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Mixed lump-kink solutions to the BKP equation

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ABSTRACT

By using the Hirota bilinear form of the (2+1)-dimensional BKP equation, ten classes of interaction solutions between lumps and kinks are constructed through Maple symbolic computations beginning with a linear combination ansatz. The resulting lump-kink solutions are reduced to lumps and kinks when the exponential function and the quadratic function disappears, respectively. Analyticity is naturally guaranteed for the presented lump-kink solution if the constant term is chosen to be positive.

1. Introduction

The Hirota bilinear method provides a powerful approach for solving integrable equations [1], and soliton solutions describe various significant nonlinear phenomena in nature [2,3]. Upon taking long wave limits, rational solutions can be generated from solitons [4,3], which include lump solutions, rationally localized solutions in all directions in space, and particularly rogue waves [5,6]. Positions and complexions are other kinds of exact solutions which integrable equations possess [7,8], and interaction solutions between two classes of exact solutions describe more diverse nonlinear phenomena [9]. Hirota bilinear forms play a crucial role in computing such exact solutions, but one often needs trial and error to present exact solutions to Hirota bilinear equations [1,10].

The KP equation

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0 \tag{1.1}$$

possesses the following class of lump solution [11]:

$$\begin{aligned} u = 2(\ln f)_{xx},\ f &= \Big(a_1x + a_2y + \frac{a_1{a_2}^2 - a_1{a_6}^2 + 2\,a_2a_5a_6}{{a_1}^2 + {a_5}^2}t + a_4\Big)^2 \\ &+ \Big(a_5x + a_6y + \frac{2a_1a_2a_6 - {a_2}^2a_5 + a_5{a_6}^2}{{a_1}^2 + {a_5}^2}t + a_8\Big)^2 + \frac{3({a_1}^2 + {a_5}^2)^3}{(a_1a_6 - a_2a_5)^2}, \end{aligned} \tag{1.2}$$

where the parameters a_i's are arbitrary but $a_1a_6 - a_2a_5 \neq 0$, which guarantees that u will be a lump solution. This class contains a subclass of lump solutions presented earlier [12]:

$$u = 4\frac{-[x + ay + (a^2 - b^2)t]^2 + b^2(y + 2at)^2 + 3/b^2}{\{[x + ay + (a^2 - b^2)t]^2 + b^2(y + 2at)^2 + 3/b^2\}^2}, \tag{1.3}$$

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involving two free parameters a and b. Many other integrable equations possess lump solutions, for example, the three-dimensional three-wave resonant interaction [13], the BKP equation [14,15], the Davey–Stewartson equation II [4], and the Ishimori-I equation [16]. Moreover, symbolic computations show that various nonintegrable equations possess lump solutions as well, and such examples include (2+1)-dimensional generalized KP, BKP and Sawada–Kotera equations [17–19].

General rational solutions to integrable equations have been studied through the Wronskian formulation, the Casoratian formulation and the Grammian or Pfaffian formulation [1,3], and physically significant equations are the KdV equation, the Boussinesq equation and the nonlinear Schrödinger equation in (1+1)-dimensions, the KP and BKP equations in (2+1)-dimensions, and the Toda and Ablowitz–Ladik type lattice equations in (0+1)-dimensions (see, e.g., [9,20–22]). Direct searches have been also conducted for general rational solutions to nonlinear partial differential equations, including generalized bilinear differential equations (see, e.g., [23–29]).

In this paper, we would like to consider the (2+1)-dimensional BKP equation [30,31]:

$$P_{BKP}(u) := (u_t + 15u_x u_{xxx} + 15{u_x}^3 - 15u_x u_y + u_{5x})_x - 5u_{xxxy} - 5u_{yy} = 0. \tag{1.4}$$

The equation is a first member in the BKP integrable hierarchy, and it is a (2+1)-dimensional generalization of the Caudrey–Dodd–Gibbon–Sawada–Kotera (CDGSK) equation [32,33]:

$$v_t + 15vv_{xxx} + 15v_x v_{xx} + 45v^2 v_x + v_{5x} = 0. \tag{1.5}$$

Evidently, if taking $v = u_x$ where u is only a function of x and t, (1.4) becomes (1.5). Like the Boussinesq equation, the underlying spectral problem associated with (1.5) is of third order:

$$-\phi_y + \phi_{xxx} + (3v - \lambda)\phi = 0, \tag{1.6}$$

which is a basis for solving the Cauchy problem of the CDGSK equation (1.5) [34,35] by the inverse scattering transform [3].

