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New soliton hierarchies associated with the real Lie algebra so$(4, \mathbb{R})$

Shundong Zhu[a], Shoufeng Shen[b*†], Yongyang Jin[b], Chunxia Li[c] and Wen-Xiu Ma[d]

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We present some new matrix spectral problems, based on the real special orthogonal Lie algebra so$(4, \mathbb{R})$, and construct corresponding soliton hierarchies by means of zero curvature equations associated with these spectral problems. With the aid of symbolic computation by Maple, new soliton hierarchies of Kaup–Newell type, Ablowitz–Kaup–Newell–Segur type and Wadati–Konno–Ichikawa type are obtained to illustrate the use of so$(4, \mathbb{R})$. Copyright © 2016 John Wiley & Sons, Ltd.

Keywords: soliton hierarchy; matrix spectral problem; Lie algebra so$(4, \mathbb{R})$; zero curvature equation; Hamiltonian structure

1. Introduction

It is well known that zero curvature equations on real Lie algebras such as sl$(2, \mathbb{R})$ and so$(3, \mathbb{R})$ lay the foundation for constructing soliton hierarchies. Usually, one starts with matrix spectral problems (or Lax pairs) associated with these algebras. Some useful soliton hierarchies containing the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy, the Kaup–Newell (KN) hierarchy, the Wadati–Konno–Ichikawa (WKI) hierarchy, the Korteweg-de Vries hierarchy, the modified Korteweg-de Vries hierarchy, the Benjamin–Ono hierarchy, the Tu hierarchy, the Dirac hierarchy, the coupled Harry–Dym hierarchy and the coupled Burgers hierarchy have been derived in References [1–38]. The so-called trace identity [8, 27, 31], or the variational identity [26, 32], provides a basic tool for generating Hamiltonian structures of soliton hierarchies. Generally speaking, the corresponding Hamiltonian pairs can generate hereditary recursion operators if there exist bi-Hamiltonian structures for a given soliton hierarchy [39].

We will use the matrix loop algebra $\widetilde{\mathrm{so}}(4, \mathbb{R})$ derived from the real Lie algebra so$(4, \mathbb{R})$, to construct new KN type hierarchy. Two soliton hierarchies associated with this algebra have been constructed in Reference [40]. The relations between the hierarchy of su$(2, \mathbb{C}) \otimes$ su$(2, \mathbb{C})$ and the hierarchy of so$(4, \mathbb{R})$ also have been obtained. The Lie algebra so$(4, \mathbb{R})$ consisting of 4×4 skew-symmetric real matrices has a basis

$$e_1 = \begin{bmatrix} 0&0&0&-1\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0&-1&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0&0&-1&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{bmatrix},$$
$$e_4 = \begin{bmatrix} 0&0&0&0\\ 0&0&0&-1\\ 0&0&0&0\\ 0&1&0&0 \end{bmatrix}, \quad e_5 = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-1\\ 0&0&1&0 \end{bmatrix}, \quad e_6 = \begin{bmatrix} 0&0&0&0\\ 0&0&-1&0\\ 0&1&0&0\\ 0&0&0&0 \end{bmatrix},$$

[a] College of Science, Lishui University, Lishui 323000, China
[b] Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou 310023, China
[c] School of Mathematical Sciences, Capital Normal University, Beijing 100048, China
[d] Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
* Correspondence to: Shoufeng Shen, Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou 310023, China.
† E-mail: mathssf@zjut.edu.cn

whose commutator relations are

$$\begin{aligned}
&[e_1,e_2]=-e_4,\ [e_1,e_3]=-e_5,\quad [e_1,e_4]=e_2,\ \ [e_1,e_5]=e_3,\quad [e_1,e_6]=0,\\
&[e_2,e_3]=e_6,\quad [e_2,e_4]=-e_1,\quad [e_2,e_5]=0,\quad [e_2,e_6]=-e_3,\\
&[e_3,e_4]=0,\quad\ \ [e_3,e_5]=-e_1,\quad [e_3,e_6]=e_2,\\
&[e_4,e_5]=e_6,\quad [e_4,e_6]=-e_5,\\
&[e_5,e_6]=e_4.
\end{aligned}$$

Then, the matrix loop algebra $\widetilde{\mathrm{so}}(4,\mathbb{R})$ we adopt in what follows is defined as

$$\widetilde{\mathrm{so}}(4,\mathbb{R})=\left\{\sum_{j\geq 0}M_j\lambda^{n-j}\,\middle|\,M_j\in \mathrm{so}(4,\mathbb{R}),\ j\geq 0,\ n\in\mathbb{Z}\right\},$$

that is, the space of all Laurent series in λ with a finite number of non-zero terms of positive powers of λ and coefficient matrices in $\mathrm{so}(4,\mathbb{R})$.

Next, we give a brief account of the procedure for building soliton hierarchies associated with $\mathrm{so}(4,\mathbb{R})$, which is called the Tu scheme.

Step 1: Usually, one need to select a suitable $U=U(u,\lambda)\in\widetilde{\mathrm{so}}(4,\mathbb{R})$ to fit the spatial spectral problem $\phi_x=U(u,\lambda)\phi$, where u denotes a column dependent variable and λ is the spectral parameter.

Step 2: We need to construct an appropriate Laurent series solution $W=W(u,\lambda)$ such as $W=\sum_{k=0}^{\infty}W_k\lambda^{-2k}$, $W_k\in\mathrm{so}(4,\mathbb{R})$ to the stationary zero curvature equation $W_x=[U,W]$.

Step3: By means of the solution obtained in the aforementioned step, we construct suitable temporal spectral problems $\phi_{t_n}=V^{[n]}\phi$, $n\geq 0$ to guarantee the zero curvature equations

$$U_{t_n}-V_x^{[n]}+\left[U,V^{[n]}\right]=0,\qquad n\geq 0 \tag{1.1}$$

can generate a soliton hierarchy $u_{t_n}=K_n(u)$, $n\geq 0$.

Step4: Finally, we construct Hamiltonian structures of the obtained soliton hierarchy

$$u_{t_n}=K_n(u)=J\frac{\delta\mathcal{H}_n}{\delta u},\qquad n\geq 0,$$

by applying the trace identity [8,27,31],

$$\frac{\delta}{\delta u}\int \mathrm{tr}\left(\frac{\partial U}{\partial\lambda}W\right)\mathrm{d}x=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\,\mathrm{tr}\left(\frac{\partial U}{\partial u}W\right),\qquad \gamma=-\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln\left|\mathrm{tr}\left(W^2\right)\right|. \tag{1.2}$$

Here, we must point out that the formula $\gamma=-\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln\left|\mathrm{tr}\left(W^2\right)\right|$ is given by Guo and Zhang in References [27,31].

In this paper, we would like to introduce some new spectral problems associated with the Lie algebra $\mathrm{so}(4,\mathbb{R})$ and construct the corresponding soliton hierarchies. New soliton hierarchies of KN type, AKNS type and WKI type will be worked out respectively in Sections 2–4, which form our main results. In our analysis, we will use software Maple to deal with some complicated computations. The last section is devoted to some conclusions and discussions.

2. New soliton hierarchy of Kaup–Newell type

In this section, we will construct new soliton hierarchy of KN type from the following spatial matrix spectral problem,

$$\phi_x=U\phi=U(u,\lambda)\phi,\qquad u=\begin{bmatrix}p_1\\p_2\\q_1\\q_2\end{bmatrix},\qquad \phi=\begin{bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{bmatrix}, \tag{2.1}$$

with the matrix U being chosen as

$$\begin{aligned}U&=\lambda^2e_1+\lambda q_1e_2+\lambda q_2e_3+\lambda p_1e_4+\lambda p_2e_5\\
&=\begin{bmatrix}0&-\lambda q_1&-\lambda q_2&-\lambda^2\\ \lambda q_1&0&0&-\lambda p_1\\ \lambda q_2&0&0&-\lambda p_2\\ \lambda^2&\lambda p_1&\lambda p_2&0\end{bmatrix}\in\widetilde{\mathrm{so}}(4,\mathbb{R}).\end{aligned} \tag{2.2}$$

We can take it as the four-component generalized form of the soliton hierarchy obtained in Reference [38].

Then, let W be of the following form

$$W = ae_1 + c_1e_2 + c_2e_3 + b_1e_4 + b_2e_5 + de_6 = \begin{bmatrix} 0 & -c_1 & -c_2 & -a \\ c_1 & 0 & -d & -b_1 \\ c_2 & d & 0 & -b_2 \\ a & b_1 & b_2 & 0 \end{bmatrix} \in \widetilde{so}(4,\mathbb{R}), \tag{2.3}$$

we solve the stationary zero curvature equation $W_x = [U, W]$, and it becomes

$$\begin{aligned} a_x &= \lambda(p_1c_1 + p_2c_2 - q_1b_1 - q_2b_2), \\ d_x &= \lambda(q_1c_2 + p_1b_2 - q_2c_1 - p_2b_1), \\ b_{1x} &= -\lambda^2c_1 + \lambda q_1a + \lambda p_2d, \\ b_{2x} &= -\lambda^2c_2 + \lambda q_2a - \lambda p_1d, \\ c_{1x} &= \lambda^2b_1 - \lambda p_1a + \lambda q_2d, \\ c_{2x} &= \lambda^2b_2 - \lambda p_2a - \lambda q_1d. \end{aligned} \tag{2.4}$$

According to the relations of the spectral parameter λ in these equations (2.4), we take the following Laurent series expansions

$$\begin{aligned} a &= \sum_{k=0}^{\infty} a^{(k)}\lambda^{-2k}, \qquad d = \sum_{k=0}^{\infty} d^{(k)}\lambda^{-2k}, \\ b_i &= \sum_{k=0}^{\infty} b_i^{(k)}\lambda^{-2k-1}, \qquad c_i = \sum_{k=0}^{\infty} c_i^{(k)}\lambda^{-2k-1}, \qquad i = 1, 2, \end{aligned} \tag{2.5}$$

namely,

$$W = \sum_{k=0}^{\infty} W_k\lambda^{-2k}, \qquad W_k = \begin{bmatrix} 0 & -c_1^{(k)}\lambda^{-1} & -c_2^{(k)}\lambda^{-1} & -a^{(k)} \\ c_1^{(k)}\lambda^{-1} & 0 & -d^{(k)} & -b_1^{(k)}\lambda^{-1} \\ c_2^{(k)}\lambda^{-1} & d^{(k)} & 0 & -b_2^{(k)}\lambda^{-1} \\ a^{(k)} & b_1^{(k)}\lambda^{-1} & b_2^{(k)}\lambda^{-1} & 0 \end{bmatrix}.$$

Substituting (2.5) into (2.4), we arrive at

$$\begin{aligned} a_x^{(k)} &= p_1c_1^{(k)} + p_2c_2^{(k)} - q_1b_1^{(k)} - q_2b_2^{(k)}, \\ d_x^{(k)} &= q_1c_2^{(k)} + p_1b_2^{(k)} - q_2c_1^{(k)} - p_2b_1^{(k)}, \\ b_1^{(0)} &= p_1a^{(0)} - q_2d^{(0)}, \\ b_2^{(0)} &= p_2a^{(0)} + q_1d^{(0)}, \\ c_1^{(0)} &= q_1a^{(0)} + p_2d^{(0)}, \\ c_2^{(0)} &= q_2a^{(0)} - p_1d^{(0)}, \\ b_1^{(k+1)} &= c_{1x}^{(k)} + p_1a^{(k+1)} - q_2d^{(k+1)}, \\ b_2^{(k+1)} &= c_{2x}^{(k)} + p_2a^{(k+1)} + q_1d^{(k+1)}, \\ c_1^{(k+1)} &= -b_{1x}^{(k)} + q_1a^{(k+1)} + p_2d^{(k+1)}, \\ c_2^{(k+1)} &= -b_{2x}^{(k)} + q_2a^{(k+1)} - p_1d^{(k+1)}, \qquad k \geq 0. \end{aligned} \tag{2.6}$$

