



Hirota bilinear equations with linear subspaces of solutions

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ABSTRACT

We explore when Hirota bilinear equations possess linear subspaces of solutions. First, we establish a sufficient and necessary criterion for the existence of linear subspaces of exponential traveling wave solutions to Hirota bilinear equations. Second, we show that multivariate polynomials whose zeros form a vector space can generate the desired Hirota bilinear equations with given linear subspaces of solutions, and formulate such multivariate polynomials by using multivariate polynomials which have one and only one zero. Third, applying an algorithm using weights, we present parameterizations of wave numbers and frequencies achieved by using one parameter to compute the desired Hirota bilinear equations.

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1. Introduction

Typical integrable equations, such as the KdV equation, the Boussinesq equation and the KP equation, possess multi-soliton solutions, generated from combinations of multiple exponential waves on the basis of their Hirota bilinear forms [1]. The multiple exp-function method provides a direct and efficient way to construct generic multiple exponential wave solutions to nonlinear equations, which is particularly powerful while using computer algebra systems [2].

Besides soliton solutions, another class of interesting multiple exponential wave solutions are linear combinations of exponential waves, which implies the existence of linear subspaces of solutions. It is shown that a kind of nonlinear equations can possess such a linear superposition principle, and a sufficient criterion for its existence was furnished for Hirota bilinear equations in [3].

We would, in this paper, like to completely characterize Hirota bilinear equations which possess linear subspaces of exponential traveling wave solutions. The involved exponential wave solutions may and may not satisfy the corresponding dispersion relation. Our resulting theory will exhibit that Hirota bilinear equations share some common features with linear equations, which explains, to some extent, why bilinear equations can be solved analytically. Based on the Hirota bilinear formulation, we will establish a condition which is both sufficient and necessary for guaranteeing the applicability of the linear superposition principle for exponential waves [3].

Interestingly, multivariate polynomials whose zeros form a vector space can generate the desired Hirota bilinear equations which possess given linear subspaces of solutions. However, it is still an open question to us how to characterize a multivariate polynomial with one and only one zero point. This problem seems more difficult than Hilbert's 17th problem. The 3 + 1 dimensional KP, Jimbo–Miwa and BKP equations will be covered as special cases of the computed illustrative examples.

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The paper is organized as follows. In Section 2, we will analyze the linear superposition principle for exponential traveling waves and establish a sufficient and necessary criterion for the existence of linear subspaces of exponential traveling wave solutions to Hirota bilinear equations.

In Section 3, upon analyzing zeros of a kind of multivariate polynomials, we will show that multivariate polynomials whose zeros form a vector space can generate the desired Hirota bilinear equations with given linear subspaces of solutions, and formulate such multivariate polynomials by using multivariate polynomials which have one and only one zero. This gives a clear idea of what the sought Hirota bilinear equations look like.

In Section 4, applying an algorithm using weights, we will present a class of parameterizations of wave numbers and frequencies achieved by using one parameter to generate the desired Hirota bilinear equations, and compute illustrative examples to shed light on the parameterizations.

In Section 5, we will give a few concluding remarks.

2. Linear superposition principle

Let $M \in \mathbb{N}$ be given. We recall Hirota's bilinear operators defined by the following rule [4]:

$$\begin{aligned} D_{x_1}^{n_1} \cdots D_{x_M}^{n_M} f \cdot g &= (\partial_{x_1} - \partial_{x'_1})^{n_1} \cdots (\partial_{x_M} - \partial_{x'_M})^{n_M} f(x_1, \dots, x_M) g(x'_1, \dots, x'_M) \big|_{x'_1=x_1, \dots, x'_M=x_M} \\ &= \partial_{x'_1}^{n_1} \cdots \partial_{x'_M}^{n_M} f(x_1 + x'_1, \dots, x_M + x'_M) g(x_1 - x'_1, \dots, x_M - x'_M) \big|_{x'_1=\dots=x'_M=0}, \end{aligned} \quad (2.1)$$

where $n_1, \dots, n_M$ are arbitrary nonnegative integers. For example, we have:

$$D_x f \cdot g = f_x g - f g_x, \quad D_x D_t f \cdot g = f_{xt} g - f_x g_t - f_t g_x + f g_{xt}.$$

Let P be a polynomial in M variables satisfying

$$P(\underbrace{0, \dots, 0}_M) = 0, \quad (2.2)$$

which means that P has no constant term. The corresponding Hirota bilinear equation reads

$$P(D_x) f \cdot f = P(D_{x_1}, \dots, D_{x_M}) f \cdot f = 0. \quad (2.3)$$

Note that a term of odd degree in P produces zero in the resulting Hirota bilinear equation, and so we assume that P is an even polynomial, i.e.,

$$P(-x_1, \dots, -x_M) = P(x_1, \dots, x_M). \quad (2.4)$$

Various equations of mathematical physics are written as Hirota bilinear forms through dependent variable transformations [1,5]. For example, the KdV equation:

$$u_t + 6uu_x + u_{xxx} = 0$$

and the Boussinesq equation:

$$u_{tt} + (u^2)_{xx} + u_{xxxx} = 0$$

are written as

$$(D_x D_t + D_x^4) f \cdot f = 0$$

under the transformation $u = 2(\ln f)_{xx}$ and

$$(D_t^2 + D_x^4) f \cdot f = 0$$

under the transformation $u = 6(\ln f)_{xx}$, respectively. Wronskian solutions, including solitons, positons and complexitons [6–8], and quasi-periodic solutions [9–11] can be presented systematically, based on Hirota bilinear forms.