In what follows, we are going to look for mixed lump-kink solutions of the (2+1)-dimensional BKP equation and compute ten classes of such interaction solutions through making symbolic computations with Maple vigorously, which exhibit diverse nonlinear phenomena. The resulting mixed lump-kind solutions also provide supplements to the existing lump and kink solutions. We will begin with the Hirota bilinear form of the (2+1)-dimensional BKP equation, to determine what linear combination functions of quadratic functions and the exponential function will solve the bilinear BKP equation. A few concluding remarks will be given in the last section.

2. Abundant lump-kink solutions

It is known that the first-order logarithmic derivative transformation

$$u = 2(\ln f)_x \tag{2.1}$$

transforms the (2+1)-dimensional BKP equation (1.4) into the following (2+1)-dimensional Hirota bilinear equation:

$$\begin{aligned} B_{BKP}(f) &:= (D_x^6 - 5D_x^3 D_y + D_x D_t - 5D_y^2) f \cdot f \\ &= -10f_{yy}f + 10{f_y}^2 + 2f_{xt}f - 2f_t f_x - 10f_{xxxy}f + 30f_{xxy}f_x - 30f_{xy}f_{xx} \\ &\quad + 10f_y f_{xxx} + 2f_{6x}f - 12f_{5x}f_x + 30f_{4x}f_{xx} - 20{f_{xxx}}^2 = 0. \end{aligned} \tag{2.2}$$

The transformation (2.1) is one of the two standard characteristic ones in Bell polynomial theories of integrable equations [36,37], and the other one is $u = 2(\ln f)_{xx}$, which can transform the KP equation into a Hirota bilinear equation. There is a clear relation between the BKP equation and the bilinear BKP equation:

$$P_{BKP}(u) = \left[\frac{B_{BKP}(f)}{f^2}\right]_x. \tag{2.3}$$

Therefore, if the function f solves the (2+1)-dimensional bilinear BKP equation (2.2), the function $u = 2(\ln f)_x$ presents a solution to the (2+1)-dimensional BKP equation (1.4).

The following analysis aims to compute interaction solutions between lumps and kinks of the (2+1)-dimensional BKP equation (1.4) through determining what linear combination functions of quadratic functions and the exponential function will solve the (2+1)-dimensional BKP equation (2.2). This supplements basic approaches to soliton solutions and dromion-type solutions, including the Hirota perturbation technique, the inverse scattering transform, and symmetry constraints (see, e.g., [38–43]).

Using the computer algebra system with Maple, we search for linear combination solutions to the bilinear BKP equation (2.2). We begin with an ansatz of taking linear combinations:

$$f = {\xi_1}^2 + {\xi_2}^2 + e^{\xi_3} + a_{13}, \tag{2.4}$$

with the three wave variables

$$\begin{cases} \xi_1 = a_1 x + a_2 y + a_3 t + a_4, \\ \xi_2 = a_5 x + a_6 y + a_7 t + a_8, \\ \xi_3 = a_9 x + a_{10} y + a_{11} t + a_{12}, \end{cases} \tag{2.5}$$

in which all the parameters a_i's are real constants to be determined later.

Upon careful classification, symbolic computations with Maple tell ten classes of nontrivial solutions for the parameters a_i's as follows. To get more solutions, we tried to remove different pairs of the parameters in the set of the variables to solve for. In each class of solutions in the following list, the parameters not expressed in the set are arbitrary, provided that all formulas in the same set are well defined.

Class 1. In terms of a_1 and a_5:

$$\Big\{a_3 = \frac{5\,a_2{}^2(a_1{}^4 - 16\,a_1{}^2a_5{}^2 - 16\,a_5{}^4)}{a_1{}^5},\ a_6 = -\frac{a_2a_5(3\,a_1{}^2 + 4\,a_5{}^2)}{a_1{}^3},$$
$$a_7 = -\frac{5\,a_2{}^2a_5(7\,a_1{}^4 - 8\,a_1{}^2a_5{}^2 - 16\,a_5{}^4)}{a_1{}^6},\ a_9 = \frac{4\,b\,a_5}{a_1}, a_{10} = -\frac{4\,a_2{}^2a_5(3\,a_1{}^2 - 4\,a_5{}^2)}{9\,b\,a_1{}^5},$$
$$a_{11} = -\frac{4\,a_2{}^3a_5(5\,a_1{}^4 - 40\,a_1{}^2a_5{}^2 + 16\,a_5{}^4)}{3\,b\,a_1{}^8},\ a_{13} = -\frac{3\,a_1(a_1{}^4 - 3\,a_1{}^2a_5{}^2 - 4\,a_5{}^4)}{16\,a_2a_5{}^2}\Big\},$$

where b solves $3\,a_1b^2 + a_2 = 0$.