To determine the sequence of $\left\{a^{(k)}, d^{(k)}, b_i^{(k)}, c_i^{(k)}, i = 1, 2\right\}$, we rewrite $a_x^{(k+1)}$ and $d_x^{(k+1)}$ as

$$\begin{aligned} a_x^{(k+1)} &= p_1c_1^{(k+1)} + p_2c_2^{(k+1)} - q_1b_1^{(k+1)} - q_2b_2^{(k+1)} \\ &= p_1\left(-b_{1x}^{(k)} + q_1a^{(k+1)} + p_2d^{(k+1)}\right) + p_2\left(-b_{2x}^{(k)} + q_2a^{(k+1)} - p_1d^{(k+1)}\right) \\ &\quad - q_1\left(c_{1x}^{(k)} + p_1a^{(k+1)} - q_2d^{(k+1)}\right) - q_2\left(c_{2x}^{(k)} + p_2a^{(k+1)} + q_1d^{(k+1)}\right) \\ &= -p_1b_{1x}^{(k)} - p_2b_{2x}^{(k)} - q_1c_{1x}^{(k)} - q_2c_{2x}^{(k)}, \\ d_x^{(k+1)} &= q_1c_2^{(k+1)} + p_1b_2^{(k+1)} - q_2c_1^{(k+1)} - p_2b_1^{(k+1)} \\ &= q_1\left(-b_{2x}^{(k)} + q_2a^{(k+1)} - p_1d^{(k+1)}\right) + p_1\left(c_{2x}^{(k)} + p_2a^{(k+1)} + q_1d^{(k+1)}\right) \\ &\quad - q_2\left(-b_{1x}^{(k)} + q_1a^{(k+1)} + p_2d^{(k+1)}\right) - p_2\left(c_{1x}^{(k)} + p_1a^{(k+1)} - q_2d^{(k+1)}\right) \\ &= q_2b_{1x}^{(k)} - q_1b_{2x}^{(k)} - p_2c_{1x}^{(k)} + p_1c_{2x}^{(k)}. \end{aligned}$$

Thus, we can obtain the following recursion relations:

$$
\begin{aligned}
b_1^{(k+1)} &= c_{1x}^{(k)} + p_1\partial^{-1}\left(-p_1b_{1x}^{(k)} - p_2b_{2x}^{(k)} - q_1c_{1x}^{(k)} - q_2c_{2x}^{(k)}\right) - q_2\partial^{-1}\left(q_2b_{1x}^{(k)} - q_1b_{2x}^{(k)} - p_2c_{1x}^{(k)} + p_1c_{2x}^{(k)}\right),\\
b_2^{(k+1)} &= c_{2x}^{(k)} + p_2\partial^{-1}\left(-p_1b_{1x}^{(k)} - p_2b_{2x}^{(k)} - q_1c_{1x}^{(k)} - q_2c_{2x}^{(k)}\right) + q_1\partial^{-1}\left(q_2b_{1x}^{(k)} - q_1b_{2x}^{(k)} - p_2c_{1x}^{(k)} + p_1c_{2x}^{(k)}\right),\\
c_1^{(k+1)} &= -b_{1x}^{(k)} + q_1\partial^{-1}\left(-p_1b_{1x}^{(k)} - p_2b_{2x}^{(k)} - q_1c_{1x}^{(k)} - q_2c_{2x}^{(k)}\right) + p_2\partial^{-1}\left(q_2b_{1x}^{(k)} - q_1b_{2x}^{(k)} - p_2c_{1x}^{(k)} + p_1c_{2x}^{(k)}\right),\\
c_2^{(k+1)} &= -b_{2x}^{(k)} + q_2\partial^{-1}\left(-p_1b_{1x}^{(k)} - p_2b_{2x}^{(k)} - q_1c_{1x}^{(k)} - q_2c_{2x}^{(k)}\right) - p_1\partial^{-1}\left(q_2b_{1x}^{(k)} - q_1b_{2x}^{(k)} - p_2c_{1x}^{(k)} + p_1c_{2x}^{(k)}\right),
\end{aligned}
$$

namely,

$$
\begin{bmatrix} b_1^{(k+1)} \\ b_2^{(k+1)} \\ c_1^{(k+1)} \\ c_2^{(k+1)} \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & L_{22} & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33} & L_{34} \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix} \begin{bmatrix} b_1^{(k)} \\ b_2^{(k)} \\ c_1^{(k)} \\ c_2^{(k)} \end{bmatrix},
$$

$$
\begin{aligned}
L_{11} &= -p_1\partial^{-1}p_1\partial - q_2\partial^{-1}q_2\partial, & L_{12} &= -p_1\partial^{-1}p_2\partial + q_2\partial^{-1}q_1\partial,\\
L_{13} &= \partial - p_1\partial^{-1}q_1\partial + q_2\partial^{-1}p_2\partial, & L_{14} &= -p_1\partial^{-1}q_2\partial - q_2\partial^{-1}p_1\partial,\\
L_{21} &= -p_2\partial^{-1}p_1\partial + q_1\partial^{-1}q_2\partial, & L_{22} &= -p_2\partial^{-1}p_2\partial - q_1\partial^{-1}q_1\partial,\\
L_{23} &= -p_2\partial^{-1}q_1\partial - q_1\partial^{-1}p_2\partial, & L_{24} &= \partial - p_2\partial^{-1}q_2\partial + q_1\partial^{-1}p_1\partial,\\
L_{31} &= -\partial - q_1\partial^{-1}p_1\partial + p_2\partial^{-1}q_2\partial, & L_{32} &= -q_1\partial^{-1}p_2\partial - p_2\partial^{-1}q_1\partial,\\
L_{33} &= -q_1\partial^{-1}q_1\partial - p_2\partial^{-1}p_2\partial, & L_{34} &= -q_1\partial^{-1}q_2\partial + p_2\partial^{-1}p_1\partial,\\
L_{41} &= -q_2\partial^{-1}p_1\partial - p_1\partial^{-1}q_2\partial, & L_{42} &= -\partial - q_2\partial^{-1}p_2\partial + p_1\partial^{-1}q_1\partial,\\
L_{43} &= -q_2\partial^{-1}q_1\partial + p_1\partial^{-1}p_2\partial, & L_{44} &= -q_2\partial^{-1}q_2\partial - p_1\partial^{-1}p_1\partial.
\end{aligned}
$$

Furthermore, we take $a^{(0)} = 1, d^{(0)} = 0$ and impose the integration conditions

$$
a^{(k)}|_{u=0} = d^{(k)}|_{u=0} = b_i^{(k)}|_{u=0} = c_i^{(k)}|_{u=0} = 0, \qquad i = 1, 2, \quad k \geq 1,
$$

which are used to guarantee the uniformity of $\left\{a^{(k)}, d^{(k)}, b_i^{(k)}, c_i^{(k)},\ i = 1, 2\right\}$. Therefore, by means of symbolic computation software Maple, the first few sets can be computed as follows:

$$
\begin{aligned}
b_1^{(0)} &= p_1, \qquad b_2^{(0)} = p_2, \qquad c_1^{(0)} = q_1, \qquad c_2^{(0)} = q_2,\\
a^{(1)} &= -\frac{1}{2}\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right), \qquad d^{(1)} = p_1q_2 - q_1p_2,\\
b_1^{(1)} &= q_{1x} - \frac{1}{2}p_1\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) - q_2(p_1q_2 - q_1p_2),\\
b_2^{(1)} &= q_{2x} - \frac{1}{2}p_2\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + q_1(p_1q_2 - q_1p_2),\\
c_1^{(1)} &= -p_{1x} - \frac{1}{2}q_1\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + p_2(p_1q_2 - q_1p_2),\\
c_2^{(1)} &= -p_{2x} - \frac{1}{2}q_2\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) - p_1(p_1q_2 - q_1p_2),\\
a^{(2)} &= -p_1q_{1x} + q_1p_{1x} - p_2q_{2x} + p_{2x}q_2 + \frac{3}{8}p_1^4 + \frac{3}{4}p_1^2p_2^2 + \frac{3}{4}p_1^2q_1^2 + \frac{9}{4}p_1^2q_2^2 - 3p_2q_1q_2p_1\\
&\quad + \frac{9}{4}p_2^2q_1^2 + \frac{3}{4}p_2^2q_2^2 + \frac{3}{8}p_2^4 + \frac{3}{4}q_1^2q_2^2 + \frac{3}{8}q_1^4 + \frac{3}{8}q_2^4,\\
d^{(2)} &= -p_1p_{2x} + p_2p_{1x} - q_1q_{2x} + q_{1x}q_2 - \frac{3}{2}q_2^3p_1 + \frac{3}{2}q_1p_1^2p_2 - \frac{3}{2}q_2p_1^3 - \frac{3}{2}p_2^2q_2p_1\\
&\quad - \frac{3}{2}q_1^2q_2p_1 + \frac{3}{2}q_1^3p_2 + \frac{3}{2}q_1q_2^2p_2 + \frac{3}{2}p_2^3q_1,\\
b_1^{(2)} &= -p_{1xx} + 3p_{2x}(p_1q_2 - p_2q_1) - \frac{3}{2}q_{1x}\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \frac{3}{8}p_1^5 + \frac{3}{4}p_1^3p_2^2 + \frac{3}{4}p_1^3q_1^2\\
&\quad + \frac{15}{4}p_1^3q_2^2 - \frac{9}{2}p_2q_1q_2p_1^2 + \frac{3}{8}p_1p_2^4 + \frac{9}{4}p_1p_2^2q_1^2 + \frac{9}{4}p_1p_2^2q_2^2 + \frac{3}{8}p_1q_1^4 + \frac{9}{4}p_1q_1^2q_2^2\\
&\quad + \frac{15}{8}p_1q_2^4 - \frac{3}{2}q_2p_2^3q_1 - \frac{3}{2}q_2q_1^3p_2 - \frac{3}{2}q_1q_2^3p_2,
\end{aligned}
$$

$$
\begin{aligned}
b_2^{(2)} &= -p_{2xx} - 3p_{1x}(p_1q_2 - p_2q_1) - \frac{3}{2}q_{2x}\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \frac{3}{8}p_2p_1^4 - \frac{3}{2}q_1q_2p_1^3 + \frac{3}{4}p_1^2p_2^3 \\
&\quad + \frac{9}{4}p_2p_1^2q_1^2 + \frac{9}{4}p_2p_1^2q_2^2 - \frac{9}{2}p_2^2q_1q_2p_1 - \frac{3}{2}q_1^3q_2p_1 - \frac{3}{2}q_1q_2^3p_1 + \frac{3}{8}p_2^5 + \frac{15}{4}p_2^3q_1^2 \\
&\quad + \frac{3}{4}p_2^3q_2^2 + \frac{15}{8}p_2q_1^4 + \frac{9}{4}p_2q_1^2q_2^2 + \frac{3}{8}p_2q_2^4, \\
c_1^{(2)} &= -q_{1xx} + 3q_{2x}(p_1q_2 - p_2q_1) + \frac{3}{2}p_{1x}\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \frac{3}{8}q_1p_1^4 - \frac{3}{2}p_2q_2p_1^3 + \frac{3}{4}p_1^2q_1^3 \\
&\quad + \frac{9}{4}q_1p_1^2p_2^2 + \frac{9}{4}q_1p_1^2q_2^2 - \frac{3}{2}p_2^3q_2p_1 - \frac{9}{2}p_2q_1^2q_2p_1 - \frac{3}{2}p_2q_2^3p_1 + \frac{15}{8}q_1p_2^4 + \frac{15}{4}p_2^2q_1^3 \\
&\quad + \frac{9}{4}q_1p_2^2q_2^2 + \frac{3}{8}q_1^5 + \frac{3}{4}q_1^3q_2^2 + \frac{3}{8}q_1q_2^4, \\
c_2^{(2)} &= -q_{2xx} - 3q_{1x}(p_1q_2 - p_2q_1) + \frac{3}{2}p_{2x}\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \frac{15}{8}q_2p_1^4 - \frac{3}{2}q_1p_1^3p_2 + \frac{9}{4}q_2p_1^2p_2^2 \\
&\quad + \frac{9}{4}q_2p_1^2q_1^2 + \frac{15}{4}p_1^2q_2^3 - \frac{3}{2}p_1p_2^3q_1 - \frac{3}{2}p_1q_1^3p_2 - \frac{9}{2}p_2q_1q_2^2p_1 + \frac{3}{8}q_2p_2^4 + \frac{9}{4}q_2p_2^2q_1^2 \\
&\quad + \frac{3}{4}p_2^2q_2^3 + \frac{3}{8}q_2q_1^4 + \frac{3}{4}q_1^2q_2^3 + \frac{3}{8}q_2^5, \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots .
\end{aligned}
$$

