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Lump-wave structures and their dynamic behavior in a refined Bogoyavlensky-Konopelchenko-like model

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A B S T R A C T

The aim of this study is to explore lump wave structures in a (2+1)-dimensional refined Bogoyavlensky-Konopelchenko model characterized by spatially balanced dispersion. The underlying nonlinear equation is derived using generalized bilinear derivatives related to the prime number three. By computing positive quadratic solutions of the bilinear form symbolically, lump solutions are obtained. These localized waves propagate along straight-line trajectories at constant velocity, with stationary centers of zero amplitude. Their formation arises from the intricate balance between nonlinearity and dispersion. The methodology of this work consists of the systematic symbolic treatment of the considered nonlinear model, together with the associated algebraic system, to derive its lump wave structures.

1. Introduction

In the study of soliton theory and integrable systems, nonlinear wave structures such as solitons, rogue waves, and lump waves emerge from the delicate balance between nonlinearity and dispersion. Finding exact solutions to these models is crucial in mathematical physics and engineering, as they reveal fundamental wave dynamics and provide analytical insights into complex behaviors. However, obtaining these solutions remains challenging, motivating both analytical and numerical approaches. Two main frameworks for this purpose are the inverse scattering transform (IST) [1] and Hirota's bilinear method [2]. The IST, a nonlinear analogue of the Fourier transform, solves integrable PDEs through Lax pairs [3] and is widely used to study soliton and dispersive dynamics [4]. Hirota's method offers a direct algebraic approach, especially effective in constructing solitons and lumps in higher-dimensional systems [5–8].

Beyond solitons, lump waves and rogue waves form another important class of localized structures [9]. Lump waves are algebraically decaying solutions that are rationally localized in space at fixed time [10]. Notably, the KPI equation supports a variety of lump solutions [11], some of which arise as long-wave limits of multi-soliton interactions [12]. These phenomena extend beyond integrable systems, with lump-type solutions also appearing in nonintegrable KP-, BKP-, and Boussinesq-type models [13], and even in linear higher-dimensional wave equations through superposition [14].

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One fundamental mathematical question in this area is to identify broader classes of PDE structures and nonlinear models that admit lump wave solutions. Not every PDE supports such solutions, and it is therefore important to systematically explore those models that do. Existing studies have considered both linear PDEs (see, e.g., [14,15]) and nonlinear PDEs (see, e.g., [16–21]).

One widely used approach is the sum-of-squares ansatz. This method has the notable advantage of ensuring the positivity of solutions to bilinear equations and, consequently, the analyticity of solutions to the corresponding nonlinear models, thereby preventing blow-up behavior. However, a key challenge lies in the appropriate choice of variables when solving the resulting nonlinear algebraic systems.

Lump solutions are commonly constructed using the sum-of-squares ansatz, where a positive quadratic function is substituted into a bilinear equation [9]. Logarithmic differentiation is then applied to yield explicit lump solutions of the original nonlinear PDE. In this study, we employ this approach to a refined Bogoyavlensky-Konopelchenko-like (rBK-like) equation in $(2+1)$ dimensions, which includes three nonlinear interactions and five dispersive terms.

The novelty of this work lies in exploring the intricate balance between nonlinearity and dispersion that sustains localized lump structures. These structures are derived and analyzed symbolically through the stationary properties of the underlying quadratic function.

2. A refined model with spatially balanced dispersions

It is well established that Hirota's bilinear derivatives can be extended to include differential operators of arbitrary order. A general class of these extensions, known as generalized bilinear differential operators, was introduced in [22] and is defined by:

$$D_{p,x}^{m} D_{p,y}^{n} D_{p,t}^{k} f\cdot f=\left(\frac{\partial}{\partial x}+\alpha_p\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial y}+\alpha_p\frac{\partial}{\partial y'}\right)^n\left(\frac{\partial}{\partial t}+\alpha_p\frac{\partial}{\partial t'}\right)^k f(x,y,t)f(x',y',t')|_{x'=x,y'=y,t'=t}, \tag{1}$$

where

$$\alpha_p^k=(-1)^{r(k)} \text{ where } k\equiv r(k) \bmod p,\ 0\le r(k)<p. \tag{2}$$