Let us now fix $N \in \mathbb{N}$ and introduce N wave variables:

$$\eta_i = k_{1,i} x_1 + \cdots + k_{M,i} x_M, \quad 1 \leq i \leq N, \quad (2.5)$$

and N exponential wave functions:

$$f_i = e^{\eta_i} = e^{k_{1,i} x_1 + \cdots + k_{M,i} x_M}, \quad 1 \leq i \leq N, \quad (2.6)$$

where the $k_{j,i}$'s are constants. Noting a bilinear identity [1]:

$$P(D_{x_1}, \dots, D_{x_M}) e^{\eta_i} \cdot e^{\eta_j} = P(k_{1,i} - k_{1,j}, \dots, k_{M,i} - k_{M,j}) e^{\eta_i + \eta_j}, \quad (2.7)$$

it follows directly from (2.2) that every of the exponential wave functions f_i , $1 \leq i \leq N$, presents a solution to the Hirota bilinear equation (2.3).

Next, we consider a linear combination:

$$f = \varepsilon_1 f_1 + \cdots + \varepsilon_N f_N = \varepsilon_1 e^{\eta_1} + \cdots + \varepsilon_N e^{\eta_N}, \quad (2.8)$$

where ε_i , $1 \leq i \leq N$, are arbitrary constants. A natural question is when this linear combination will still present a solution to the Hirota bilinear equation (2.3).

To answer this question, we make the following computation by using (2.7), (2.2) and (2.4):

$$\begin{aligned} P(D_{x_1}, \dots, D_{x_M}) f \cdot f &= \sum_{i,j=1}^N \varepsilon_i \varepsilon_j P(D_{x_1}, \dots, D_{x_M}) e^{\eta_i} \cdot e^{\eta_j} = \sum_{i,j=1}^N \varepsilon_i \varepsilon_j P(k_{1,i} - k_{1,j}, \dots, k_{M,i} - k_{M,j}) e^{\eta_i + \eta_j} \\ &= \sum_{1 \leq i < j \leq N} \varepsilon_i \varepsilon_j [P(k_{1,i} - k_{1,j}, \dots, k_{M,i} - k_{M,j}) + P(k_{1,j} - k_{1,i}, \dots, k_{M,j} - k_{M,i})] e^{\eta_i + \eta_j} \\ &= \sum_{1 \leq i < j \leq N} 2\varepsilon_i \varepsilon_j P(k_{1,i} - k_{1,j}, \dots, k_{M,i} - k_{M,j}) e^{\eta_i + \eta_j}. \end{aligned}$$

This will play a crucial role in establishing the linear superposition principle for the exponential waves e^{η_i} , $1 \leq i \leq N$.

It now follows that a linear combination function f defined by (2.8) solves the Hirota bilinear equation (2.3) if and only if the condition of:

$$P(k_{1,i} - k_{1,j}, \dots, k_{M,i} - k_{M,j}) = 0, \quad 1 \leq i < j \leq N, \quad (2.9)$$

is satisfied. The condition (2.9) gives us a system of nonlinear algebraic equations on the wave related numbers k_{ij} 's, as soon as the polynomial P is given. We will see that there are more opportunities to get solutions for the variables k_{ij} 's in higher dimensional cases, because there are more parameters to select in the system of algebraic equations resulting from (2.9).

We summarize our result as follows.

Theorem 2.1 (Criterion for linear superposition principle). *Let $P(x_1, \dots, x_M)$ be an even polynomial satisfying $P(0, \dots, 0) = 0$ and the N wave variables η_i , $1 \leq i \leq N$, be defined by $\eta_i = k_{1,i}x_1 + \cdots + k_{M,i}x_M$, $1 \leq i \leq N$, where the k_{ij} 's are all constants. Then any linear combination of the exponential waves e^{η_i} , $1 \leq i \leq N$, solves the Hirota bilinear equation $P(D_{x_1}, \dots, D_{x_M})f \cdot f = 0$ if and only if $P(k_{1,i} - k_{1,j}, \dots, k_{M,i} - k_{M,j}) = 0$, $1 \leq i < j \leq N$.*

This tells precisely when a linear superposition of exponential wave solutions still solves a given Hirota bilinear equation. The theorem describes the interrelation between Hirota bilinear equations and the linear superposition principle for exponential waves more elaborately than that given in [3]. It also paves a way of constructing N -wave solutions to Hirota bilinear equations. The system (2.9) is a resonance condition we need to deal with (see [12] for resonance of 2-solitons). Once we get a solution of the wave related numbers k_{ij} 's by solving the system (2.9), we can present an N -wave solution, formed by (2.8), to the considered Hirota bilinear equation.

Below we just list two special examples in $3 + 1$ dimensions with

$$\eta_i = k_i x + l_i y + m_i z + \omega_i t, \quad 1 \leq i \leq N, \quad (2.10)$$

to give an idea about what the wave related numbers should be taken. More general examples will be created later in Section 4.

