain $\gamma = 0$, and further, we have

$$\frac{\delta}{\delta u} \int \left(-\frac{2a_{m+1} + pb_m + qc_m}{2m} \right) dx = \begin{bmatrix} b_m \\ c_m \end{bmatrix}, \quad m \geq 1. \quad (3.17)$$

This way, we obtain Hamiltonian structures for the reduced D-Kaup–Newell hierarchy (3.15) associated with $\text{so}(3, \mathbb{R})$:

$$u_{t_m} = K_m = J \begin{bmatrix} b_m \\ c_m \end{bmatrix} = J \frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0, \quad (3.18)$$

where the Hamiltonian functionals are given by

$$\begin{aligned} \mathcal{H}_0 &= \int \frac{1}{2} (p^2 + q^2) dx, \\ \mathcal{H}_m &= \int \left(-\frac{2a_{m+1} + pb_m + qc_m}{2m} \right) dx, \quad m \geq 1. \end{aligned} \quad (3.19)$$

Introduce a second Hamiltonian operator

$$M = J\Psi = \Psi^\dagger J = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad (3.20)$$

where all elements can be explicitly worked out:

$$\begin{cases} M_{11} = -2\alpha\partial + \alpha^2 q\partial^{-1}q + \alpha\partial p\partial^{-1}q - \alpha q\partial^{-1}p\partial - \partial p\partial^{-1}p\partial, \\ M_{12} = -\alpha^2 + \partial^2 - \alpha^2 q\partial^{-1}p - \alpha q\partial^{-1}q\partial - \alpha\partial p\partial^{-1}p - \partial p\partial^{-1}q\partial, \\ M_{21} = \alpha^2 - \partial^2 - \alpha^2 p\partial^{-1}q + \alpha\partial q\partial^{-1}q + \alpha p\partial^{-1}p\partial - \partial q\partial^{-1}p\partial, \\ M_{22} = -2\alpha\partial + \alpha^2 p\partial^{-1}p - \alpha\partial q\partial^{-1}p + \alpha p\partial^{-1}q\partial - \partial q\partial^{-1}q\partial. \end{cases} \quad (3.21)$$

A direct computation by Maple can show that J and M constitute a Hamiltonian pair. Thus once proving $L_{K_0}\Phi = 0$, we can show by the same argument as in the previous section that the operator

$$\Phi = \Psi^\dagger = \begin{bmatrix} -\alpha - \partial p\partial^{-1}p - \alpha q\partial^{-1}p & \partial - \partial p\partial^{-1}q - \alpha q\partial^{-1}q \\ -\partial + \alpha p\partial^{-1}p - \partial q\partial^{-1}p & -\alpha + \alpha p\partial^{-1}q - \partial q\partial^{-1}q \end{bmatrix} \quad (3.22)$$

is a common hereditary recursion operator for the whole hierarchy (3.15), and the hierarchy (3.15) is bi-Hamiltonian:

$$u_{t_m} = K_m = J \frac{\delta \mathcal{H}_m}{\delta u} = M \frac{\delta \mathcal{H}_{m-1}}{\delta u}, \quad m \geq 1. \quad (3.23)$$

All this shows that the reduced D-Kaup–Newell hierarchy (3.15) associated with $so(3, \mathbb{R})$ is Liouville integrable [26].

When $m = 0$, we get a linear system

$$u_{t_0} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_0} = K_0 = \begin{bmatrix} p_x + \alpha q \\ -\alpha p + q_x \end{bmatrix} = J \frac{\delta \mathcal{H}_0}{\delta u}. \quad (3.24)$$

When $m = 1$, we obtain a nonlinear system of bi-Hamiltonian equations:

$$\begin{aligned} u_{t_1} &= \begin{bmatrix} p \\ q \end{bmatrix}_{t_1} = K_1 \\ &= \begin{bmatrix} -\alpha^2 q - 2\alpha p_x + q_{xx} - \frac{1}{2}(\alpha q + p_x)(p^2 + q^2) - p(pp_x + qq_x) \\ \alpha^2 p - 2\alpha q_x - p_{xx} + \frac{1}{2}(\alpha p + q_x)(p^2 + q^2) - q(pp_x + qq_x) \end{bmatrix} \\ &= J \frac{\delta \mathcal{H}_1}{\delta u} = M \frac{\delta \mathcal{H}_0}{\delta u}, \end{aligned} \quad (3.25)$$

where $\mathcal{H}_1$ can also be explicitly given by

$$\mathcal{H}_1 = \int \left[\frac{1}{2}(-\alpha p^2 - \alpha q^2 - p_x q + p q_x) - \frac{1}{8}(p^2 + q^2)^2 \right] dx. \quad (3.26)$$