Class 2. In terms of a_1 and a_5 when $a_{10} = 0$:

$$\Big\{a_2 = \frac{1}{2}\,b\,a_9{}^2,\ a_3 = \frac{5}{2}\,a_9{}^4(b - 2\,a_1),\ a_6 = \frac{a_5a_9{}^2(b - 4\,a_1)}{2\,(b - a_1)},$$
$$a_7 = -\frac{5\,a_9{}^4(b\,a_1 - a_1{}^2 + a_5{}^2)}{2\,a_5},\ a_{10} = 0,\ a_{11} = -a_9{}^5,\ a_{13} = -\frac{2\,(a_1{}^2 + a_5{}^2)}{a_9{}^2}\Big\},$$

where b satisfies $b^2 - 2\,a_1b + a_1{}^2 - 3\,a_5{}^2 = 0$.

Class 3. Solution 1 in terms of a_2 and a_6:

$$\Big\{a_1 = \frac{2}{3}\,\frac{a_6}{a_9{}^2},\ a_3 = -\frac{15}{2}\,a_9{}^2a_6,\ a_5 = -\frac{2}{3}\,\frac{a_2}{a_9{}^2},$$
$$a_7 = \frac{15}{2}\,a_2a_9{}^2,\ a_{10} = -\frac{1}{2}\,a_9{}^3,\ a_{11} = -\frac{9}{4}\,a_9{}^5,\ a_{13} = 0\Big\}.$$

Class 4. Solution 2 in terms of a_2 and a_6:

$$\Big\{a_1 = \frac{1}{3}\,\frac{b(a_2 - a_6)}{a_9{}^2},\ a_3 = \frac{15}{2}\,\frac{a_9{}^2\,[(6625109\,b + 2744210)(a_2 + a_6)]}{15994428\,b + 6625109},$$
$$a_5 = \frac{1}{3}\,\frac{b(a_2 + a_6)}{a_9{}^2},\ a_7 = -\frac{15}{2}\,\frac{a_9{}^2\,[(5741\,b + 2378)(a_2 - a_6)]}{13860\,b + 5741},$$
$$a_{10} = \frac{1}{2}\,\frac{a_9{}^3(3\,b + 1)}{12\,b + 5},\ a_{11} = -\frac{9}{2}\,\frac{a_9{}^5(29\,b + 12)}{408\,b + 169},\ a_{13} = -\frac{4}{9}\,\frac{(a_2{}^2 + a_6{}^2)(5\,b + 2)}{a_9{}^6}\Big\},$$

where b solves $b^2 - 2\,b - 1 = 0$.

Class 5. Solution 1 in terms of a_2 and a_6 when $a_{10} = 0$:

$$\Big\{a_1 = -\frac{1}{2}\,\frac{b\,a_6 - a_2{}^2 - a_6{}^2}{a_2a_9{}^2},\ a_3 = \frac{5}{2}\,\frac{a_9{}^2(b\,a_6 + a_2{}^2 - a_6{}^2)}{a_2},\ a_5 = \frac{1}{2}\,\frac{b}{a_9{}^2},$$
$$a_7 = -\frac{5}{2}\,b\,a_9{}^2 + 5\,a_6a_9{}^2,\ a_{10} = 0,\ a_{11} = -a_9{}^5,\ a_{13} = -\frac{2\,(a_2{}^2 + a_6{}^2)}{a_9{}^6}\Big\},$$

where b solves $b^2 - 2\,a_6b - 3\,a_2{}^2 + a_6{}^2 = 0$.