We point out that the localness of the first two sets of $\left\{a^{(k)}, d^{(k)}, b_i^{(k)}, c_i^{(k)},\ i = 1, 2\right\}$ is not an accident, and in fact, the functions $\left\{a^{(k)}, d^{(k)}, b_i^{(k)}, c_i^{(k)},\ i = 1, 2\right\}$ are all local. We prove this fact as follows, first from $W_x = [U, W]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{tr}\left(W^2\right) = 2\mathrm{tr}(WW_x) = 2\mathrm{tr}\left(W[U, W]\right) = 0,$$

and so, due to $\mathrm{tr}(W^2) = -2\left(a^2 + d^2 + b_1^2 + b_2^2 + c_1^2 + c_2^2\right)$, we can obtain

$$a^2 + d^2 + b_1^2 + b_2^2 + c_1^2 + c_2^2 = \left(a^2 + d^2 + b_1^2 + b_2^2 + c_1^2 + c_2^2\right)\big|_{u=0} = 1,$$

the last step of which follows from the initial data. Then, by using the Laurent expansions (2.5), a balance of coefficients of λ^k for each $k \geq 1$ tells that

$$a^{(k+1)} + d^{(k+1)} = -\frac{1}{2}\left[\sum_{\substack{i+j=k+1\\ i,j\geq 1}}\left(a^{(i)}a^{(j)} + d^{(i)}d^{(j)}\right) + \sum_{i+j=k}\left(b_1^{(i)}b_1^{(j)} + b_2^{(i)}b_2^{(j)} + c_1^{(i)}c_1^{(j)} + c_2^{(i)}c_2^{(j)}\right)\right]. \tag{2.7}$$

Similarly, for the determinant $|W| = (ad - c_2b_1 + c_1b_2)^2$, we can compute

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(ad - c_2b_1 + c_1b_2) &= a_xd + ad_x - c_{2x}b_1 - c_2b_{1x} + c_{1x}b_2 + c_1b_{2x} \\
&= \lambda(p_1c_1 + p_2c_2 - q_1b_1 - q_2b_2)d + a\lambda(q_1c_2 + p_1b_2 - q_2c_1 - p_2b_1) \\
&\quad - (\lambda^2b_2 - \lambda p_2a - \lambda q_1d)b_1 - c_2(-\lambda^2c_1 + \lambda q_1a + \lambda p_2d) \\
&\quad + (\lambda^2b_1 - \lambda p_1a + \lambda q_2d)b_2 + c_1(-\lambda^2c_2 + \lambda q_2a - \lambda p_1d) \\
&= 0.
\end{aligned}
$$

Thus, we arrive at

$$d^{(k+1)} = -\sum_{\substack{i+j=k+1\\ i,j\geq 1}} a^{(i)}d^{(j)} + \sum_{i+j=k} c_2^{(i)}b_1^{(j)} - \sum_{i+j=k} c_1^{(i)}b_2^{(j)}. \tag{2.8}$$

Based on these two recursion relations and the last four recursion relations in (2.22), an application of the mathematical induction finally shows that all functions $\left\{a^{(k)}, d^{(k)}, b_i^{(k)}, c_i^{(k)},\ i = 1, 2\right\}$ are differential polynomials in u, and so, they are all local. This method to prove the localness is also applicable to the new hierarchies of AKNS type and WKI type which will be proposed in Sections 3 and 4, respectively.

Now, taking

$$
\begin{aligned}
V^{[n]} &= \lambda(\lambda^{2n+1}W)_+ = \lambda^{2n+2}W_0 + \lambda^{2n}W_1 + \cdots + \lambda^2W_n \\
&= \begin{bmatrix}
0 & -\sum_{k=0}^{n}\lambda^{2(n-k)+1}c_1^{(k)} & -\sum_{k=0}^{n}\lambda^{2(n-k)+1}c_2^{(k)} & -\sum_{k=0}^{n}\lambda^{2(n-k)+2}a^{(k)} \\
\sum_{k=0}^{n}\lambda^{2(n-k)+1}c_1^{(k)} & 0 & -\sum_{k=0}^{n}\lambda^{2(n-k)+2}d^{(k)} & -\sum_{k=0}^{n}\lambda^{2(n-k)+1}b_1^{(k)} \\
\sum_{k=0}^{n}\lambda^{2(n-k)+1}c_2^{(k)} & \sum_{k=0}^{n}\lambda^{2(n-k)+2}d^{(k)} & 0 & -\sum_{k=0}^{n}\lambda^{2(n-k)+1}b_2^{(k)} \\
\sum_{k=0}^{n}\lambda^{2(n-k)+2}a^{(k)} & \sum_{k=0}^{n}\lambda^{2(n-k)+1}b_1^{(k)} & \sum_{k=0}^{n}\lambda^{2(n-k)+1}b_2^{(k)} & 0
\end{bmatrix},
\end{aligned}
$$

the zero curvature equations (1.1)

$$U_{t_n} - V_x^{[n]} + \left[U, V^{[n]}\right] = 0, \qquad n \geq 0$$

engender a new four-component soliton hierarchy of KN type

$$\begin{bmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{bmatrix}_{t_n} = K_n = \begin{bmatrix} b_{1x}^{(n)} \\ b_{2x}^{(n)} \\ c_{1x}^{(n)} \\ c_{2x}^{(n)} \end{bmatrix} = \Phi^n \begin{bmatrix} b_{1x}^{(0)} \\ b_{2x}^{(0)} \\ c_{1x}^{(0)} \\ c_{2x}^{(0)} \end{bmatrix} = \Phi^n \begin{bmatrix} p_{1x} \\ p_{2x} \\ q_{1x} \\ q_{2x} \end{bmatrix}, \qquad n \geq 0. \tag{2.9}$$

Here, Φ can be determined by the aforementioned recursion relations,

$$\begin{aligned}
&\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} & \Phi_{24} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} & \Phi_{34} \\ \Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44} \end{bmatrix}, \\
&\Phi_{11} = -\partial p_1 \partial^{-1} p_1 - \partial q_2 \partial^{-1} q_2, \qquad \Phi_{12} = -\partial p_1 \partial^{-1} p_2 + \partial q_2 \partial^{-1} q_1, \\
&\Phi_{13} = \partial - \partial p_1 \partial^{-1} q_1 + \partial q_2 \partial^{-1} p_2, \qquad \Phi_{14} = -\partial p_1 \partial^{-1} q_2 - \partial q_2 \partial^{-1} p_1, \\
&\Phi_{21} = -\partial p_2 \partial^{-1} p_1 + \partial q_1 \partial^{-1} q_2, \qquad \Phi_{22} = -\partial p_2 \partial^{-1} p_2 - \partial q_1 \partial^{-1} q_1, \\
&\Phi_{23} = -\partial p_2 \partial^{-1} q_1 - \partial q_1 \partial^{-1} p_2, \qquad \Phi_{24} = \partial - \partial p_2 \partial^{-1} q_2 + \partial q_1 \partial^{-1} p_1, \\
&\Phi_{31} = -\partial - \partial q_1 \partial^{-1} p_1 + \partial p_2 \partial^{-1} q_2, \qquad \Phi_{32} = -\partial q_1 \partial^{-1} p_2 - \partial p_2 \partial^{-1} q_1, \\
&\Phi_{33} = -\partial q_1 \partial^{-1} q_1 - \partial p_2 \partial^{-1} p_2, \qquad \Phi_{34} = -\partial q_1 \partial^{-1} q_2 + \partial p_2 \partial^{-1} p_1, \\
&\Phi_{41} = -\partial q_2 \partial^{-1} p_1 - \partial p_1 \partial^{-1} q_2, \qquad \Phi_{42} = -\partial - \partial q_2 \partial^{-1} p_2 + \partial p_1 \partial^{-1} q_1, \\
&\Phi_{43} = -\partial q_2 \partial^{-1} q_1 + \partial p_1 \partial^{-1} p_2, \qquad \Phi_{44} = -\partial q_2 \partial^{-1} q_2 - \partial p_1 \partial^{-1} p_1.
\end{aligned}$$

Next, we consider Hamiltonian structures by using the trace identity (1.2). The calculation of this part is very direct, without any additional mathematical skills. In the following sections, we will omit these contents for simplification. It is direct to see

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 0 & -q_1 & -q_2 & -2\lambda \\ q_1 & 0 & 0 & -p_1 \\ q_2 & 0 & 0 & -p_2 \\ 2\lambda & p_1 & p_2 & 0 \end{bmatrix}, \qquad \mathrm{tr}\left(W \frac{\partial U}{\partial \lambda}\right) = -4\lambda a - 2(q_1 c_1 + q_2 c_2 + p_1 b_1 + p_2 b_2).$$

Similarly, we have

$$\mathrm{tr}\left(W \frac{\partial U}{\partial p_i}\right) = -2\lambda b_i, \qquad \mathrm{tr}\left(W \frac{\partial U}{\partial q_i}\right) = -2\lambda c_i, \qquad i = 1, 2.$$

Now, the corresponding trace identity becomes

$$\frac{\delta}{\delta u} \int (2\lambda a + p_1 b_1 + p_2 b_2 + q_1 c_1 + q_2 c_2)\, \mathrm{d}x = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \begin{bmatrix} \lambda b_1 \\ \lambda b_2 \\ \lambda c_1 \\ \lambda c_2 \end{bmatrix}.$$

Balancing coefficients of each power of λ in the aforementioned equality, we have

$$\frac{\delta}{\delta u} \int \left(2a^{(n+1)} + p_1 b_1^{(n)} + p_2 b_2^{(n)} + q_1 c_1^{(n)} + q_2 c_2^{(n)}\right) \mathrm{d}x = (\gamma - 2n) \begin{bmatrix} b_1^{(n)} \\ b_2^{(n)} \\ c_1^{(n)} \\ c_2^{(n)} \end{bmatrix}, \qquad n \geq 0.$$

Checking a particular case with $n = 1$ yields $\gamma = 0$, and thus, we obtain

$$\frac{\delta}{\delta u} \int \left(-\frac{2a^{(n+1)} + p_1 b_1^{(n)} + p_2 b_2^{(n)} + q_1 c_1^{(n)} + q_2 c_2^{(n)}}{2n}\right) \mathrm{d}x = \begin{bmatrix} b_1^{(n)} \\ b_2^{(n)} \\ c_1^{(n)} \\ c_2^{(n)} \end{bmatrix}, \qquad n \geq 1.$$

Consequently, we obtain the following Hamiltonian structure for the soliton hierarchy of KN type

$$\begin{bmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{bmatrix}_{t_n} = K_n = \begin{bmatrix} b_{1x}^{(n)} \\ b_{2x}^{(n)} \\ c_{1x}^{(n)} \\ c_{2x}^{(n)} \end{bmatrix} = J \begin{bmatrix} b_1^{(n)} \\ b_2^{(n)} \\ c_1^{(n)} \\ c_2^{(n)} \end{bmatrix} = J\frac{\delta \mathcal{H}_n}{\delta u}, \qquad n \geq 0, \tag{2.10}$$

with the Hamiltonian operator

$$J = \begin{bmatrix} \partial & 0 & 0 & 0 \\ 0 & \partial & 0 & 0 \\ 0 & 0 & \partial & 0 \\ 0 & 0 & 0 & \partial \end{bmatrix},$$

and the Hamiltonian functionals

$$\begin{aligned} \mathcal{H}_0 &= \int \frac{1}{2}\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) dx, \\ \mathcal{H}_n &= \int \left(-\frac{2a^{(n+1)} + p_1 b_1^{(n)} + p_2 b_2^{(n)} + q_1 c_1^{(n)} + q_2 c_2^{(n)}}{2n}\right) dx, \qquad n \geq 1. \end{aligned}$$