For example, when $p=3$, the sequence of coefficients is given by:

$$\alpha_3=-1,\ \alpha_3^2=\alpha_3^3=1,\ \alpha_3^4=-1,\ \alpha_3^5=\alpha_3^6=1,\ \cdots. \tag{3}$$

This follows a repeating block of length 3: $-1,1,1$. When $p=5$, the sequence takes the form:

$$\alpha_5=-1,\ \alpha_5^2=1,\ \alpha_5^3=-1,\ \alpha_5^4=\alpha_5^5=1,\ \alpha_5^6=-1,\ \alpha_5^7=1,\ \alpha_5^8=-1,\ \alpha_5^9=\alpha_5^{10}=1,\ \cdots. \tag{4}$$

This corresponds to a repeating block of length 5: $-1,1,-1,1,1$.

2.1. A refined bilinear model

For the case $p=3$, we ntroduce the following refined Bogoyavlensky-Konopelchenko-like (rBK-like) bilinear equation:

$$\begin{aligned}
F_{\text{rBK-like}}(f) :=& \left(\sigma_1 D_{3,x}^4+\sigma_2 D_{3,x}^3 D_{3,y}+\sigma_3 D_{3,x}^2 D_{3,y}+\delta_1 D_{3,t} D_{3,x}\right.\\
&\left.+\delta_2 D_{3,t} D_{3,y}+\delta_3 D_{3,x}^2+\delta_4 D_{3,y}^2+\delta_5 D_{3,t}^2\right) f\cdot f\\
=& 2[3\sigma_1 f_{xx}^2+3\sigma_2 f_{xx} f_{xy}+\sigma_3 f f_{xxy}+\delta_1(f_{tx} f-f_t f_x)+\delta_2(f_{ty} f-f_t f_y)\\
&+\delta_3(f_{xx} f-f_x^2)+\delta_4(f_{yy} f-f_y^2)+\delta_5(f_{tt} f-f_t^2)]=0,
\end{aligned} \tag{5}$$

where $D_{3,x}$, $D_{3,y}$ and $D_{3,t}$ represent the generalized bilinear derivatives, while σ_i for $1\le i\le 3$ and δ_j for $1\le j\le 5$ are arbitrary constants. This proposed construction includes five spatially balanced second-order derivative terms, which account for linear dispersions, and extends the classical Bogoyavlensky-Konopelchenko equation. The case for $p=2$ was previously analyzed in [23].

2.2. Nonlinear form of the model equation

By redefining the dependent variables as

$$u=2(\ln f)_x,\ v=2(\ln f)_y, \tag{6}$$

we obtain the refined BK-like (rBK-like) model in its nonlinear form:

$$\begin{aligned}
P_{\text{rBK-like}}(u,v) :=& \sigma_1\left(3u_x u_{xx}+3uu_x^2+\frac{3}{2}u^3 u_x+\frac{3}{2}u^2 u_{xx}\right)\\
&+\sigma_2\left(\frac{9}{8}u^2 u_x v+\frac{3}{8}u^3 u_y+\frac{3}{4}uu_{xx}v+\frac{3}{4}u_x^2 v+\frac{3}{4}u^2 u_{xy}+\frac{9}{4}uu_x u_y+\frac{3}{2}u_{xx}u_y+\frac{3}{2}u_x u_{xy}\right)\\
&+\sigma_3\left(\frac{1}{2}uvu_x+\frac{1}{4}u^2 u_y+\frac{1}{2}u_{xx}v+\frac{3}{2}u_x u_y+uu_{xy}+u_{xxy}\right)\\
&+\delta_1 u_{tx}+\delta_2 u_{ty}+\delta_3 u_{xx}+\delta_4 u_{yy}+\delta_5 u_{tt}=0,
\end{aligned} \tag{7}$$

subject to the compatibility condition $v_x = u_y$. This equation includes three sets of nonlinear terms along with five dispersive contributions. Reductions of interest arise when only three dispersive coefficients are non-zero. Specifically, if $\delta_1 = \delta_4 = \delta_5 = 1$ and the other coefficients vanish, the refined Eq. (7) reduces to a rBK-like equation involving the u_{tx}, u_{yy} and u_{tt} terms:

$$\begin{aligned}&\sigma_1\left(3u_xu_{xx}+3uu_x^2+\frac{3}{2}u^3u_x+\frac{3}{2}u^2u_{xx}\right)\\&+\sigma_2\left(\frac{9}{8}u^2u_xv+\frac{3}{8}u^3u_y+\frac{3}{4}uu_{xx}v+\frac{3}{4}u_x^2v+\frac{3}{4}u^2u_{xy}+\frac{9}{4}uu_xu_y+\frac{3}{2}u_{xx}u_y+\frac{3}{2}u_xu_{xy}\right)\\&+\sigma_3\left(\frac{1}{2}uvu_x+\frac{1}{4}u^2u_y+\frac{1}{2}u_{xx}v+\frac{3}{2}u_xu_y+uu_{xy}+u_{xxy}\right)+u_{tx}+u_{yy}+u_{tt}=0.\end{aligned} \tag{8}$$

If, instead, $\delta_2 = \delta_3 = \delta_5 = 1$ and the other coefficients vanish, the equation simplifies to a rBK-like equation containing the u_{ty}, u_{xx} and u_{tt} terms:

$$\begin{aligned}&\sigma_1\left(3u_xu_{xx}+3uu_x^2+\frac{3}{2}u^3u_x+\frac{3}{2}u^2u_{xx}\right)\\&+\sigma_2\left(\frac{9}{8}u^2u_xv+\frac{3}{8}u^3u_y+\frac{3}{4}uu_{xx}v+\frac{3}{4}u_x^2v+\frac{3}{4}u^2u_{xy}+\frac{9}{4}uu_xu_y+\frac{3}{2}u_{xx}u_y+\frac{3}{2}u_xu_{xy}\right)\\&+\sigma_3\left(\frac{1}{2}uvu_x+\frac{1}{4}u^2u_y+\frac{1}{2}u_{xx}v+\frac{3}{2}u_xu_y+uu_{xy}+u_{xxy}\right)+u_{ty}+u_{xx}+u_{tt}=0.\end{aligned} \tag{9}$$

Here, $v_x = u_y$ is defined as before.

2.3. From bilinear to nonlinear form

The bilinear form (5) and the nonlinear Eq. (7) are related through the equation

$$P_{\mathrm{r}BK-like}(u,v)=\left[\frac{F_{\mathrm{r}BK-like}(f)}{f^2}\right]_x. \tag{10}$$

Thus, if the tau-function f satisfies the bilinear equation, then the fields u and v, as defined in (6), satisfy the corresponding nonlinear rBK-like model.

This generalized formulation naturally raises questions about the integrability of the system, especially regarding the existence of lump solutions, a key feature of integrable (2+1)-dimensional models. In the next section, we explore such lump wave solutions that arise from the nonlinear and dispersive contributions of the model.

3. Lump waves shaped by nonlinearity and dispersion

We now turn to our main task: explicitly constructing lump wave solutions for the rBK-like Eq. (7) by analyzing its generalized bilinear form (5) using symbolic computation. Specifically, we will highlight the critical roles of the nonlinear and dispersive terms in facilitating the formation of lump structures and explore the stationary points of the corresponding quadratic function to examine the dynamic behavior of the lump waves.

3.1. Ansatz based on the sum of squares

A standard approach for deriving lump solutions in higher-dimensional nonlinear evolution equations is the sum-of-squares ansatz, originally introduced in [11]. The key idea is to express the dependent variable as a logarithmic derivative of a positive quadratic function. In particular, we consider the quadratic form

$$f=\zeta_1^2+\zeta_2^2+a_9,\ \zeta_1=a_1x+a_2y+a_3t+a_4,\ \zeta_2=a_5x+a_6y+a_7t+a_8, \tag{11}$$

which ensures rational localization in both spatial directions. Substituting (11) into the bilinear Eq. (5) reduces the problem to solving an algebraic system for the parameters a_i and the coefficients δ_i. Symbolic computation then yields explicit expressions for a_9, δ_1, and δ_2, while the remaining parameters and coefficients are left unspecified. These expressions are:

$$a_9=-\frac{3(a_1^2+a_5^2)(a_3^2+a_7^2)[\sigma_1(a_1^2+a_5^2)+\sigma_2(a_1a_2+a_5a_6)]}{\delta_3(a_1a_7-a_3a_5)^2+\delta_4(a_2a_7-a_3a_6)^2}, \tag{12}$$

$$\begin{aligned}\delta_1=&-\frac{1}{(a_3^2+a_7^2)(a_1a_6-a_2a_5)}\left\{\delta_3\left[(a_1^2-a_5^2)(a_2a_7+a_3a_6)-2a_1a_5(a_2a_3-a_6a_7)\right]\right.\\&\left.+\delta_4(a_2^2+a_6^2)(a_2a_7-a_3a_6)-\delta_5(a_3^2+a_7^2)(a_2a_7-a_3a_6)\right\},\end{aligned} \tag{13}$$

and

$$\begin{aligned}\delta_2=&\frac{1}{(a_3^2+a_7^2)(a_1a_6-a_2a_5)}\left\{\delta_3(a_1^2+a_5^2)(a_1a_7-a_3a_5)+\delta_4\left[(a_2^2-a_6^2)(a_1a_7+a_3a_5)\right.\right.\\&\left.\left.-2a_2a_6(a_1a_3-a_5a_7)\right]-\delta_5(a_3^2+a_7^2)(a_1a_7-a_3a_5)\right\}.\end{aligned} \tag{14}$$

The resulting expressions, (13), (14), and (12), encapsulate the dispersion relations and structural conditions that underlie the lump solutions. In particular, a_9 reflects a delicate balance among the wave numbers and dispersion coefficients, while δ_1 and δ_2 correspond to mixed dispersive coefficients that generate dispersive effects. Similar dispersion relations have been observed in lump solutions related to the second flow of the KP hierarchy and in generalized KP-type models (see, e.g., [24–28]). It is also noteworthy that σ_3 does not appear in the expressions of a_9, δ_1, and δ_2.

Well-posedness of the solution requires two essential non-degeneracy conditions. The first condition is as follows:

$$\delta_3(a_1a_7 - a_3a_5)^2 + \delta_4(a_2a_7 - a_3a_6)^2 \neq 0, \tag{15}$$

which implies $\delta_3^2 + \delta_4^2 \neq 0$, ensuring nontrivial dispersive contributions. The second condition is

$$(a_3^2 + a_7^2)(a_1a_6 - a_2a_5) \neq 0, \tag{16}$$

which guarantees that

$$a_1a_6 - a_2a_5 \neq 0, \tag{17}$$

thereby ensuring that the solutions u and v, defined through the logarithmic derivative transformation (6), decay to zero as $x^2 + y^2 \to \infty$, confirming spatial localization.

The positivity of f, and hence the analyticity of the resulting lump waves u and v, is guaranteed by the following necessary and sufficient mixed condition involving the wave numbers and dispersion parameters:

$$\frac{\sigma_1(a_1^2 + a_5^2) + \sigma_2(a_1a_2 + a_5a_6)}{\delta_3(a_1a_7 - a_3a_5)^2 + \delta_4(a_2a_7 - a_3a_6)^2} < 0. \tag{18}$$

This can be ensured by either of the two sufficient conditions:

$$\frac{\sigma_1}{\delta_3(a_1a_7 - a_3a_5)^2 + \delta_4(a_2a_7 - a_3a_6)^2} \leq 0, \ \frac{\sigma_2(a_1a_2 + a_5a_6)}{\delta_3(a_1a_7 - a_3a_5)^2 + \delta_4(a_2a_7 - a_3a_6)^2} < 0. \tag{19}$$

or

$$\frac{\sigma_1}{\delta_3(a_1a_7 - a_3a_5)^2 + \delta_4(a_2a_7 - a_3a_6)^2} < 0, \ \frac{\sigma_2(a_1a_2 + a_5a_6)}{\delta_3(a_1a_7 - a_3a_5)^2 + \delta_4(a_2a_7 - a_3a_6)^2} \leq 0. \tag{20}$$

Condition (18) ensures that $a_9 > 0$, as seen from (12), thereby guaranteeing that f, defined in (11), is strictly positive. Consequently, the solutions u and v are analytic throughout the (x, y, t)-domain.