Class 6. Solution 2 in terms of a_2 and a_6 when $a_{10} = 0$:

$$\Big\{a_1 = \frac{1}{2}\,\frac{b}{a_9{}^2},\ a_3 = -\frac{5}{2}\,(b - 2\,a_2)a_9{}^2,\ a_5 = \frac{1}{2}\,\frac{a_6(b - 4\,a_2)}{a_9{}^2(b - a_2)},$$
$$a_7 = \frac{5}{2}\,\frac{a_6a_9{}^2(b + 2\,a_2)}{b - a_2},\ a_{10} = 0,\ a_{11} = -a_9{}^5,\ a_{13} = -\frac{2\,(a_2{}^2 + a_6{}^2)}{a_9{}^6}\Big\},$$

where b solves $b^2 - 2\,a_2b + a_2{}^2 - 3\,a_6{}^2 = 0$.

Class 7. Solution 1 in terms of a_1 and a_6:

$$\Big\{a_2 = \frac{1}{12}\,\frac{9\,a_1{}^2a_9{}^4 - 4\,a_6{}^2}{a_1a_9{}^2},\ a_3 = \frac{405\,a_1{}^4a_9{}^8 - 1080\,a_1{}^2a_6{}^2a_9{}^4 + 80\,a_6{}^4}{144\,a_1{}^3a_9{}^4},$$
$$a_5 = 0,\ a_7 = \frac{5}{6}\,\frac{a_6\left(9\,a_1{}^2a_9{}^4 - 4\,a_6{}^2\right)}{a_1{}^2a_9{}^2},\ a_{10} = \frac{1}{12}\,\frac{3\,a_1{}^2a_9{}^4 - 4\,a_6{}^2}{a_9a_1{}^2},$$
$$a_{11} = \frac{81\,a_1{}^4a_9{}^8 - 360\,a_1{}^2a_6{}^2a_9{}^4 + 80\,a_6{}^4}{144\,a_9{}^3a_1{}^4},\ a_{13} = -\frac{1}{4}\,\frac{a_1{}^2\left(9\,a_1{}^2a_9{}^4 - 4\,a_6{}^2\right)}{a_9{}^2a_6{}^2}\Big\}.$$

Class 8. Solution 2 in terms of a_1 and a_6:

$$\Big\{a_2 = -\frac{a_6a_5}{a_1},\ a_3 = -\frac{5\,a_6{}^2}{a_1},\ a_7 = -\frac{5\,a_5a_6{}^2}{a_1{}^2},$$
$$a_9 = b,\ a_{10} = -\frac{1}{3}\,\frac{b\,a_6}{a_1},\ a_{11} = -\frac{b\,a_6{}^2}{a_1{}^2},\ a_{13} = 0\Big\},$$

where b solves $3\,a_1b^2 - 2\,a_6 = 0$.

Class 9. Solution 1 in terms of a_1 and a_9:

$$\Big\{a_2 = \frac{9\,a_1{}^2a_9{}^4 - 4\,a_6{}^2}{12\,a_1a_9{}^2},\ a_3 = \frac{5\,(81\,a_1{}^4a_9{}^8 - 216\,a_1{}^2a_6{}^2a_9{}^4 + 16\,a_6{}^4)}{144\,a_1{}^3a_9{}^4},$$
$$a_5 = 0,\ a_7 = \frac{5\,a_6(9\,a_1{}^2a_9{}^4 - 4\,a_6{}^2)}{6\,a_1{}^2a_9{}^2},\ a_{10} = \frac{3\,a_1{}^2a_9{}^4 - 4\,a_6{}^2}{12\,a_1{}^2a_9},$$
$$a_{11} = \frac{81\,a_1{}^4a_9{}^8 - 360\,a_1{}^2a_6{}^2a_9{}^4 + 80\,a_6{}^4}{144\,a_1{}^4a_9{}^3},\ a_{13} = -\frac{a_1{}^2\left(9\,a_1{}^2a_9{}^4 - 4\,a_6{}^2\right)}{4\,a_6{}^2a_9{}^2}\Big\}.$$