The one example is the following polynomial:

$$P(x, y, z, t) = x^3 y - tx + ty - z^2. \quad (2.11)$$

The corresponding condition (2.9) reads

$$\begin{aligned} P(k_i - k_j, l_i - l_j, m_i - m_j, \omega_i - \omega_j) &= k_i^3 l_i - k_j^3 l_j - 3k_i^2 k_j l_i + 3k_i^2 k_j l_j + 3k_i k_j^2 l_i - 3k_i k_j^2 l_j - k_j^3 l_i + k_j^3 l_j + \omega_i l_i - \omega_j l_j \\ &\quad - \omega_j l_i + \omega_j l_j - \omega_i k_i + \omega_i k_j + \omega_j k_i - \omega_j k_j - m_i^2 + 2m_i m_j - m_j^2 = 0, \end{aligned}$$

and the resulting Hirota bilinear equation is

$$(D_x^3 D_y - D_t D_x + D_t D_y - D_z^2) f \cdot f = 0, \quad (2.12)$$

namely,

$$(f_{xxx} - f_{tx} + f_{ty} - f_{zz})f - 3f_{xy}f_x + 3f_{xy}f_{xx} - f_y f_{xxx} + f_x f_x - f_t f_y + f_z^2 = 0.$$

Under the transformation $u = 2(\ln f)_x$, this equation is mapped into

$$u_{xxx} + 3(u_x u_y)_x - u_{tx} + u_{ty} - u_{zz} = 0. \quad (2.13)$$

Based on the linear superposition principle for exponential waves in Theorem 2.1, solving the above system on the wave related numbers leads to an N -wave solution to the nonlinear Eq. (2.13):

$$u = 2(\ln f)_x, \quad f = \sum_{i=1}^N \varepsilon_i f_i = \sum_{i=1}^N \varepsilon_i e^{k_i x - (1/3)a^2 k_i y + a k_i^2 z - 4[a^2/(a^2+3)]k_i^3 t}, \quad (2.14)$$

where the ε_i 's and k_i 's are arbitrary constants. Each exponential wave f_i in the solution f satisfies the corresponding nonlinear dispersion relation, i.e., we have

$$P(k_i, l_i, m_i, \omega_i) = 0, \quad 1 \leq i \leq N.$$

The other example is the following polynomial:

$$P(x, y, z, t) = ty - x^3y + 3x^2 + 3z^2. \quad (2.15)$$

The corresponding condition (2.9) reads

$$P(k_i - k_j, l_i - l_j, m_i - m_j, \omega_i - \omega_j) = \omega_i l_i - \omega_i l_j - \omega_j l_i + \omega_j l_j - k_i^3 l_i + k_i^3 l_j + 3k_i^2 k_j l_i - 3k_i^2 k_j l_j - 3k_i k_j^2 l_i + 3k_i k_j^2 l_j + k_j^3 l_i - k_j^3 l_j + 3m_i^2 - 6m_i m_j + 3m_j^2 + 3k_i^2 - 6k_i k_j + 3k_j^2 = 0,$$

and the resulting Hirota bilinear equation is:

$$(D_t D_y - D_x^3 D_y + 3D_x^2 + 3D_z^2) f \cdot f = 0, \quad (2.16)$$

namely,

$$(f_{ty} - f_{xxx} f_y + 3f_{xx} + 3f_{zz})f - f_t f_y + f_{xxx} f_y + 3f_{xx} f_x - 3f_{xx} f_{xy} - 3f_x^2 - 3f_z^2 = 0.$$

Under the transformation $u = 2(\ln f)_x$, this equation is mapped into

$$u_{yt} - u_{xxx} - 3(u_x u_y)_x + 3u_{xx} + 3u_{zz} = 0. \quad (2.17)$$

Based on the linear superposition principle for exponential waves in Theorem 2.1, solving the above system on the wave related numbers leads to an N -wave solution to the nonlinear Eq. (2.17):

$$u = 2(\ln f)_x, \quad f = \sum_{i=1}^N \varepsilon_i f_i = \sum_{i=1}^N \varepsilon_i e^{k_i x + (a^2 + 1)k_i^{-1} y + a k_i z + k_i^3 t}, \quad (2.18)$$

where a , the ε_i 's and the k_i 's are arbitrary constants. However, each exponential wave f_i in the solution f does not satisfy the corresponding nonlinear dispersion relation, i.e., we have

$$P(k_i, l_i, m_i, \omega_i) \neq 0, \quad 1 \leq i \leq N.$$

It is also direct to prove that

$$P(D_x, D_y, D_z, D_t)(e^\xi f) \cdot (e^\eta g) = e^{\xi + \eta} P(D_x + k_1 - k_2, D_y + l_1 - l_2, D_z + m_1 - m_2, D_t - \omega_1 + \omega_2) f \cdot g, \quad (2.19)$$

where $\xi = k_1 x + l_1 y + m_1 z - \omega_1 t$, $\eta = k_2 x + l_2 y + m_2 z - \omega_2 t$, and P is a polynomial in the indicated variables. Taking

$$\xi = \eta = \xi_0 = k_0 x + l_0 y + m_0 z - \omega_0 t,$$

the above identity yields

$$P(D_x, D_y, D_z, D_t)(e^{\xi_0} f) \cdot (e^{\xi_0} g) = e^{2\xi_0} P(D_x, D_y, D_z, D_t) f \cdot g. \quad (2.20)$$

Therefore, we can get a new class of multiple exponential wave solutions by $f' = e^{\xi_0} f$, where f is an original multiple exponential wave solution like any one in (2.14) and (2.18); and such solutions form a new linear subspace of solutions.