It is now a direct computation that this new soliton hierarchy is bi-Hamiltonian,

$$u_{t_n} = K_n = J\frac{\delta \mathcal{H}_n}{\delta u} = M\frac{\delta \mathcal{H}_{n-1}}{\delta u}, \qquad n \geq 1, \tag{2.11}$$

where the second Hamiltonian operator M is given by

$$M = \Phi J = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix},$$

$$\begin{aligned}
M_{11} &= -\partial p_1 \partial^{-1} p_1 \partial - \partial q_2 \partial^{-1} q_2 \partial, & M_{12} &= -\partial p_1 \partial^{-1} p_2 \partial + \partial q_2 \partial^{-1} q_1 \partial, \\
M_{13} &= \partial^2 - \partial p_1 \partial^{-1} q_1 \partial + \partial q_2 \partial^{-1} p_2 \partial, & M_{14} &= -\partial p_1 \partial^{-1} q_2 \partial - \partial q_2 \partial^{-1} p_1 \partial, \\
M_{21} &= -\partial p_2 \partial^{-1} p_1 \partial + \partial q_1 \partial^{-1} q_2 \partial, & M_{22} &= -\partial p_2 \partial^{-1} p_2 \partial - \partial q_1 \partial^{-1} q_1 \partial, \\
M_{23} &= -\partial p_2 \partial^{-1} q_1 \partial - \partial q_1 \partial^{-1} p_2 \partial, & M_{24} &= \partial^2 - \partial p_2 \partial^{-1} q_2 \partial + \partial q_1 \partial^{-1} p_1 \partial, \\
M_{31} &= -\partial^2 - \partial q_1 \partial^{-1} p_1 \partial + \partial p_2 \partial^{-1} q_2 \partial, & M_{32} &= -\partial q_1 \partial^{-1} p_2 \partial - \partial p_2 \partial^{-1} q_1 \partial, \\
M_{33} &= -\partial q_1 \partial^{-1} q_1 \partial - \partial p_2 \partial^{-1} p_2 \partial, & M_{34} &= -\partial q_1 \partial^{-1} q_2 \partial + \partial p_2 \partial^{-1} p_1 \partial, \\
M_{41} &= -\partial q_2 \partial^{-1} p_1 \partial - \partial p_1 \partial^{-1} q_2 \partial, & M_{42} &= -\partial^2 - \partial q_2 \partial^{-1} p_2 \partial + \partial p_1 \partial^{-1} q_1 \partial, \\
M_{43} &= -\partial q_2 \partial^{-1} q_1 \partial + \partial p_1 \partial^{-1} p_2 \partial, & M_{44} &= -\partial q_2 \partial^{-1} q_2 \partial - \partial p_1 \partial^{-1} p_1 \partial.
\end{aligned}$$

Thus, the new soliton hierarchy of KN type (2.9) is Liouville integrable (see [41] for definition), that is, it possesses infinitely many conserved functionals and symmetries which form Abelian algebras:

$$\begin{aligned} \{\mathcal{H}_l, \mathcal{H}_m\}_J &= \int \left(\frac{\delta \mathcal{H}_l}{\delta u}\right)^T J \frac{\delta \mathcal{H}_m}{\delta u} dx = 0, \qquad l, m \geq 0, \\ \{\mathcal{H}_l, \mathcal{H}_m\}_M &= \int \left(\frac{\delta \mathcal{H}_l}{\delta u}\right)^T M \frac{\delta \mathcal{H}_m}{\delta u} dx = 0, \qquad l, m \geq 0, \end{aligned}$$

and

$$[K_l, K_m] = K_l'(u)[K_m] - K_m'(u)[K_l] = 0, \qquad l, m \geq 0.$$

These commuting relations are also consequences of the Virasoro algebras of Lax operators.

Remark 1
If we let the spectral matrix U be

$$\begin{aligned} U &= (\lambda^2 + s)e_1 + \lambda q_1 e_2 + \lambda q_2 e_3 + \lambda p_1 e_4 + \lambda p_2 e_5 + r e_6 \\ &= \begin{bmatrix} 0 & -\lambda q_1 & -\lambda q_2 & -\lambda^2 - s \\ \lambda q_1 & 0 & -r & -\lambda p_1 \\ \lambda q_2 & r & 0 & -\lambda p_2 \\ \lambda^2 + s & \lambda p_1 & \lambda p_2 & 0 \end{bmatrix}, \end{aligned}$$

and

$$r = \beta_1\left(p_1^2+p_2^2+q_1^2+q_2^2\right)+\beta_2(p_1q_2-p_2q_1),$$
$$s = \beta_3\left(p_1^2+p_2^2+q_1^2+q_2^2\right)+\beta_4(p_1q_2-p_2q_1),$$

following the procedure of constructing the aforementioned soliton hierarchy of KN type, we can obtain a generalized soliton hierarchy of KN type.

Although, we can consider the following $(N+2)\times(N+2)$ matrix spectral problem,

$$\phi_x = U\phi, \qquad U = \begin{bmatrix} 0 & -\lambda q & -m\lambda^2 \\ \lambda q^T & 0 & -\lambda p \\ m\lambda^2 & \lambda p^T & 0 \end{bmatrix}, \qquad u = \begin{bmatrix} p \\ q^T \end{bmatrix}, \tag{2.12}$$

where $p=(p_1,p_2,\cdots,p_N)^T$, $q=(q_1,q_2,\cdots,q_N)$ and λ is the spectral parameter. When $N=m=1$, (2.12) is nothing but the one considered by Ma in Reference [38], and when $N=2, m=1$, it corresponds to (2.2), but we can only solve this problem formally because the localness can not be proved. Firstly, the stationary zero curvature equation $W_x=[U,W]$ becomes

$$\begin{aligned} a_x &= \lambda(cp-qb),\\ b_x &= -m\lambda^2c^T+\lambda aq^T-\lambda dp,\\ c_x &= m\lambda^2b^T-\lambda ap^T+\lambda qd,\\ d_x &= \lambda\left(-q^Tc-pb^T+c^Tq+bp^T\right), \end{aligned} \tag{2.13}$$

if we assume that W is of the form

$$W = \begin{bmatrix} 0 & -c & -a \\ c^T & d & -b \\ a & b^T & 0 \end{bmatrix}, \tag{2.14}$$

where a is a scalar, c^T and b are N-dimensional column vectors, and generally speaking d is an $N\times N$ non-zero matrix. To consider furthermore, we let

$$W=\sum_{k=0}^{\infty}W_k\lambda^{-2k}, \qquad W_k = \begin{bmatrix} 0 & -c^{(k)}\lambda^{-1} & -a^{(k)} \\ c^{(k)T}\lambda^{-1} & d^{(k)} & -b^{(k)}\lambda^{-1} \\ a^{(k)} & b^{(k)T}\lambda^{-1} & 0 \end{bmatrix}. \tag{2.15}$$

Thus, the system (2.13) gives rise to

$$\begin{aligned} b^{(0)} &= p,\\ c^{(0)} &= q,\\ a_x^{(k)} &= c^{(k)}p-qb^{(k)},\\ d_x^{(k)} &= -q^Tc^{(k)}-pb^{(k)T}+c^{(k)T}q+b^{(k)}p^T,\\ b_x^{(k)} &= -mc^{(k+1)T}+a^{(k+1)}q^T-d^{(k+1)}p,\\ c_x^{(k)T} &= mb^{(k+1)}-a^{(k+1)}p-d^{(k+1)}q^T, \qquad k\ge 0. \end{aligned} \tag{2.16}$$

So substituting

$$\begin{aligned} a^{(k+1)} &= \partial^{-1}\left(c^{(k+1)}p-qb^{(k+1)}\right),\\ d^{(k+1)} &= \partial^{-1}\left(-q^Tc^{(k+1)}-pb^{(k+1)T}+c^{(k+1)T}q+b^{(k+1)}p^T\right), \end{aligned}$$

into the last two equations in the aforementioned system (2.16), we have

$$\begin{aligned} \begin{bmatrix} b_x^{(k)} \\ c_x^{(k)T} \end{bmatrix} &= \mathcal{L}\begin{bmatrix} b^{(k+1)} \\ c^{(k+1)T} \end{bmatrix},\\ \mathcal{L} &= \begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix}+\begin{bmatrix} -q^T\partial^{-1}q & q^T\partial^{-1}p^T \\ p\partial^{-1}q & -p\partial^{-1}p^T \end{bmatrix}\\ &\quad+\begin{bmatrix} \left(\partial^{-1}p(\cdot)^T\right)p-\left(\partial^{-1}(\cdot)p^T\right)p & \left(\partial^{-1}q^T(\cdot)\right)p-\left(\partial^{-1}(\cdot)^Tq\right)p \\ \left(\partial^{-1}p(\cdot)^T\right)q^T-\left(\partial^{-1}(\cdot)p^T\right)q^T & \left(\partial^{-1}q^T(\cdot)\right)q^T-\left(\partial^{-1}(\cdot)^Tq\right)q^T \end{bmatrix}. \end{aligned}$$

Finally, taking

$$\begin{aligned} V^{[n]} &= \lambda(\lambda^{2n+1}W)_+\\ &= \lambda^{2n+2}W_0+\lambda^{2n}W_1+\cdots+\lambda^2W_n, \end{aligned}$$

the zero curvature Eqs (1.1) generate a new multi-component soliton hierarchy of KN type formally

$$\begin{aligned}\begin{bmatrix} p_x \\ q_x^T \end{bmatrix} &= \begin{bmatrix} b_x^{(0)} \\ c_x^{(0)T} \end{bmatrix} = \mathcal{L}\sigma \begin{bmatrix} b_x^{(1)} \\ c_x^{(1)T} \end{bmatrix} \\ &= (\mathcal{L}\sigma)^2 \begin{bmatrix} b_x^{(2)} \\ c_x^{(2)T} \end{bmatrix} = \cdots = (\mathcal{L}\sigma)^n \begin{bmatrix} b_x^{(n)} \\ c_x^{(n)T} \end{bmatrix} \\ &= (\mathcal{L}\sigma)^n \begin{bmatrix} p \\ q^T \end{bmatrix}_{t_n} = \Psi^n \begin{bmatrix} p \\ q^T \end{bmatrix}_{t_n}, \end{aligned} \tag{2.17}$$

namely,

$$\begin{bmatrix} p \\ q^T \end{bmatrix}_{t_n} = K_n = \begin{bmatrix} b_x^{(n)} \\ c_x^{(n)T} \end{bmatrix} = \Phi^n \begin{bmatrix} p_x \\ q_x^T \end{bmatrix}, \qquad n \geq 0, \tag{2.18}$$

where $\sigma = \begin{bmatrix} \partial^{-1} & 0 \\ 0 & \partial^{-1} \end{bmatrix}$, $\Phi = \Psi^{-1} = \begin{bmatrix} \partial & 0 \\ 0 & \partial \end{bmatrix} \mathcal{L}^{-1}$.

Remark 2
In fact, we can take another form of matrix spectral problem (2.1) with spectral matrix,

$$\begin{aligned} U &= \lambda^2 r e_1 + \lambda q_1 e_2 + \lambda q_2 e_3 + \lambda p_1 e_4 + \lambda p_2 e_5 \\ &= \begin{bmatrix} 0 & -\lambda q_1 & -\lambda q_2 & -\lambda^2 r \\ \lambda q_1 & 0 & 0 & -\lambda p_1 \\ \lambda q_2 & 0 & 0 & -\lambda p_2 \\ \lambda^2 r & \lambda p_1 & \lambda p_2 & 0 \end{bmatrix} \in \widetilde{\mathrm{so}}(4, \mathbb{R}), \end{aligned} \tag{2.19}$$

and

$$r = \beta_1 \left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \beta_2 (p_1 q_2 - p_2 q_1), \qquad \beta_1^2 + \beta_2^2 \neq 0. \tag{2.20}$$

Then, let W be of the form (2.3), we solve the stationary zero curvature equation $W_x = [U, W]$, and it becomes

$$\begin{aligned} a_x &= \lambda (p_1 c_1 + p_2 c_2 - q_1 b_1 - q_2 b_2), \\ d_x &= \lambda (q_1 c_2 + p_1 b_2 - q_2 c_1 - p_2 b_1), \\ b_{1x} &= -\lambda^2 r c_1 + \lambda q_1 a + \lambda p_2 d \\ b_{2x} &= -\lambda^2 r c_2 + \lambda q_2 a - \lambda p_1 d \\ c_{1x} &= \lambda^2 r b_1 - \lambda p_1 a + \lambda q_2 d \\ c_{2x} &= \lambda^2 r b_2 - \lambda p_2 a - \lambda q_1 d. \end{aligned} \tag{2.21}$$