Class 10. Solution 2 in terms of a_1 and a_9:

$$\Big\{a_3 = -\frac{5\,(a_1a_9{}^6 - 2\,a_1a_9{}^3a_{10} - a_2a_9{}^4 + a_1a_{10}{}^2 - 2\,a_2a_9a_{10})}{a_9{}^2},$$
$$a_5 = \frac{b\,(a_1a_9{}^3 + 2\,a_1a_{10} - 2\,a_2a_9)}{a_9},$$
$$a_6 = \frac{1}{3}\,\frac{2\,a_1a_9{}^6 - 4\,a_1a_9{}^3a_{10} - a_2a_9{}^4 + 2\,a_1a_{10}{}^2 - 2\,a_2a_9a_{10}}{b\,(a_9{}^3 - 4\,a_{10})a_9{}^3},$$
$$a_7 = \frac{5}{3}\,\frac{a_1a_9{}^9 + a_2a_9{}^7 - 3\,a_1a_9{}^3a_{10}{}^2 - 8\,a_2a_9{}^4a_{10} + 2\,a_1a_{10}{}^3 - 2\,a_2a_9a_{10}{}^2}{b\,a_9{}^4(a_9{}^3 - 4\,a_{10})},$$
$$a_{11} = -\frac{a_9{}^6 - 5\,a_9{}^3a_{10} - 5\,a_{10}{}^2}{a_9},$$
$$a_{13} = -\frac{8}{3}\,\frac{(a_9{}^3 + 2\,a_{10})(a_1{}^2a_9{}^6 - 2\,a_1{}^2a_9{}^3a_{10} - a_1a_2a_9{}^4 + a_1{}^2a_{10}{}^2 - 2\,a_1a_2a_9a_{10} + a_2{}^2a_9{}^2)}{(a_9{}^3 - 4\,a_{10})^2a_9{}^5}\Big\},$$

where b solves $(3\,a_9{}^4 - 12\,a_9a_{10})b^2 - 1 = 0$.

Obviously, the equations for b above always have real solutions when we choose the parameters properly, some of which definitely have solutions. To generate more solutions for the parameters, we tried to solve the resulting systems of nonlinear algebraic equations in terms of other combinations of the parameters, but failed to get any new non-trivial solutions.

Those sets of solutions for the parameters engender ten classes of linear combination solutions to the bilinear BKP equation (2.2), defined by (2.4) and (2.5), and then the resulting combination solutions engender ten classes of mixed lump-kink solutions to the (2+1)-dimensional BKP equation (1.4), under the transformation (2.1). The analyticity of the interactions solutions is definitely guaranteed, when we require a_{13} to be positive. These lump-kink solutions reduce to the kinks when the quadratic function disappears, and the lumps when the exponential function disappears. It is recognized that the resulting lump-kink solutions do not tend to zero in all directions in space due to a kink wave, and they form a peak at finite times generated by the involved lump wave.

3. Concluding remarks

Based on the Hirota form of the (2+1)-dimensional BKP equation, we worked out ten classes of mixed lump-kink solutions to the (2+1)-dimensional BKP equation explicitly by symbolic computations with Maple, and the resulting ten classes of interaction solutions, supplementing the existing solutions in the literature, are reduced to lump and kink solutions under reductions.

The obtained mixed lump-kink solutions are different kinds of exact solutions generated from other linear combinations only consisting of exponential functions [44–47]. Note that the KP and BKP equations can be solved by the Wronskian technique [48,49]. This inspires us to compute a novel kind of Wronskian solutions by introducing different kinds of entries through linear combinations.

It is also realized that one needs to choose a set of parameters to solve for to get nontrivial solutions to systems of nonlinear algebraic equations resulted from integrable equation. It is not clear to us how to solve generally, though we have some experiences on choosing what parameters to solve for.

If we replace the Hirota derivatives in (2.2) with generalized bilinear derivatives [50], all computations made above will be different, indeed. For the BKP-like equations determined by the generalized bilinear derivatives with two special values 3, 5:

$$(D_{3,x}^6 - 5D_{3,x}^3 D_{3,y} + D_{3,x} D_{3,t} - 5D_{3,y}^2) f \cdot f = 0,$$
$$(D_{5,x}^6 - 5D_{5,x}^3 D_{5,y} + D_{5,x} D_{5,t} - 5D_{5,y}^2) f \cdot f = 0,$$

we will have different lump-kink solutions, but lump solutions derived from quadratic functions remain the same (see [18,27] for more discussions).

Moreover, it is interesting to study linear combination solutions to general bilinear and tri-linear differential equations involving generalized bilinear derivatives [50]. The corresponding mixed interaction solutions will be different from resonant solutions generated by using the linear superposition principle [51,52].

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