3. Bilinear equations with given linear subspaces of solutions

Taking one of the wave variables η_i , $1 \leq i \leq N$, to be a constant, say, taking

$$\eta_{i_0} = \varepsilon_{i_0}, \quad \text{i.e.,} \quad k_{j, i_0} = 0, \quad 1 \leq j \leq M, \quad (3.1)$$

where $1 \leq i_0 \leq N$ is fixed, the N -wave solution condition (2.9) requires that all other wave related numbers need to satisfy the dispersion relation of the Hirota bilinear equation (2.3):

$$P(k_{1,i}, \dots, k_{M,i}) = 0, \quad 1 \leq i \leq N, \quad i \neq i_0. \quad (3.2)$$

The resulting solution corresponds to a specific case of N -soliton solutions by the Hirota perturbation technique [1], truncated at the second-order perturbation term.

Combining the dispersion relation (3.2) with the N -wave solution condition (2.9) yields the following sufficient condition on P for the corresponding Hirota bilinear equation (2.3) to satisfy the linear superposition principle for exponential waves:

$$P(\mathbf{k}) = P(\mathbf{l}) = 0 \Rightarrow P(\mathbf{k} - \mathbf{l}) = 0, \quad (3.3)$$

where $\mathbf{k}$ and $\mathbf{l}$ are M dimensional vectors. Let us prove a property of zeros of such multivariate polynomials as follows.

Theorem 3.1 (Vector space of zeros). Let $P(x_1, \dots, x_M)$ be a real (or complex) multivariate polynomial (which could contain terms of both even and odd degree). Suppose P has at least one zero and satisfies that if $P(\mathbf{k}) = P(\mathbf{l}) = 0$, then $P(\mathbf{k} - \mathbf{l}) = 0$. Then all zeros of P form a real (or complex) vector space.

Proof. Let S be the set of all zeros of P and $\mathbf{k}_0$ be a zero of P . By the property (3.3), we know

$$P(\mathbf{0}) = P(\mathbf{k}_0 - \mathbf{k}_0) = 0, \quad P(-\mathbf{k}_0) = P(\mathbf{0} - \mathbf{k}_0) = 0. \quad (3.4)$$

Therefore, S contains the null vector $\mathbf{0}$ and negative vectors of zeros.

Now let $\mathbf{k} \in S$ and α be a constant. We want to prove $\alpha\mathbf{k} \in S$. If $\mathbf{k} = \mathbf{0}$, obviously we have $P(\alpha\mathbf{k}) = P(\mathbf{0}) = 0$. Assume $\mathbf{k} \neq \mathbf{0}$. Since $-\mathbf{k} \in S$, the property (3.3) implies that

$$P(2\mathbf{k}) = P(\mathbf{k} - (-\mathbf{k})) = 0, \dots, P(n\mathbf{k}) = P((n-1)\mathbf{k} - (-\mathbf{k})) = 0, \dots$$

By induction, we know $P(n\mathbf{k}) = 0$ for all $n \in \mathbb{N}$. Note that $P(\gamma\mathbf{k})$ is a polynomial in γ , and so it should have only finitely many zeros. But all $n\mathbf{k}$, $n \in \mathbb{N}$, present infinitely many zeros since $\mathbf{k} \neq \mathbf{0}$. Therefore

$$P(\gamma\mathbf{k}) = 0 \quad \forall \gamma,$$

and this tells $\alpha\mathbf{k} \in S$.

If $\mathbf{k}, \mathbf{l} \in S$ and α and β are two constants, then $\alpha\mathbf{k} \in S$ and $\beta\mathbf{l} \in S$. Further we have $-\beta\mathbf{l} \in S$ and:

$$P(\alpha\mathbf{k} + \beta\mathbf{l}) = P(\alpha\mathbf{k} - (-\beta\mathbf{l})) = 0. \quad (3.5)$$

This says that $\alpha\mathbf{k} + \beta\mathbf{l} \in S$. It then follows that the set S of zeros of P forms a vector space. $\square$

Given an n -dimensional linear subspace V_n of $\mathbb{R}^M$, let us introduce a constant matrix $A = (a_{ij})_{M'M}$ of rank n for any $M' \in \mathbb{N}$ such that the solution space of a linear system:

$$A\mathbf{x} = \mathbf{0}, \quad \mathbf{x} = (x_1, \dots, x_M)^T, \quad (3.6)$$

defines the n -dimensional subspace V_n . Let $Q(y_1, \dots, y_{M'})$ be a multivariate polynomial in

$$\mathbf{y} = (y_1, \dots, y_{M'})^T \quad (3.7)$$

and possess only one zero: $\mathbf{y} = \mathbf{y}_0$. Then

$$P(x_1, \dots, x_M) = Q((A\mathbf{x} + \mathbf{y}_0)^T) \quad (3.8)$$

gives us a multivariate polynomial satisfying the property (3.3), and the resulting Hirota bilinear equation possesses the linear subspace of exponential wave solutions defined by

$$f = \sum_{i=1}^N \varepsilon_i e^{k_{1,i}x_1 + \dots + k_{M,i}x_M}, \quad N \geq 1, \quad (3.9)$$

where $A(k_{1,i} - k_{1,j}, \dots, k_{M,i} - k_{M,j})^T = 0$, $1 \leq i \neq j \leq N$, and the ε_i 's are arbitrary constants. In generating examples of such Hirota bilinear equations, one remaining question for us is how to determine if a multivariate polynomial has one and only one zero point, which seems more difficult than Hilbert's 17th problem.