Further setting (2.5), the aforementioned system (2.21) gives rise to

$$\begin{aligned} a_x^{(k)} &= p_1 c_1^{(k)} + p_2 c_2^{(k)} - q_1 b_1^{(k)} - q_2 b_2^{(k)}, \\ d_x^{(k)} &= q_1 c_2^{(k)} + p_1 b_2^{(k)} - q_2 c_1^{(k)} - p_2 b_1^{(k)}, \\ b_1^{(0)} &= \frac{1}{r}\left(p_1 a^{(0)} - q_2 d^{(0)}\right), \\ b_2^{(0)} &= \frac{1}{r}\left(p_2 a^{(0)} + q_1 d^{(0)}\right), \\ c_1^{(0)} &= \frac{1}{r}\left(q_1 a^{(0)} + p_2 d^{(0)}\right), \\ c_2^{(0)} &= \frac{1}{r}\left(q_2 a^{(0)} - p_1 d^{(0)}\right), \\ b_1^{(k+1)} &= \frac{1}{r}\left(c_{1x}^{(k)} + p_1 a^{(k+1)} - q_2 d^{(k+1)}\right), \\ b_2^{(k+1)} &= \frac{1}{r}\left(c_{2x}^{(k)} + p_2 a^{(k+1)} + q_1 d^{(k+1)}\right), \\ c_1^{(k+1)} &= \frac{1}{r}\left(-b_{1x}^{(k)} + q_1 a^{(k+1)} + p_2 d^{(k+1)}\right), \\ c_2^{(k+1)} &= \frac{1}{r}\left(-b_{2x}^{(k)} + q_2 a^{(k+1)} - p_1 d^{(k+1)}\right), \qquad k \geq 0. \end{aligned} \tag{2.22}$$

To construct recursion relations, we rewrite

$$\begin{aligned}
a_x^{(k+1)} &= p_1c_1^{(k+1)} + p_2c_2^{(k+1)} - q_1b_1^{(k+1)} - q_2b_2^{(k+1)} \\
&= \frac{p_1}{r}\left(-b_{1x}^{(k)} + q_1a^{(k+1)} + p_2d^{(k+1)}\right) + \frac{p_2}{r}\left(-b_{2x}^{(k)} + q_2a^{(k+1)} - p_1d^{(k+1)}\right) \\
&\quad - \frac{q_1}{r}\left(c_{1x}^{(k)} + p_1a^{(k+1)} - q_2d^{(k+1)}\right) - \frac{q_2}{r}\left(c_{2x}^{(k)} + p_2a^{(k+1)} + q_1d^{(k+1)}\right) \\
&= \frac{1}{r}\left(-p_1b_{1x}^{(k)} - p_2b_{2x}^{(k)} - q_1c_{1x}^{(k)} - q_2c_{2x}^{(k)}\right), \\
d_x^{(k+1)} &= q_1c_2^{(k+1)} + p_1b_2^{(k+1)} - q_2c_1^{(k+1)} - p_2b_1^{(k+1)} \\
&= \frac{q_1}{r}\left(-b_{2x}^{(k)} + q_2a^{(k+1)} - p_1d^{(k+1)}\right) + \frac{p_1}{r}\left(c_{2x}^{(k)} + p_2a^{(k+1)} + q_1d^{(k+1)}\right) \\
&\quad - \frac{q_2}{r}\left(-b_{1x}^{(k)} + q_1a^{(k+1)} + p_2d^{(k+1)}\right) - \frac{p_2}{r}\left(c_{1x}^{(k)} + p_1a^{(k+1)} - q_2d^{(k+1)}\right) \\
&= \frac{1}{r}\left(q_2b_{1x}^{(k)} - q_1b_{2x}^{(k)} - p_2c_{1x}^{(k)} + p_1c_{2x}^{(k)}\right).
\end{aligned}$$

Thus, we arrive at

$$\begin{aligned}
b_1^{(k+1)} &= c_{1x}^{(k)} + p_1\partial^{-1}\frac{1}{r}\left(-p_1b_{1x}^{(k)} - p_2b_{2x}^{(k)} - q_1c_{1x}^{(k)} - q_2c_{2x}^{(k)}\right) - q_2\partial^{-1}\frac{1}{r}\left(q_2b_{1x}^{(k)} - q_1b_{2x}^{(k)} - p_2c_{1x}^{(k)} + p_1c_{2x}^{(k)}\right), \\
b_2^{(k+1)} &= c_{2x}^{(k)} + p_2\partial^{-1}\frac{1}{r}\left(-p_1b_{1x}^{(k)} - p_2b_{2x}^{(k)} - q_1c_{1x}^{(k)} - q_2c_{2x}^{(k)}\right) + q_1\partial^{-1}\frac{1}{r}\left(q_2b_{1x}^{(k)} - q_1b_{2x}^{(k)} - p_2c_{1x}^{(k)} + p_1c_{2x}^{(k)}\right), \\
c_1^{(k+1)} &= -b_{1x}^{(k)} + q_1\partial^{-1}\frac{1}{r}\left(-p_1b_{1x}^{(k)} - p_2b_{2x}^{(k)} - q_1c_{1x}^{(k)} - q_2c_{2x}^{(k)}\right) + p_2\partial^{-1}\frac{1}{r}\left(q_2b_{1x}^{(k)} - q_1b_{2x}^{(k)} - p_2c_{1x}^{(k)} + p_1c_{2x}^{(k)}\right), \\
c_2^{(k+1)} &= -b_{2x}^{(k)} + q_2\partial^{-1}\frac{1}{r}\left(-p_1b_{1x}^{(k)} - p_2b_{2x}^{(k)} - q_1c_{1x}^{(k)} - q_2c_{2x}^{(k)}\right) - p_1\partial^{-1}\frac{1}{r}\left(q_2b_{1x}^{(k)} - q_1b_{2x}^{(k)} - p_2c_{1x}^{(k)} + p_1c_{2x}^{(k)}\right).
\end{aligned}$$

Combined with the aforementioned recursion relations and a set of appropriate initial values

$$a^{(0)} = 1, \qquad d^{(0)} = 0, \qquad b_1^{(0)} = \frac{p_1}{r}, \qquad b_2^{(0)} = \frac{p_2}{r}, \qquad c_1^{(0)} = \frac{q_1}{r}, \qquad c_2^{(0)} = \frac{q_2}{r},$$

we can determine the sequence of $\left\{a^{(k)}, d^{(k)}, b_i^{(k)}, c_i^{(k)},\ i = 1, 2\right\}$ uniquely, where the integration conditions

$$a^{(k)}\Big|_{u=0} = d^{(k)}\Big|_{u=0} = b_i^{(k)}\Big|_{u=0} = c_i^{(k)}\Big|_{u=0} = 0, \qquad i = 1, 2, \quad k \geq 1$$

also need be imposed. For readability, these tedious expressions are omitted, and one can obtain them by using symbolic computation software Maple.

Now, we let $V^{[n]}$ be the following form:

$$\begin{aligned}
V^{[n]} &= \lambda(\lambda^{2n+1}W)_+ + \Delta_n \\
&= \lambda^{2n+2}W_0 + \lambda^{2n}W_1 + \cdots + \lambda^2W_n + \begin{bmatrix} 0 & -\lambda h_{1,n} & -\lambda h_{2,n} & -\lambda^2 f_n \\ \lambda h_{1,n} & 0 & 0 & -\lambda g_{1,n} \\ \lambda h_{2,n} & 0 & 0 & -\lambda g_{2,n} \\ \lambda^2 f_n & \lambda g_{1,n} & \lambda g_{2,n} & 0 \end{bmatrix},
\end{aligned} \tag{2.23}$$

where functions $f_n, g_{i,n}, h_{i,n},\ i = 1, 2$ need be determined. Thus, the zero curvature equations $U_{t_n} - V_x^{[n]} + \left[U, V^{[n]}\right] = 0,\ n \geq 0$ give

$$\begin{aligned}
g_{1,n} &= \frac{p_1f_n}{r}, \qquad g_{2,n} = \frac{p_2f_n}{r}, \qquad h_{1,n} = \frac{q_1f_n}{r}, \qquad h_{2,n} = \frac{q_2f_n}{r}, \\
f_{nx} &= r_{t_n} + p_1h_{1,n} + p_2h_{2,n} - q_1g_{1,n} - q_2g_{2,n} = r_{t_n}, \\
p_{1t_n} &= b_{1x}^{(n)} + g_{1,nx} = b_{1x}^{(n)} + \left(\frac{p_1f_n}{r}\right)_x, \\
p_{2t_n} &= b_{2x}^{(n)} + g_{2,nx} = b_{2x}^{(n)} + \left(\frac{p_2f_n}{r}\right)_x, \\
q_{1t_n} &= c_{1x}^{(n)} + h_{1,nx} = c_{1x}^{(n)} + \left(\frac{q_1f_n}{r}\right)_x, \\
q_{2t_n} &= c_{2x}^{(n)} + h_{2,nx} = c_{2x}^{(n)} + \left(\frac{q_2f_n}{r}\right)_x.
\end{aligned}$$

So we can obtain

$$
\begin{aligned}
f_{nx} &= r_{t_n}\\
&= 2\beta_1\left(p_1p_{1t_n}+p_2p_{2t_n}+q_1q_{1t_n}+q_2q_{2t_n}\right)+\beta_2\left(p_{1t_n}q_2+p_1q_{2t_n}-p_{2t_n}q_1-p_2q_{1t_n}\right)\\
&= 2\beta_1\left\{p_1\left[b_{1x}^{(n)}+\left(\frac{p_1f_n}{r}\right)_x\right]+p_2\left[b_{2x}^{(n)}+\left(\frac{p_2f_n}{r}\right)_x\right]+q_1\left[c_{1x}^{(n)}+\left(\frac{q_1f_n}{r}\right)_x\right]+q_2\left[c_{2x}^{(n)}+\left(\frac{q_2f_n}{r}\right)_x\right]\right\}\\
&\quad+\beta_2\left\{q_2\left[b_{1x}^{(n)}+\left(\frac{p_1f_n}{r}\right)_x\right]+p_1\left[c_{2x}^{(n)}+\left(\frac{q_2f_n}{r}\right)_x\right]-q_1\left[b_{2x}^{(n)}+\left(\frac{p_2f_n}{r}\right)_x\right]-p_2\left[c_{1x}^{(n)}+\left(\frac{q_1f_n}{r}\right)_x\right]\right\}\\
&= -2\beta_1\left(-p_1b_{1x}^{(n)}-p_2b_{2x}^{(n)}-q_1c_{1x}^{(n)}-q_2c_{2x}^{(n)}\right)+\beta_2\left(q_2b_{1x}^{(n)}-q_1b_{2x}^{(n)}-p_2c_{1x}^{(n)}+p_1c_{2x}^{(n)}\right)+2f_{nx}-2\frac{r_x}{r}f_n.
\end{aligned}
$$

This equation can be changed to

$$
\left(\frac{f_n}{r}\right)_x = 2\beta_1 a_x^{(n+1)}-\beta_2 d_x^{(n+1)} \quad\Rightarrow\quad f_n=\left(2\beta_1a^{(n+1)}-\beta_2d^{(n+1)}\right)r. \tag{2.24}
$$

This means that another new soliton hierarchy of KN type

$$
\begin{bmatrix} p_1\\ p_2\\ q_1\\ q_2\end{bmatrix}_{t_n} = K_n = \begin{bmatrix} b_{1x}^{(n)}+2\beta_1(p_1a^{(n+1)})_x-\beta_2(p_1d^{(n+1)})_x\\ b_{2x}^{(n)}+2\beta_1(p_2a^{(n+1)})_x-\beta_2(p_2d^{(n+1)})_x\\ c_{1x}^{(n)}+2\beta_1(q_1a^{(n+1)})_x-\beta_2(q_1d^{(n+1)})_x\\ c_{2x}^{(n)}+2\beta_1(q_2a^{(n+1)})_x-\beta_2(q_2d^{(n+1)})_x\end{bmatrix} \tag{2.25}
$$

has been derived.

3. New soliton hierarchy of Ablowitz–Kaup–Newell–Segur type

To construct new soliton hierarchy of AKNS type, let us introduce the matrix spectral problem (2.1) with the spectral matrix U being chosen as

$$
\begin{aligned}
U &= \lambda e_1+q_1e_2+q_2e_3+p_1e_4+p_2e_5+re_6\\
&= \begin{bmatrix} 0 & -q_1 & -q_2 & -\lambda\\ q_1 & 0 & -r & -p_1\\ q_2 & r & 0 & -p_2\\ \lambda & p_1 & p_2 & 0\end{bmatrix} \in \widetilde{\mathrm{so}}(4,\mathbb{R}),
\end{aligned} \tag{3.1}
$$

and

$$
r=\beta_1\left(p_1^2+p_2^2+q_1^2+q_2^2\right)+\beta_2(p_1q_2-p_2q_1). \tag{3.2}
$$

This spectral problem is of the same type as the soliton hierarchies of AKNS type associated with $\mathrm{sl}(2,\mathbb{R})$ or $\mathrm{so}(3,\mathbb{R})$ [1, 37, 42], but its underlying loop algebra is different.