In summary, the construction of lump wave solutions via the logarithmic derivative transformation relies on two essential conditions (17) and (18). The first condition, the determinant condition (17), guarantees spatial localization in all directions, while the seond ensures well-posedness and analyticity of u and v across the spatiotemporal domain. The sufficient conditions, (19) and (20), provide explicit criteria for analytic regularity. Under these requirements, the resulting u and v indeed represent valid lump wave solutions.

3.2. Locus of stationary points

The dynamical behavior of the lump waves can be further explored by analyzing the stationary points of f. Setting the spatial derivatives to zero, $f_x = f_y = 0$, leads to the linear system

$$a_1\zeta_1 + a_5\zeta_2 = 0, \ a_2\zeta_1 + a_6\zeta_2 = 0, \tag{21}$$

which, under the non-degeneracy condition (17), simplifies to: $\zeta_1 = 0$ and $\zeta_2 = 0$, while ζ_1, ζ_2 are defined in (11). Solving this system yields explicit linear relations for $x(t)$ and $y(t)$:

$$x(t) = \frac{[(a_1^2 + a_5^2)\delta_3 - (a_2^2 + a_6^2)\delta_4]\delta_1 + 2(a_1a_2 + a_5a_6)\delta_2\delta_3}{(a_1\delta_1 + a_2\delta_2)^2 + (a_5\delta_1 + a_6\delta_2)^2} t + \frac{a_2a_8 - a_4a_6}{a_1a_6 - a_2a_5}, \tag{22}$$

$$y(t) = \frac{2(a_1a_2 + a_5a_6)\delta_1\delta_4 - [(a_1^2 + a_5^2)\delta_3 - (a_2^2 + a_6^2)\delta_4]\delta_2}{(a_1\delta_1 + a_2\delta_2)^2 + (a_5\delta_1 + a_6\delta_2)^2} t - \frac{a_1a_8 - a_4a_5}{a_1a_6 - a_2a_5}. \tag{23}$$

These parametric equations describe a linear trajectory in the (x, y)-plane, along which the lump's amplitude becomes exactly zero. Along this path, the lump effectively vanishes; however, it remains rationally localized and non-vanishing in the surrounding spatial domain.

4. Concluding remarks

A $(2+1)$-dimensional refined Bogoyavlensky-Konopelchenko-type model is formulated and analyzed, with lump wave solutions studied using symbolic computation and computer algebra systems. The interplay between nonlinearity and dispersion plays a crucial role in the formation of lump waves. These lump waves vanish along a characteristic trajectory, which is determined by the stationary points of the associated quadratic function.

It is worth noting that the refined Bogoyavlensky-Konopelchenko-type model exhibits a specific characteristic property. The resulting algebraic system determining the lump waves can be solved for the dispersive coefficients δ_1 and δ_2, together with the constant coefficient a_9, whereas it can not be solved for the frequency coefficients a_3 and a_7, together with the constant coefficient a_9. In contrast, in the general computation of lump waves using the sum-of-squares ansatz, the coefficients a_3, a_7 and a_9 can always be uniquely determined (see, e.g., [11]).

Lump waves appear in a wide range of physical and mathematical contexts, demonstrating their versatility and highlighting the challenges in modeling nonlinear dispersive phenomena. They have been explored in systems across different dimensions: linear [14], nonlinear, and nonintegrable systems in $(2+1)$ [29–35], $(3+1)$ [24,36,37], and even $(4+1)$ dimensions [38]. The construction of these waves often relies on Hirota's bilinear method and its extensions, which are well-suited for capturing localized coherent structures [9].

Moreover, nonlinear integrable models that exhibit lump waves may also admit soliton and rogue wave solutions [39–42]. Lump waves can further display rich interactions with other nonlinear excitations, making their existence, structure, and dynamics in generalized integrable systems an active and significant area of ongoing research.

CRediT authorship contribution statement

Jin-Yun Yang: Writing – review & editing, Visualization, Investigation; **Wen-Xiu Ma:** Writing – review & editing, Writing – original draft, Methodology, Formal analysis, Conceptualization.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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