We conclude our analysis in the following theorem.

Theorem 3.2 (Structure of Hirota bilinear equations). Let $M, M', n \in \mathbb{N}$ and $A = (a_{ij})_{M'M}$ be a constant matrix of rank n . Suppose that $Q(y_1, \dots, y_{M'})$ is a multivariate polynomial in $\mathbf{y} = (y_1, \dots, y_{M'})^T$ and possesses only one zero: $\mathbf{y} = \mathbf{y}_0$. Then

$$P(x_1, \dots, x_M) = Q((A\mathbf{x} + \mathbf{y}_0)^T), \quad \mathbf{x} = (x_1, \dots, x_M)^T,$$

is a multivariate polynomial whose zeros form an n -dimensional subspace, and the corresponding Hirota bilinear equation

$$P(D_{x_1}, \dots, D_{x_M})f \cdot f = Q((AD\mathbf{x} + \mathbf{y}_0)^T)f \cdot f = 0$$

possesses a linear subspace of solutions defined by

$$f = \sum_{i=1}^N \varepsilon_i e^{k_{1,i}x_1 + \dots + k_{M,i}x_M}, \quad N \geq 1,$$

where $A(k_{1,i} - k_{1,j}, \dots, k_{M,i} - k_{M,j})^T = 0$, $1 \leq i \neq j \leq N$, and the ε_i 's are arbitrary constants.

In what follows, we will present two examples to shed light on the algorithm in Theorem 3.2.

The first example has

$$Q(y_1, y_2) = y_1^2 + 2y_2^2, \quad \mathbf{y}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix}, \quad \mathbf{x} = (x, y, z, t)^T. \quad (3.10)$$

Then the associated multivariate polynomial is

$$P(x, y, z, t) = x^2 - 6xz + 17z^2 + 2y^2 + 8yz - 4yt - 8zt + 2t^2, \quad (3.11)$$

and the corresponding Hirota bilinear equation reads

$$(D_x^2 - 6D_x D_z + 17D_z^2 + 2D_y^2 + 8D_y D_z - 4D_y D_t - 8D_z D_t + 2D_t^2)f \cdot f = 0. \quad (3.12)$$

This bilinear equation possesses the linear subspace of solutions defined by

$$f = \sum_{i=1}^N \varepsilon_i f_i = e^{k_0 x + l_0 y + m_0 z - \omega_0 t} \sum_{i=1}^N \varepsilon_i e^{3m_i x - (2m_i + \omega_i)y + m_i z - \omega_i t}, \quad N \geq 1, \quad (3.13)$$

where the ε_i 's, m_i 's and ω_i 's are arbitrary constants but k_0, l_0, m_0 and ω_0 are arbitrary fixed constants. Obviously, all exponential wave f_i in the solution f satisfy the corresponding nonlinear dispersion relation iff $e^{k_0 x + l_0 y + m_0 z - \omega_0 t}$ satisfies the corresponding nonlinear dispersion relation.

The second example has

$$Q(y_1, y_2) = y_1^4 + (y_2 - 1)^2, \quad \mathbf{y}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix}, \quad \mathbf{x} = (x, y, z, t)^T. \quad (3.14)$$

Then the associated multivariate polynomial is

$$P(x, y, z, t) = x^4 - 8x^3 z + 24x^2 z^2 - 32xz^3 + 16z^4 + x^2 + 6xy + 9y^2 - 2xt - 6yt + t^2, \quad (3.15)$$

and the corresponding Hirota bilinear equation reads

$$(D_x^4 - 8D_x^3 D_z + 24D_x^2 D_z^2 - 32D_x D_z^3 + 16D_z^4 + D_x^2 + 6D_x D_y + 9D_y^2 - 2D_x D_t - 6D_y D_t + D_t^2)f \cdot f = 0. \quad (3.16)$$

This bilinear equation possesses the linear subspace of solutions defined by

$$f = \sum_{i=1}^N \varepsilon_i f_i = e^{k_0 x + l_0 y + m_0 z - \omega_0 t} \sum_{i=1}^N \varepsilon_i e^{2m_i x + l_i y + m_i z + (2m_i + 3l_i)t}, \quad N \geq 1, \quad (3.17)$$

where the ε_i 's, l_i 's, and m_i 's are arbitrary constants but k_0, l_0, m_0 and ω_0 are arbitrary fixed constants. Obviously, all exponential wave f_i in the solution f satisfy the corresponding nonlinear dispersion relation iff $e^{k_0 x + l_0 y + m_0 z - \omega_0 t}$ satisfies the corresponding nonlinear dispersion relation.