We point out that, no bi-Hamiltonian structure presented for the second non-reduced D-Kaup–Newell soliton hierarchy associated with $so(3, \mathbb{R})$ [27]. Only one of the two differential operators in the recursive structure is Hamiltonian, and the other is quasi-Hamiltonian for the non-reduced D-Kaup–Newell soliton hierarchy in [27].

4. Concluding Remarks

In this paper, we have introduced two reduced D-Kaup–Newell spectral problems associated with $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{so}(3, \mathbb{R})$ and generated their hierarchies of bi-Hamiltonian equations via the zero curvature formulation. The Liouville integrability of the resulting soliton hierarchies has been shown upon establishing bi-Hamiltonian structures through the trace identity.

Compared to the D-Kaup–Newell soliton hierarchies [19, 27], our soliton hierarchies present different features such as bi-Hamiltonian structures, though our spectral problems involve less potentials. Moreover, the newly presented Hamiltonian pairs display different recursion operator structures from the known Kaup–Newell recursion operators associated with $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{so}(3, \mathbb{R})$ [3, 24].

Different choices of spectral matrices from matrix loop algebras bring us various soliton hierarchies with hereditary recursion operators, and any study on integrable couplings will help in classifying multicomponent integrable systems associated with nonsemisimple Lie algebras.

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References

- [1] P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.* **21** (1968) 467–490.
- [2] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems, *Stud. Appl. Math.* **53** (1974) 249–315.
- [3] D. J. Kaup and A. C. Newell, An exact solution for a derivative nonlinear Schrödinger equation, *J. Math. Phys.* **19** (1978) 798–801.
- [4] V. S. Gerdzhikov, M. I. Ivanov and P. P. Kulish, Quadratic bundle and nonlinear equations, *Theoret. Math. Phys.* **44** (1980) 784–795.
- [5] M. Wadati, K. Konno and Y. H. Ichikawa, New integrable nonlinear evolution equations, *J. Phys. Soc. Jpn.* **47** (1979) 1698–1700.
- [6] M. Boiti, F. Pempinelli and G. Z. Tu, Canonical structure of soliton equations via isospectral eigenvalue problems, *Nuovo Cimento B* **79** (1984) 231–265.
- [7] H. Grosse, New solitons connected to the Dirac equation, *Phys. Rep.* **134** (1986) 297–304.
- [8] W. X. Ma and R. G. Zhou, A coupled AKNS-Kaup-Newell soliton hierarchy, *J. Math. Phys.* **40** (1999) 4419–4428.