Then, let W be of the form (2.3), we solve the stationary zero curvature equation $W_x=[U,W]$, and it becomes

$$
\begin{aligned}
a_x &= p_1c_1+p_2c_2-q_1b_1-q_2b_2,\\
d_x &= q_1c_2+p_1b_2-q_2c_1-p_2b_1,\\
b_{1x} &= -\lambda c_1+q_1a+p_2d-rb_2\\
b_{2x} &= -\lambda c_2+q_2a-p_1d+rb_1\\
c_{1x} &= \lambda b_1-p_1a+q_2d-rc_2\\
c_{2x} &= \lambda b_2-p_2a-q_1d+rc_1.
\end{aligned} \tag{3.3}
$$

Further setting

$$
\begin{aligned}
a &= \sum_{k=0}^{\infty}a^{(k)}\lambda^{-k}, \qquad d=\sum_{k=0}^{\infty}d^{(k)}\lambda^{-k},\\
b_i &= \sum_{k=0}^{\infty}b_i^{(k)}\lambda^{-k}, \qquad c_i=\sum_{k=0}^{\infty}c_i^{(k)}\lambda^{-k}, \qquad i=1,2,
\end{aligned} \tag{3.4}
$$

namely,

$$
W=\sum_{k=0}^{\infty}W_k\lambda^{-k}, \qquad W_k=\begin{bmatrix} 0 & -c_1^{(k)} & -c_2^{(k)} & -a^{(k)}\\ c_1^{(k)} & 0 & -d^{(k)} & -b_1^{(k)}\\ c_2^{(k)} & d^{(k)} & 0 & -b_2^{(k)}\\ a^{(k)} & b_1^{(k)} & b_2^{(k)} & 0\end{bmatrix},
$$

system (3.3) gives rise to

$$\begin{aligned} a_x^{(k)} &= p_1 c_1^{(k)} + p_2 c_2^{(k)} - q_1 b_1^{(k)} - q_2 b_2^{(k)}, \\ d_x^{(k)} &= q_1 c_2^{(k)} + p_1 b_2^{(k)} - q_2 c_1^{(k)} - p_2 b_1^{(k)}, \\ b_1^{(0)} &= b_2^{(0)} = c_1^{(0)} = c_2^{(0)} = 0, \\ b_1^{(k+1)} &= c_{1x}^{(k)} + p_1 a^{(k)} - q_2 d^{(k)} + r c_2^{(k)}, \\ b_2^{(k+1)} &= c_{2x}^{(k)} + p_2 a^{(k)} + q_1 d^{(k)} - r c_1^{(k)}, \\ c_1^{(k+1)} &= -b_{1x}^{(k)} + q_1 a^{(k)} + p_2 d^{(k)} - r b_2^{(k)}, \\ c_2^{(k+1)} &= -b_{2x}^{(k)} + q_2 a^{(k)} - p_1 d^{(k)} + r b_1^{(k)}, \qquad k \geq 0. \end{aligned} \tag{3.5}$$

We take $a^{(0)} = 1, d^{(0)} = 0$ and impose the integration conditions

$$a^{(k)}|_{u=0} = d^{(k)}|_{u=0} = b_i^{(k)}|_{u=0} = c_i^{(k)}|_{u=0} = 0, \qquad i = 1, 2, \quad k \geq 1$$

to determine the sequence of $\left\{a^{(k)}, d^{(k)}, b_i^{(k)}, c_i^{(k)},\ i = 1, 2\right\}$ uniquely. Therefore, by means of symbolic computation software Maple, the first few sets can be computed as follows:

$$\begin{aligned}
b_1^{(1)} &= p_1, \qquad b_2^{(1)} = p_2, \qquad c_1^{(1)} = q_1, \qquad c_2^{(1)} = q_2, \\
a^{(1)} &= 0, \qquad d^{(1)} = 0, \\
b_1^{(2)} &= q_{1x} + \left[\beta_1 \left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \beta_2 \left(p_1 q_2 - p_2 q_1\right)\right] q_2, \\
b_2^{(2)} &= q_{2x} - \left[\beta_1 \left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \beta_2 \left(p_1 q_2 - p_2 q_1\right)\right] q_1, \\
c_1^{(2)} &= -p_{1x} - \left[\beta_1 \left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \beta_2 \left(p_1 q_2 - p_2 q_1\right)\right] p_2, \\
c_2^{(2)} &= -p_{2x} + \left[\beta_1 \left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \beta_2 \left(p_1 q_2 - p_2 q_1\right)\right] p_1, \\
a^{(2)} &= -\frac{1}{2}\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right), \qquad d^{(2)} = p_1 q_2 - p_2 q_1, \\
b_1^{(3)} &= -p_{1xx} - \left[2\beta_1 \left(p_1 p_{1x} + p_2 p_{2x} + q_1 q_{1x} + q_2 q_{2x}\right) + \beta_2 \left(p_{1x} q_2 + p_1 q_{2x} - p_{2x} q_1 - p_2 q_{1x}\right)\right] p_2 \\
&\quad - 2\left[\beta_1 \left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \beta_2 \left(p_1 q_2 - p_2 q_1\right)\right] p_{2x} + \left[\beta_1 \left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \beta_2 \left(p_1 q_2 - p_2 q_1\right)\right]^2 p_1 \\
&\quad - \frac{1}{2} p_1 \left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) - q_2 (p_1 q_2 - p_2 q_1), \\
b_2^{(3)} &= -p_{2xx} + \left[2\beta_1 \left(p_1 p_{1x} + p_2 p_{2x} + q_1 q_{1x} + q_2 q_{2x}\right) + \beta_2 (p_{1x} q_2 + p_1 q_{2x} - p_{2x} q_1 - p_2 q_{1x})\right] p_1 \\
&\quad + 2\left[\beta_1 \left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \beta_2 (p_1 q_2 - p_2 q_1)\right] p_{1x} + \left[\beta_1 \left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \beta_2 (p_1 q_2 - p_2 q_1)\right]^2 p_2 \\
&\quad - \frac{1}{2} p_2 \left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + q_1 (p_1 q_2 - p_2 q_1), \\
c_1^{(3)} &= -q_{1xx} - \left[2\beta_1 \left(p_1 p_{1x} + p_2 p_{2x} + q_1 q_{1x} + q_2 q_{2x}\right) + \beta_2 (p_{1x} q_2 + p_1 q_{2x} - p_{2x} q_1 - p_2 q_{1x})\right] q_2 \\
&\quad - 2\left[\beta_1 \left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \beta_2 (p_1 q_2 - p_2 q_1)\right] q_{2x} + \left[\beta_1 \left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \beta_2 (p_1 q_2 - p_2 q_1)\right]^2 q_1 \\
&\quad - \frac{1}{2}\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) q_1 + p_2 (p_1 q_2 - p_2 q_1), \\
c_2^{(3)} &= -q_{2xx} + \left[2\beta_1 (p_1 p_{1x} + p_2 p_{2x} + q_1 q_{1x} + q_2 q_{2x}) + \beta_2 (p_{1x} q_2 + p_1 q_{2x} - p_{2x} q_1 - p_2 q_{1x})\right] q_1 \\
&\quad + 2\left[\beta_1 \left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \beta_2 (p_1 q_2 - p_2 q_1)\right] q_{1x} + \left[\beta_1 \left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \beta_2 (p_1 q_2 - p_2 q_1)\right]^2 q_2 \\
&\quad - \frac{1}{2}\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) q_2 - p_1 (p_1 q_2 - p_2 q_1), \\
a^{(3)} &= -p_1 q_{1x} + p_{1x} q_1 - q_{2x} p_2 + p_{2x} q_2 - 2(p_1 q_2 - p_2 q_1)\left[\beta_1 \left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \beta_2 (p_1 q_2 - p_2 q_1)\right], \\
d^{(3)} &= -p_{2x} p_1 + p_2 p_{1x} - q_{2x} q_1 + q_{1x} q_2 + \left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right)\left[\beta_1 \left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \beta_2 (p_1 q_2 - p_2 q_1)\right].
\end{aligned}$$

Now, taking

$$\begin{aligned} V^{[n]} &= (\lambda^n W)_+ + \Delta_n \\ &= (\lambda^n W)_+ + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -f_n & 0 \\ 0 & f_n & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned} \tag{3.6}$$

the zero curvature equations $U_{t_n} - V_x^{[n]} + \left[U, V^{[n]}\right] = 0,\ n \geq 0$ give

$$\begin{aligned}
f_{nx} = r_{t_n} &= 2\beta_1\left(p_1 p_{1t_n} + p_2 p_{2t_n} + q_1 q_{1t_n} + q_2 q_{2t_n}\right)\\
&\quad + \beta_2\left(p_{1t_n} q_2 + p_1 q_{2t_n} - p_{2t_n} q_1 - p_2 q_{1t_n}\right),\\
p_{1t_n} &= -c_1^{(n+1)} - p_2 f_n,\\
p_{2t_n} &= -c_2^{(n+1)} + p_1 f_n,\\
q_{1t_n} &= b_1^{(n+1)} - q_2 f_n,\\
q_{2t_n} &= b_2^{(n+1)} + q_1 f_n.
\end{aligned}$$

Here, P_+ denotes the polynomial part of P of λ, and the modification terms Δ_n aims at guarantee the zero curvature equations can engender a soliton hierarchy. Thus, we can obtain

$$\begin{aligned}
f_{nx} &= 2\beta_1\left(p_1 p_{1t_n} + p_2 p_{2t_n} + q_1 q_{1t_n} + q_2 q_{2t_n}\right) + \beta_2\left(p_{1t_n} q_2 + p_1 q_{2t_n} - p_{2t_n} q_1 - p_2 q_{1t_n}\right)\\
&= 2\beta_1\left[p_1\left(-c_1^{(n+1)} - p_2 f_n\right) + p_2\left(-c_2^{(n+1)} + p_1 f_n\right) + q_1\left(b_1^{(n+1)} - q_2 f_n\right) + q_2\left(b_2^{(n+1)} + q_1 f_n\right)\right]\\
&\quad + \beta_2\left[q_2\left(-c_1^{(n+1)} - p_2 f_n\right) + p_1\left(b_2^{(n+1)} + q_1 f_n\right) - q_1\left(-c_2^{(n+1)} + p_1 f_n\right) - p_2\left(b_1^{(n+1)} - q_2 f_n\right)\right]\\
&= -2\beta_1\left(p_1 c_1^{(n+1)} + p_2 c_2^{n+1} - q_1 b_1^{(n+1)} - q_2 b_2^{(n+1)}\right) + \beta_2\left(q_1 c_2^{(n+1)} + p_1 b_2^{(n+1)} - q_2 c_1^{(n+1)} - p_2 b_1^{(n+1)}\right)\\
&= -2\beta_1 a_x^{(n+1)} + \beta_2 d_x^{(n+1)},
\end{aligned}$$

so we can set

$$f_n = -2\beta_1 a^{(n+1)} + \beta_2 d^{(n+1)}. \tag{3.7}$$

This means that a new soliton hierarchy of AKNS type

$$\begin{bmatrix} p_1\\ p_2\\ q_1\\ q_2 \end{bmatrix}_{t_n} = K_n = \begin{bmatrix} -c_1^{(n+1)} + 2\beta_1 p_2 a^{(n+1)} - \beta_2 p_2 d^{(n+1)}\\ -c_2^{(n+1)} - 2\beta_1 p_1 a^{(n+1)} + \beta_2 p_1 d^{(n+1)}\\ b_1^{(n+1)} + 2\beta_1 q_2 a^{(n+1)} - \beta_2 q_2 d^{(n+1)}\\ b_2^{(n+1)} - 2\beta_1 q_1 a^{(n+1)} + \beta_2 q_1 d^{(n+1)} \end{bmatrix}$$

has been derived. In fact, we can rewrite the aforementioned soliton hierarchy to more useful form by a matrix operator Φ. Next, we will construct a matrix operator Φ that satisfies

$$K_n = \Phi K_{n-1} = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14}\\ \Phi_{21} & \Phi_{22} & \Phi_{23} & \Phi_{24}\\ \Phi_{31} & \Phi_{32} & \Phi_{33} & \Phi_{34}\\ \Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44} \end{bmatrix} K_{n-1},$$

namely,

$$\begin{bmatrix} p_1\\ p_2\\ q_1\\ q_2 \end{bmatrix}_{t_n} = K_n = \Phi^n \begin{bmatrix} -c_1^{(1)}\\ -c_2^{(1)}\\ b_1^{(1)}\\ b_2^{(1)} \end{bmatrix} = \Phi^n \begin{bmatrix} -q_1\\ -q_2\\ p_1\\ p_2 \end{bmatrix}. \tag{3.8}$$