4. Parameterizations of wave numbers and frequencies

Let us now consider how to compute Hirota bilinear equations with linear subspaces of solutions by using parameterizations of wave numbers and frequencies. The problem is how to construct a multivariate polynomial $P(x_1, \dots, x_M)$ with no constant term such that

$$P(k_{1,1} - k_{1,2}, \dots, k_{M,1} - k_{M,2}) = 0, \quad (4.1)$$

for two sets of constants $k_{1,i}, \dots, k_{M,i}, i = 1, 2$.

Let us first introduce the weights for the independent variables:

$$(w(x_1), \dots, w(x_M)) = (n_1, \dots, n_M), \quad (4.2)$$

where each weight $w(x_i) = n_i$ is an integer, and then form a polynomial $P(x_1, \dots, x_M)$ being homogeneous in some weight (see also [3]). Second, for $i = 1, 2$, we parameterize the constants $k_{1,i}, \dots, k_{M,i}$, consisting of wave numbers and frequencies, using a parameter k_i as follows:

$$k_{j,i} = b_j k_i^{n_i}, \quad 1 \leq j \leq M, \quad (4.3)$$

where the b_j 's are constants to be determined, to balance the system (4.1). Then, plugging the parameterized constants into (4.1), we collect terms by powers of the parameters k_1 and k_2 , and set the coefficient of each power to zero, to obtain algebraic equations on the constants b_j 's and the coefficients of the polynomial $P(x_1, \dots, x_M)$. Finally, solve the resulting algebraic equations to determine the polynomial $P(x_1, \dots, x_M)$ and the parameterization.

Now, based on (2.20), the resulting parameterization obviously tells that the obtained Hirota bilinear equation possesses the linear subspace of solutions defined by

$$f = \sum_{i=1}^N \varepsilon_i f_i = e^{k_{1,0}x_1 + \dots + k_{M,0}x_M} \sum_{i=1}^N \varepsilon_i e^{b_1 k_i^{n_1} x_1 + \dots + b_M k_i^{n_M} x_M}, \quad N \geq 1, \quad (4.4)$$

where the ε_i 's and k_i 's are arbitrary constants but the $k_{i,0}$'s are arbitrary fixed constants.

The following list four illustrative examples in $3 + 1$ dimensions, which apply the above parameterization achieved by using one parameter.

4.1. Examples with the dispersion relation

Example 1. Weights $(w(x), w(y), w(z), w(t)) = (1, 1, 2, 3)$:

Let us introduce the weights of independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, 1, 2, 3). \quad (4.5)$$

Then, a general even polynomial being homogeneous in weight 4 reads

$$P = c_1 x^4 + c_2 y^4 + c_3 x^3 y + c_4 x y^3 + c_5 x^2 y^2 + c_6 x t + c_7 y t + c_8 z^2, \quad (4.6)$$

a special case of which is the polynomial in (2.11). Following the parameterization of wave numbers and frequency in (4.3), the wave variables read

$$\eta_i = k_i x + b_1 k_i y + b_2 k_i^2 z + b_3 k_i^3 t, \quad 1 \leq i \leq N,$$

where k_i , $1 \leq i \leq N$, are arbitrary constants, but b_1, b_2 and b_3 are constants to be determined. In this example, the corresponding Hirota bilinear equation $P(D_x, D_y, D_z, D_t) f \cdot f = 0$ has the linear subspace of N -wave solutions defined by

$$f = \sum_{i=1}^N c_i f_i = e^{k_0 x + l_0 y + m_0 z - \omega_0 t} \sum_{i=1}^N c_i e^{k_i x + b_1 k_i y + b_2 k_i^2 z + b_3 k_i^3 t}, \quad (4.7)$$

where k_0, l_0, m_0 and ω_0 are arbitrary fixed constants. There are three cases to determine the involved constants b_1, b_2 and b_3 as follows.

(a) The case of $c_8 \neq 0$:

A direct computation yields

$$b_3 = -\frac{4(c_2 b_1^4 + c_4 b_1^3 + c_5 b_1^2 + c_3 b_1 + c_1)}{c_7 b_1 + c_6}, \quad b_2^2 = -\frac{3b_3(c_7 b_1 + c_6)}{4c_8}, \quad (4.8)$$

where b_1 needs to satisfy

$$c_7 b_1 + c_6 \neq 0, \quad c_2 b_1^4 + c_4 b_1^3 + c_5 b_1^2 + c_3 b_1 + c_1 \neq 0, \quad (4.9)$$

to keep the non-triviality $b_1 b_2 b_3 \neq 0$.

(b) The case of $c_8 = 0$ but $c_7 \neq 0$:

In this case, we automatically have $c_6 \neq 0$, to keep the non-triviality $b_1 b_2 b_3 \neq 0$. A similar direct computation tells that

$$b_1 = -\frac{c_6}{c_7} \quad (4.10)$$

and b_2 is arbitrary, and that

$$c_2 c_6^4 - c_4 c_6^3 c_7 + c_5 c_6^2 c_7^2 - c_3 c_6 c_7^3 + c_1 c_7^4 = 0. \quad (4.11)$$

This condition on the Hirota bilinear equation considered is equivalent to

$$c_2 b_1^4 + c_4 b_1^3 + c_5 b_1^2 + c_3 b_1 + c_1 = 0. \quad (4.12)$$

(c) The case of $c_7 = c_8 = 0$:

In this case, we automatically have $c_6 = 0$, to keep the non-triviality $b_1 b_2 b_3 \neq 0$. This time, b_1 must satisfy (4.12), but b_2 can be arbitrary.