- [9] T. M. Alberty, T. Koikawa and R. Sasaki, Canonical structure of soliton equations I, *Phys. D* **5** (1982) 43–65.
- [10] A. P. Fordy and M. Antonowicz, Coupled KdV Equations with multi-Hamiltonian structures, *Phys. D* **28** (1987) 345–357.
- [11] W. X. Ma and Z. X. Zhou, Coupled integrable systems associated with a polynomial spectral problem and their Virasoro symmetry algebras, *Progr. Theor. Phys.* **96** (1996) 449–457.
- [12] W. X. Ma and B. Fuchssteiner, Integrable theory of the perturbation equations, *Chaos Solitons Fractals* **7** (1996) 1227–1250.
- [13] W. X. Ma, X. X. Xu and Y. F. Zhang, Semi-direct sums of Lie algebras and continuous integrable couplings, *Phys. Lett. A* **351** (2006) 125–130.
- [14] W. X. Ma, J. H. Meng and H. Q. Zhang, Integrable couplings, variational identities and Hamiltonian formulations, *Global J. Math. Sci.* **1** (2012) 1–17.
- [15] W. X. Ma, Integrable couplings and matrix loop algebras, *AIP Conf. Proc.* **1562** (2013) 105–122.
- [16] V. G. Drinfeld and V. V. Sokolov, Equations of Korteweg-de Vries type and simple Lie algebras, *Sov. Math. Dokl.* **23** (1981) 457–462.
- [17] M. A. Ablowitz and P. A. Clarkson, Solitons, *Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, Cambridge, 1991).
- [18] D. Y. Chen, *Introduction to Solitons* (Science Press, Beijing, 2006).
- [19] X. G. Geng and W. X. Ma, A generalized Kaup-Newell spectral problem, soliton equations and finite-dimensional integrable systems, *Il Nuovo Cimento A* **108** (1995) 477–486.
- [20] W. X. Ma, B. Fuchssteiner and W. Oevel, A three-by-three matrix spectral problem for AKNS hierarchy and its binary nonlinearization, *Phys. A* **233** (1996) 331–354.
- [21] E. Fan, Integrable evolution systems based on Gerdjikov-Ivanov equations, bi-Hamiltonian structure, finite-dimensional integrable systems and N-fold Darboux transformation, *J. Math. Phys.* **41** (2000) 7769–7782.
- [22] W. X. Ma, J. S. He and Z. Y. Qin, A supertrace identity and its applications to superintegrable systems, *J. Math. Phys.* **49** (2008) 033511.
- [23] W. X. Ma, A soliton hierarchy associated with $so(3, \mathbb{R})$, *Appl. Math. Comput.* **220** (2013) 117–122.
- [24] W. X. Ma, A spectral problem based on $so(3, \mathbb{R})$ and its associated commuting soliton equations, *J. Math. Phys.* **54** (2013) 8; Article ID: 103509.
- [25] W. X. Ma, S. Manukure and H. C. Zheng, A counterpart of the WKI soliton hierarchy associated with $so(3, \mathbb{R})$, *Z. Naturforsch.* **69** (2014) 411–419.
- [26] W. X. Ma, An integrable counterpart of the D-AKNS soliton hierarchy from $so(3, \mathbb{R})$, *Phys. Lett. A* **378** (2014) 1717–1720.
- [27] C. G. Shi, W. X. Ma and M. McAnally, Integrable counterparts of the D-Kaup-Newell soliton hierarchy, *Appl. Math. Comput.* **248** (2014) 463–469.
- [28] W. X. Ma, A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction, *Chin. Ann. Math. A* **13** (1992) 115–123; *Chin. J. Contemp. Math.* **13** (1992) 79–89.
- [29] G. Z. Tu, On Liouville integrability of zero-curvature equations and the Yang hierarchy, *J. Phys. A: Math. Gen.* **22** (1989) 2375–2392.
- [30] W. X. Ma and M. Chen, Hamiltonian and quasi-Hamiltonian structures associated with semi-direct sums of Lie algebras, *J. Phys. A: Math. Gen.* **39** (2006) 10787–10801.
- [31] W. X. Ma, Variational identities and Hamiltonian structures, *AIP Conf. Proc.* **1212** (2010) 1–27.

- [32] W. X. Ma, C. G. Shi, E. A. Appiah, C. X. Li and S. F. Shen, An integrable generalization of the Kaup-Newell soliton hierarchy, *Phys. Scr.* **89** (2014) 1–8.
- [33] P. J. Olver, Applications of Lie groups to differential equations, *Graduate Texts in Mathematics*, Vol. 107 (Springer-Verlag, New York, 1986).
- [34] B. Fuchssteiner and A. S. Fokas, Symplectic structures, their Backlund transformations and hereditary symmetries, *Phys. D* **4** (1981) 47–66.
- [35] F. Magri, A simple model of the integrable Hamiltonian equation, *J. Math. Phys.* **19** (1978) 1156–1162.
- [36] B. Fuchssteiner, Application of hereditary symmetries to nonlinear evolution equations, *Nonlinear Anal. Theor. Methods Appl.* **3** (1979) 849–862.
- [37] C. X. Li, S. F. Shen, W. X. Ma and S. M. Yu, Two new matrix spectral problems associated with $\mathfrak{so}(3, \mathbb{R})$ and their corresponding integrable soliton hierarchies, preprint, 2014, arXiv: 1406.1326.