Firstly, for (3.5), we have

$$\begin{aligned}
a_x^{(n+1)} &= p_1 c_1^{(n+1)} + p_2 c_2^{(n+1)} - q_1 b_1^{(n+1)} - q_2 b_2^{(n+1)}\\
&= p_1\left(-b_{1x}^{(n)} + q_1 a^{(n)} + p_2 d^{(n)} - r b_2^{(n)}\right) + p_2\left(-b_{2x}^{(n)} + q_2 a^{(n)} - p_1 d^{(n)} + r b_1^{(n)}\right)\\
&\quad - q_1\left(c_{1x}^{(n)} + p_1 a^{(n)} - q_2 d^{(n)} + r c_2^{(n)}\right) - q_2\left(c_{2x}^{(n)} + p_2 a^{(n)} + q_1 d^{(n)} - r c_1^{(n)}\right)\\
&= -p_1 b_{1x}^{(n)} - p_2 b_{2x}^{(n)} - q_1 c_{1x}^{(n)} - q_2 c_{2x}^{(n)} - p_1 r b_2^{(n)} + p_2 r b_1^{(n)} - q_1 r c_2^{(n)} + q_2 r c_1^{(n)}\\
&= \left[q_1\partial - q_2 r - 2\beta_1(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_1 - \beta_2(p_1\partial q_2 - p_2\partial q_1\right.\\
&\quad \left.-q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_2\right]\left(-c_1^{(n)} + 2\beta_1 p_2 a^{(n)} - \beta_2 p_2 d^{(n)}\right)\\
&\quad + \left[q_2\partial + q_1 r - 2\beta_1(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_2 + \beta_2(p_1\partial q_2 - p_2\partial q_1\right.\\
&\quad \left.-q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_1\right]\left(-c_2^{(n)} - 2\beta_1 p_1 a^{(n)} + \beta_2 p_1 d^{(n)}\right)\\
&\quad + \left[-p_1\partial + p_2 r - 2\beta_1(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_1 + \beta_2(p_1\partial q_2 - p_2\partial q_1\right.\\
&\quad \left.-q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_2\right]\left(b_1^{(n)} + 2\beta_1 q_2 a^{(n)} - \beta_2 q_2 d^{(n)}\right)\\
&\quad + \left[-p_2\partial - p_1 r - 2\beta_1(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_2 - \beta_2(p_1\partial q_2 - p_2\partial q_1\right.\\
&\quad \left.-q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_1\right]\left(b_2^{(n)} - 2\beta_1 q_1 a^{(n)} + \beta_2 q_1 d^{(n)}\right)
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
d_x^{(n+1)} &= q_1c_2^{(n+1)} + p_1b_2^{(n+1)} - q_2c_1^{(n+1)} - p_2b_1^{(n+1)} \\
&= q_1\left(-b_{2x}^{(n)} + q_2a^{(n)} - p_1d^{(n)} + rb_1^{(n)}\right) + p_1\left(c_{2x}^{(n)} + p_2a^{(n)} + q_1d^{(n)} - rc_1^{(n)}\right) \\
&\quad - q_2\left(-b_{1x}^{(n)} + q_1a^{(n)} + p_2d^{(n)} - rb_2^{(n)}\right) - p_2\left(c_{1x}^{(n)} + p_1a^{(n)} - q_2d^{(n)} + rc_2^{(n)}\right) \\
&= -q_1b_{2x}^{(n)} + p_1c_{2x}^{(n)} + q_2b_{1x}^{(n)} - p_2c_{1x}^{(n)} + q_1rb_1^{(n)} - p_1rc_1^{(n)} + q_2rb_2^{(n)} - p_2rc_2^{(n)} \\
&= \big[p_2\partial + p_1r + 2\beta_1(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_1 + \beta_2(q_1\partial q_1 + p_1\partial p_1 \\
&\quad + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_2\big]\left(-c_1^{(n)} + 2\beta_1p_2a^{(n)} - \beta_2p_2d^{(n)}\right) \\
&\quad + \big[-p_1\partial + p_2r + 2\beta_1(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_2 - \beta_2(q_1\partial q_1 + p_1\partial q_1 \\
&\quad + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_1\big]\left(-c_2^{(n)} - 2\beta_1p_1a^{(n)} + \beta_2p_1d^{(n)}\right) \\
&\quad + \big[q_2\partial + q_1r + 2\beta_1(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_1 - \beta_2(q_1\partial q_1 + p_1\partial p_1 \\
&\quad + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_2\big]\left(b_1^{(n)} + 2\beta_1q_2a^{(n)} - \beta_2q_2d^{(n)}\right) \\
&\quad + \big[-q_1\partial + q_2r + 2\beta_1(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_2 + \beta_2(q_1\partial q_1 + p_1\partial p_1 \\
&\quad + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_1\big]\left(b_2^{(n)} - 2\beta_1q_1a^{(n)} + \beta_2q_1d^{(n)}\right).
\end{aligned} \tag{3.10}$$

By means of (3.5) again, we have

$$\begin{aligned}
-c_1^{(n+1)} + 2\beta_1p_2a^{(n+1)} - \beta_2p_2d^{(n+1)} &= \Phi_{11}\left(-c_1^{(n)} + 2\beta_1p_2a^{(n)} - \beta_2p_2d^{(n)}\right) \\
&\quad + \Phi_{12}\left(-c_2^{(n)} - 2\beta_1p_1a^{(n)} + \beta_2p_1d^{(n)}\right) \\
&\quad + \Phi_{13}\left(b_1^{(n)} + 2\beta_1q_2a^{(n)} - \beta_2q_2d^{(n)}\right) \\
&\quad + \Phi_{14}\left(b_2^{(n)} - 2\beta_1q_1a^{(n)} + \beta_2q_1d^{(n)}\right),
\end{aligned}$$

with

$$\begin{aligned}
\Phi_{11} &= 2\beta_1p_2\partial^{-1}q_1\partial - 2\beta_1p_2\partial^{-1}q_2r - 4\beta_1^2p_2\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_1 \\
&\quad - 2\beta_1\beta_2p_2\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_2 - \beta_2p_2\partial^{-1}p_2\partial + \beta_2p_2\partial^{-1}p_1r \\
&\quad - 2\beta_1\beta_2p_2\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_1 + (2\beta_1\partial q_2 - 2\beta_1rq_1 + q_1)\partial^{-1}p_1 \\
&\quad - \beta_2^2p_2\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_2 + (\beta_2\partial q_2 - \beta_2rq_1 - p_2)\partial^{-1}q_2, \\
\Phi_{12} &= 2\beta_1p_2\partial^{-1}q_2\partial + 2\beta_1p_2\partial^{-1}q_1r - 4\beta_1^2p_2\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_2 \\
&\quad + 2\beta_1\beta_2p_2\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_1 + \beta_2p_2\partial^{-1}p_1\partial - \beta_2p_2\partial^{-1}p_2r \\
&\quad - 2\beta_1\beta_2p_2\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_2 + (2\beta_1\partial q_2 - 2\beta_1rq_1 + q_1)\partial^{-1}p_2 \\
&\quad + \beta_2^2p_2\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_1 - (\beta_2\partial q_2 - \beta_2rq_1 - p_2)\partial^{-1}q_1, \\
\Phi_{13} &= \partial - 2\beta_1p_2\partial^{-1}p_1\partial + 2\beta_1p_2\partial^{-1}p_2r - 4\beta_1^2p_2\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_1 \\
&\quad + 2\beta_1\beta_2p_2\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_1 - \beta_2p_2\partial^{-1}q_2\partial - \beta_2p_2\partial^{-1}q_1r \\
&\quad - 2\beta_1\beta_2p_2\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_1 + (2\beta_1\partial q_2 - 2\beta_1rq_1 + q_1)\partial^{-1}q_1 \\
&\quad + \beta_2^2p_2\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_2 - (\beta_2\partial q_2 - \beta_2rq_1 - p_2)\partial^{-1}p_2, \\
\Phi_{14} &= r - 2\beta_1p_2\partial^{-1}p_2\partial + 2\beta_1p_2\partial^{-1}p_1r - 4\beta_1^2p_2\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_2 \\
&\quad - 2\beta_1\beta_2p_2\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_2 + \beta_2p_2\partial^{-1}q_1\partial - \beta_2p_2\partial^{-1}q_2r \\
&\quad + 2\beta_1\beta_2p_2\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_2 + (2\beta_1\partial q_2 - 2\beta_1rq_1 + q_1)\partial^{-1}q_2 \\
&\quad - \beta_2^2p_2\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_1 + (\beta_2\partial q_2 - \beta_2rq_1 - p_2)\partial^{-1}p_1.
\end{aligned}$$

Through a similar calculation, we can arrive at

$$\begin{aligned}
\Phi_{21} &= -2\beta_1p_1\partial^{-1}q_1\partial - 2\beta_1p_1\partial^{-1}q_2r + 4\beta_1^2p_1\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_1 \\
&\quad + 2\beta_1\beta_2p_1\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_2 + \beta_2p_1\partial^{-1}p_2\partial - \beta_2p_1\partial^{-1}p_1r \\
&\quad + 2\beta_1\beta_2p_1\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_1 - (2\beta_1\partial q_1 + 2\beta_1rq_2 - q_2)\partial^{-1}p_1 \\
&\quad + \beta_2^2p_1\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_2 - (\beta_2\partial q_1 + \beta_2rq_2 - p_1)\partial^{-1}q_2, \\
\Phi_{22} &= -2\beta_1p_1\partial^{-1}q_2\partial - 2\beta_1p_1\partial^{-1}q_1r + 4\beta_1^2p_1\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_2 \\
&\quad - 2\beta_1\beta_2p_1\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_1 - \beta_2p_1\partial^{-1}p_1\partial + \beta_2p_1\partial^{-1}p_2r \\
&\quad + 2\beta_1\beta_2p_1\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_2 - (2\beta_1\partial q_1 + 2\beta_1rq_2 - q_2)\partial^{-1}p_2 \\
&\quad - \beta_2^2p_1\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_1 + (\beta_2\partial q_1 + \beta_2rq_2 - p_1)\partial^{-1}q_1,
\end{aligned}$$