Example 2. Weights $(w(x), w(y), w(z), w(t)) = (1, 2, 2, 3)$:

Let us introduce the weights of independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, 2, 2, 3). \quad (4.13)$$

Then, a general even polynomial being homogeneous in weight 4 reads

$$P = c_1 x^4 + c_2 x t + c_3 y^2 + c_4 y z + c_5 z^2. \quad (4.14)$$

Following the parameterization of wave numbers and frequency in (4.3), the N wave variables read

$$\eta_i = k_i x + b_1 k_i^2 y + b_2 k_i^2 z + b_3 k_i^3 t, \quad 1 \leq i \leq N,$$

where k_i , $1 \leq i \leq N$, are arbitrary constants, but b_1, b_2 and b_3 are constants to be determined.

This way, a direct computation by Maple tells that the corresponding Hirota bilinear equation $P(D_x, D_y, D_z, D_t)f \cdot f = 0$ possesses the linear subspace of N -wave solutions defined by

$$f = \sum_{i=1}^N \varepsilon_i f_i = e^{k_0 x + l_0 y + m_0 z - \omega_0 t} \sum_{i=1}^N \varepsilon_i e^{k_i x + b_1 k_i^2 y + b_2 k_i^2 z + b_3 k_i^3 t}, \quad (4.15)$$

where the ε_i 's and k_i 's are arbitrary, but k_0, l_0, m_0 and ω_0 are arbitrary fixed constants and b_1, b_2 and b_3 satisfy

$$c_2 b_3 + 4c_1 = 0, \quad c_3 b_1^2 + c_5 b_2^2 + c_4 b_1 b_2 = 3c_1. \quad (4.16)$$

The first equation above determines b_3 uniquely if $c_2 \neq 0$ or does not present any condition on b_3 if $c_1 = c_2 = 0$. There are two cases to determine b_1 and b_2 , which are depicted as follows. Since c_3 and c_5 are symmetric in P , we only consider the case of $c_3 \neq 0$.

When $c_3 \neq 0$, then b_2 needs to satisfy

$$(c_4^2 - 4c_3 c_5) b_2^2 + 12c_1 c_3 \geq 0, \quad (4.17)$$

to have a real b_1 , and b_1 is determined in terms of b_2 by the second equation in (4.16). The value of b_2 may have lots of chooses. Concretely speaking, if

$$c_1 c_3 \geq 0, \quad c_4^2 - 4c_3 c_5 \geq 0,$$

then b_2 is arbitrary; if

$$c_1 c_3 \geq 0, \quad c_4^2 - 4c_3 c_5 < 0,$$

then b_2 must be

$$-\frac{2\sqrt{3c_1 c_3 (4c_3 c_5 - c_4^2)}}{4c_3 c_5 - c_4^2} \leq b_2 \leq \frac{2\sqrt{3c_1 c_3 (4c_3 c_5 - c_4^2)}}{4c_3 c_5 - c_4^2};$$

if

$$c_1 c_3 < 0, \quad c_4^2 - 4c_3 c_5 > 0,$$

then b_2 must be

$$b_2 \leq -\frac{2\sqrt{3c_1 c_3 (4c_3 c_5 - c_4^2)}}{c_4^2 - 4c_3 c_5} \quad \text{or} \quad b_2 \geq \frac{2\sqrt{3c_1 c_3 (4c_3 c_5 - c_4^2)}}{c_4^2 - 4c_3 c_5};$$

and if

$$c_1 c_3 < 0, \quad c_4^2 - 4c_3 c_5 \leq 0,$$

then there is no value for b_2 to choose.

We point out that if $c_1 c_3 c_4 c_5 \neq 0$, then there will definitely be non-zero b_1 and b_2 . This is because we have

$$c_5 b_2^2 = 3c_1, \quad (c_4^2 - 4c_3 c_5) b_2^2 + 12c_1 c_3 = 0,$$

which leads to $c_4 = 0$, if, for example, b_1 has to be zero.

4.2. Examples without the dispersion relation

Example 1. Weights $(w(x), w(y), w(z), w(t)) = (1, -1, -2, 3)$:

Let us introduce the weights of independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, -1, -2, 3). \quad (4.18)$$

Then, an even polynomial being homogeneous in weight 2 reads

$$P = c_1 t y + c_2 x^3 y + c_3 x^4 y^2 + c_4 x^2. \quad (4.19)$$

Following the parameterization of wave numbers and frequency in (4.3), the wave variables read

$$\eta_i = k_i x + b_1 k_i^{-1} y + b_2 k_i^{-2} z + b_3 k_i^3 t, \quad 1 \leq i \leq N,$$

where k_i , $1 \leq i \leq N$, are arbitrary constants, but b_1, b_2 and b_3 are constants to be determined.

Now, a similar direct computation tells that the corresponding Hirota bilinear equation

$$(c_1 D_t D_y + c_2 D_x^3 D_y + c_3 D_x^4 D_y^2 + c_4 D_x^2) f \cdot f = 0 \quad (4.20)$$

possesses the linear subspace of N -wave solutions defined by

$$f = \sum_{i=1}^N \varepsilon_i f_i = e^{k_0 x + l_0 y + m_0 z - \omega_0 t} \sum_{i=1}^N \varepsilon_i e^{k_i x + b_1 k_i^{-1} y + b_2 k_i^{-2} z + b_3 k_i^3 t}, \quad (4.21)$$

where the ε_i 's and k_i 's and b_2 are arbitrary, but k_0, l_0, m_0 and ω_0 are arbitrary fixed constants and b_1 and b_3 satisfy

$$3c_2 b_1 + c_6 = 0, \quad c_1 b_1 b_3 + c_2 b_1 = 0, \quad c_3 b_1^2 = 0. \quad (4.22)$$

Then, taking $c_3 = 0$ tells that the Hirota bilinear equation

$$(c_1 D_t D_y + c_2 D_x^3 D_y + c_4 D_x^2) f \cdot f = 0 \quad (4.23)$$

has the N -wave solution defined by (4.21) with

$$b_1 = -\frac{c_4}{3c_2}, \quad b_3 = -\frac{c_2}{c_1}. \quad (4.24)$$

Example 2. Weights $(w(x), w(y), w(z), w(t)) = (1, -1, 1, 3)$:

Let us now introduce the weights of independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, -1, 1, 3). \quad (4.25)$$

Then, an even polynomial being homogeneous in weight 2 reads

$$P = c_1 x^2 + c_2 z^2 + c_3 xz + c_4 ty + c_5 x^3 y + c_6 yz^3, \quad (4.26)$$

a special case of which is the polynomial in (2.15). Following the parameterization of wave numbers and frequency in (4.3), the wave variables read

$$\eta_i = k_i x + b_1 k_i^{-1} y + b_2 k_i z + b_3 k_i^3 t, \quad 1 \leq i \leq N,$$

where k_i , $1 \leq i \leq N$, are arbitrary constants, but b_1, b_2 and b_3 are constants to be determined.

By Maple, a similar direct computation tells that the corresponding Hirota bilinear equation

$$(c_1 D_x^2 + c_2 D_z^2 + c_3 D_x D_z + c_4 D_t D_y + c_5 D_x^3 D_y + c_6 D_y D_z^3) f \cdot f = 0 \quad (4.27)$$

possesses the linear subspace of N -wave solutions defined by

$$f = \sum_{i=1}^N \varepsilon_i f_i = e^{k_0 x + l_0 y + m_0 z - \omega_0 t} \sum_{i=1}^N \varepsilon_i e^{k_i x + b_1 k_i^{-1} y + b_2 k_i z + b_3 k_i^3 t}, \quad (4.28)$$

where the ε_i 's and k_i 's and b_2 are arbitrary, but k_0, l_0, m_0 and ω_0 are arbitrary fixed constants and b_1 and b_3 are defined by

$$b_1 = -\frac{c_1 + c_3 b_2 + c_2 b_2^2}{3(c_5 + c_6 b_2^3)}, \quad b_3 = -\frac{c_5 + c_6 b_2^3}{c_4}. \quad (4.29)$$

5. Concluding remarks

We analyzed when Hirota bilinear equations possess the linear superposition principle for exponential traveling waves and explored how to construct multivariate polynomials which generate such Hirota bilinear equations. Particularly, we proved that multivariate polynomials whose zeros form a vector space could engender the desired Hirota bilinear equations. The associated multivariate polynomials were formulated by using multivariate polynomials which have one and only one zero. Another approach was presented by adopting a class of parameterizations of wave numbers and frequencies achieved by using one parameter, and a few illustrative examples with N -waves satisfying or not satisfying the nonlinear dispersion relation were computed by Maple.

Theorem 2.1 presents a condition which is both sufficient and necessary for guaranteeing the applicability of the linear superposition principle for exponential waves. It also follows from **Theorem 2.1** that if we begin with an arbitrary multivariate polynomial $P(x_1, \dots, x_M)$, then a sufficient and necessary condition to guarantee the applicability of the discussed linear superposition principle is

$$P_{\text{even}}(k_{1,i} - k_{1,j}, \dots, k_{M,i} - k_{M,j}) = 0, \quad 1 \leq i < j \leq N,$$

where P_{even} is the part of terms of even order of P . The whole polynomial P does not need to satisfy the above condition, unlike in [3]. The four examples in the previous section were also analyzed in [3] using a sufficient condition presented there, but our analysis in this paper is complete.

Though the linear superposition principle does not apply to nonlinear differential equations in general, we would like to emphasize that Hirota bilinear equations still can possess the linear superposition principle among exponential wave solutions. This guarantees the existence of linear subspaces of solutions and amends diversity of exact solutions by other direct methods [13–17]. The resulting theory also explains, to some extent, why Hirota bilinear equations are solvable analytically [18], particularly by determinant and pfaffian techniques [19,20].

Future research problems for us include how to achieve parameterizations of wave numbers and frequencies by using several parameters and how to create multivariate polynomials whose Hirota bilinear equations possess the linear subspaces of other specific solutions such as periodic solutions and special function solutions. It also remains open how to characterize multivariate polynomials which have one and only one zero. This seems more difficult than Hilbert's 17th problem and sounds a bit iffy to us.

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