$$\begin{aligned}
\Phi_{23} = & -r + 2\beta_1 p_1\partial^{-1}p_1\partial - 2\beta_1 p_1\partial^{-1}p_2 r + 4\beta_1^2 p_1\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_1\\
& - 2\beta_1\beta_2 p_1\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_2 + \beta_2 p_1\partial^{-1}q_2\partial + \beta_2 p_1\partial^{-1}q_1 r\\
& + 2\beta_1\beta_2 p_1\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_1 - (2\beta_1\partial q_1 + 2\beta_1 r q_2 - q_2)\partial^{-1}q_1\\
& - \beta_2^2 p_1\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_2 + (\beta_2\partial q_1 + \beta_2 r q_2 - p_1)\partial^{-1}p_2,\\
\Phi_{24} = & \,\partial + 2\beta_1 p_1\partial^{-1}p_2\partial + 2\beta_1 p_1\partial^{-1}p_1 r + 4\beta_1^2 p_1\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_2\\
& + 2\beta_1\beta_2 p_1\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_1 - \beta_2 p_1\partial^{-1}q_1\partial + \beta_2 p_1\partial^{-1}q_2 r\\
& + 2\beta_1\beta_2 p_1\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_2 - (2\beta_1\partial q_1 + 2\beta_1 r q_2 - q_2)\partial^{-1}q_2\\
& + \beta_2^2 p_1\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_1 - (\beta_2\partial q_1 + \beta_2 r q_2 - p_1)\partial^{-1}p_1,\\
\Phi_{31} = & -\partial + 2\beta_1 q_2\partial^{-1}q_1\partial - 2\beta_1 q_2\partial^{-1}q_2 r - 4\beta_1^2 q_2\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_1\\
& - 2\beta_1\beta_2 q_2\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_2 - \beta_2 q_2\partial^{-1}p_2\partial + \beta_2 q_2\partial^{-1}p_1 r\\
& - 2\beta_1\beta_2 q_2\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_1 - (2\beta_1\partial p_2 - 2\beta_1 r p_1 + p_1)\partial^{-1}p_1\\
& - \beta_2^2 q_2\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_2 - (\beta_2\partial p_2 - \beta_2 r p_1 + q_2)\partial^{-1}q_2,\\
\Phi_{32} = & -r + 2\beta_1 q_2\partial^{-1}q_2\partial + 2\beta_1 q_2\partial^{-1}q_1 r - 4\beta_1^2 q_2\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_2\\
& + 2\beta_1\beta_2 q_2\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_1 + \beta_2 q_2\partial^{-1}p_1\partial - \beta_2 q_2\partial^{-1}p_2 r\\
& - 2\beta_1\beta_2 q_2\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_2 - (2\beta_1\partial p_2 - 2\beta_1 r p_1 + p_1)\partial^{-1}p_2\\
& + \beta_2^2 q_2\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_1 + (\beta_2\partial p_2 - \beta_2 r p_1 + q_2)\partial^{-1}q_1,\\
\Phi_{33} = & -2\beta_1 q_2\partial^{-1}p_1\partial + 2\beta_1 q_2\partial^{-1}p_2 r - 4\beta_1^2 q_2\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_1\\
& + 2\beta_1\beta_2 q_2\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_2 - \beta_2 q_2\partial^{-1}q_2\partial - \beta_2 q_2\partial^{-1}q_1 r\\
& - 2\beta_1\beta_2 q_2\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_1 - (2\beta_1\partial p_2 - 2\beta_1 r p_1 + p_1)\partial^{-1}q_1\\
& + \beta_2^2 q_2\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_2 + (\beta_2\partial p_2 - \beta_2 r p_1 + q_2)\partial^{-1}p_2,\\
\Phi_{34} = & -2\beta_1 q_2\partial^{-1}p_2\partial - 2\beta_1 q_2\partial^{-1}p_1 r - 4\beta_1^2 q_2\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_2\\
& - 2\beta_1\beta_2 q_2\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_1 + \beta_2 q_2\partial^{-1}q_1\partial - \beta_2 q_2\partial^{-1}q_2 r\\
& - 2\beta_1\beta_2 q_2\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_2 - (2\beta_1\partial p_2 - 2\beta_1 r p_1 + p_1)\partial^{-1}q_2\\
& - \beta_2^2 q_2\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_1 - (\beta_2\partial p_2 - \beta_2 r p_1 + q_2)\partial^{-1}p_1,\\
\Phi_{41} = & \, r - 2\beta_1 q_1\partial^{-1}q_1\partial + 2\beta_1 q_1\partial^{-1}q_2 r + 4\beta_1^2 q_1\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_1\\
& + 2\beta_1\beta_2 q_1\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_2 + \beta_2 q_1\partial^{-1}p_2\partial - \beta_2 q_1\partial^{-1}p_1 r\\
& + 2\beta_1\beta_2 q_1\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_1 + (2\beta_1\partial p_1 + 2\beta_1 r p_2 - p_2)\partial^{-1}p_1\\
& + \beta_2^2 q_1\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_2 + (\beta_2\partial p_1 + \beta_2 r p_2 + q_1)\partial^{-1}q_2,\\
\Phi_{42} = & -\partial - 2\beta_1 q_1\partial^{-1}q_2\partial - 2\beta_1 q_1\partial^{-1}q_1 r + 4\beta_1^2 q_1\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_2\\
& - 2\beta_1\beta_2 q_1\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_1 - \beta_2 q_1\partial^{-1}p_1\partial + \beta_2 q_1\partial^{-1}p_2 r\\
& + 2\beta_1\beta_2 q_1\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_2 + (2\beta_1\partial p_1 + 2\beta_1 r p_2 - p_2)\partial^{-1}p_2\\
& - \beta_2^2 q_1\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_1 - (\beta_2\partial p_1 + \beta_2 r p_2 + q_1)\partial^{-1}q_1,\\
\Phi_{43} = & \, 2\beta_1 q_1\partial^{-1}p_1\partial - 2\beta_1 q_1\partial^{-1}p_2 r + 4\beta_1^2 q_1\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_1\\
& - 2\beta_1\beta_2 q_1\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_2 + \beta_2 q_1\partial^{-1}q_2\partial + \beta_2 q_1\partial^{-1}q_1 r\\
& + 2\beta_1\beta_2 q_1\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_1 + (2\beta_1\partial p_1 + 2\beta_1 r p_2 - p_2)\partial^{-1}q_1\\
& - \beta_2^2 q_1\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_2 - (\beta_2\partial p_1 + \beta_2 r p_2 + q_1)\partial^{-1}p_2,\\
\Phi_{44} = & \, 2\beta_1 q_1\partial^{-1}p_2\partial + 2\beta_1 q_1\partial^{-1}p_1 r + 4\beta_1^2 q_1\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}q_2\\
& + 2\beta_1\beta_2 q_1\partial^{-1}(p_1\partial q_2 - p_2\partial q_1 - q_1\partial p_2 + q_2\partial p_1)\partial^{-1}p_1 - \beta_2 q_1\partial^{-1}q_1\partial + \beta_2 q_1\partial^{-1}q_2 r\\
& + 2\beta_1\beta_2 q_1\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}q_2 + (2\beta_1\partial p_1 + 2\beta_1 r p_2 - p_2)\partial^{-1}q_2\\
& + \beta_2^2 q_1\partial^{-1}(q_1\partial q_1 + p_1\partial p_1 + q_2\partial q_2 + p_2\partial p_2)\partial^{-1}p_1 + (\beta_2\partial p_1 + \beta_2 r p_2 + q_1)\partial^{-1}p_1.
\end{aligned}$$

Remark 3
If we let the spectral matrix U be

$$\begin{aligned}
U &= (\lambda + s)e_1 + q_1 e_2 + q_2 e_3 + p_1 e_4 + p_2 e_5 + r e_6\\
&= \begin{bmatrix} 0 & -q_1 & -q_2 & -\lambda - s\\ q_1 & 0 & -r & -p_1\\ q_2 & r & 0 & -p_2\\ \lambda + s & p_1 & p_2 & 0 \end{bmatrix}
\end{aligned}$$

and

$$
\begin{aligned}
r &= \beta_1\left(p_1^2+p_2^2+q_1^2+q_2^2\right)+\beta_2(p_1q_2-p_2q_1),\\
s &= \beta_3\left(p_1^2+p_2^2+q_1^2+q_2^2\right)+\beta_4(p_1q_2-p_2q_1),
\end{aligned}
$$

following the procedure of constructing the aforementioned soliton hierarchy of AKNS type, we can obtain a generalized soliton hierarchy of AKNS type.

4. New soliton hierarchy of Wadati–Konno–Ichikawa type

In this section, we will construct new soliton hierarchy of WKI type from the matrix spectral problem (2.1) with the spectral matrix U being chosen as

$$
\begin{aligned}
U &= \lambda e_1+\lambda q_1 e_2+\lambda q_2 e_3+\lambda p_1 e_4+\lambda p_2 e_5\\
&= \begin{bmatrix} 0 & -\lambda q_1 & -\lambda q_2 & -\lambda \\ \lambda q_1 & 0 & 0 & -\lambda p_1 \\ \lambda q_2 & 0 & 0 & -\lambda p_2 \\ \lambda & \lambda p_1 & \lambda p_2 & 0 \end{bmatrix} \in \widetilde{\mathrm{so}}(4,\mathbb{R}).
\end{aligned}
\tag{4.1}
$$

This spectral problem is of the same type as the WKI hierarchies associated with $\mathrm{sl}(2,\mathbb{R})$ [3], but its underlying loop algebra is different.

Let W be of the form (2.3), we solve the stationary zero curvature equation $W_x=[U,W]$, and it becomes

$$
\begin{aligned}
a_x &= \lambda(p_1c_1+p_2c_2-q_1b_1-q_2b_2),\\
d_x &= \lambda(q_1c_2+p_1b_2-q_2c_1-p_2b_1),\\
b_{1x} &= -\lambda c_1+\lambda q_1 a+\lambda p_2 d\\
b_{2x} &= -\lambda c_2+\lambda q_2 a-\lambda p_1 d\\
c_{1x} &= \lambda b_1-\lambda p_1 a+\lambda q_2 d\\
c_{2x} &= \lambda b_2-\lambda p_2 a-\lambda q_1 d.
\end{aligned}
\tag{4.2}
$$

Further setting W be (3.4), the aforementioned system (4.2) gives rise to

$$
\begin{aligned}
a_x^{(k)} &= p_1c_1^{(k+1)}+p_2c_2^{(k+1)}-q_1b_1^{(k+1)}-q_2b_2^{(k+1)},\\
d_x^{(k)} &= q_1c_2^{(k+1)}+p_1b_2^{(k+1)}-q_2c_1^{(k+1)}-p_2b_1^{(k+1)},\\
b_1^{(k+1)} &= c_{1x}^{(k)}+p_1a^{(k+1)}-q_2d^{(k+1)},\\
b_2^{(k+1)} &= c_{2x}^{(k)}+p_2a^{(k+1)}+q_1d^{(k+1)},\\
c_1^{(k+1)} &= -b_{1x}^{(k)}+q_1a^{(k+1)}+p_2d^{(k+1)},\\
c_2^{(k+1)} &= -b_{2x}^{(k)}+q_2a^{(k+1)}-p_1d^{(k+1)}, \qquad k\geq 0,
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
p_1c_1^{(0)}+p_2c_2^{(0)}-q_1b_1^{(0)}-q_2b_2^{(0)} &= 0,\\
q_1c_2^{(0)}+p_1b_2^{(0)}-q_2c_1^{(0)}-p_2b_1^{(0)} &= 0,\\
b_1^{(0)} &= p_1a^{(0)}-q_2d^{(0)},\\
b_2^{(0)} &= p_2a^{(0)}+q_1d^{(0)},\\
c_1^{(0)} &= q_1a^{(0)}+p_2d^{(0)},\\
c_2^{(0)} &= q_2a^{(0)}-p_1d^{(0)}.
\end{aligned}
\tag{4.4}
$$

To determine the sequence of $\left\{a^{(k)},d^{(k)},b_i^{(k)},c_i^{(k)},\ i=1,2\right\}$, we rewrite $a_x^{(k)}$ to the following form:

$$
\begin{aligned}
a_x^{(k)} &= p_1c_1^{(k+1)}+p_2c_2^{(k+1)}-q_1b_1^{(k+1)}-q_2b_2^{(k+1)}\\
&= p_1\left(-b_{1x}^{(k)}+q_1a^{(k+1)}+p_2d^{(k+1)}\right)+p_2\left(-b_{2x}^{(k)}+q_2a^{(k+1)}-p_1d^{(k+1)}\right)\\
&\quad -q_1\left(c_{1x}^{(k)}+p_1a^{(k+1)}-q_2d^{(k+1)}\right)-q_2\left(c_{2x}^{(k)}+p_2a^{(k+1)}+q_1d^{(k+1)}\right)\\
&= -p_1b_{1x}^{(k)}-p_2b_{2x}^{(k)}-q_1c_{1x}^{(k)}-q_2c_{2x}^{(k)}.
\end{aligned}
$$

Thus, $a_x^{(k+1)}$ becomes

$$
\begin{aligned}
a_x^{(k+1)} &= -p_1 b_{1x}^{(k+1)} - p_2 b_{2x}^{(k+1)} - q_1 c_{1x}^{(k+1)} - q_2 c_{2x}^{(k+1)} \\
&= -p_1 \left[c_{1xx}^{(k)} + \left(p_1 a^{(k+1)}\right)_x - \left(q_2 d^{(k+1)}\right)_x \right] - p_2 \left[c_{2xx}^{(k)} + \left(p_2 a^{(k+1)}\right)_x + \left(q_1 d^{(k+1)}\right)_x \right] \\
&\quad - q_1 \left[-b_{1xx}^{(k)} + \left(q_1 a^{(k+1)}\right)_x + \left(p_2 d^{(k+1)}\right)_x \right] - q_2 \left[-b_{2xx}^{(k)} + \left(q_2 a^{(k+1)}\right)_x - \left(p_1 d^{(k+1)}\right)_x \right],
\end{aligned}
$$

namely,

$$
\begin{aligned}
&(p_1^2 + p_2^2 + q_1^2 + q_2^2 + 1) a_x^{(k+1)} + (p_1 p_{1x} + p_2 p_{2x} + q_1 q_{1x} + q_2 q_{2x})\, a^{(k+1)} \\
&\qquad = 2(p_1 q_2 - p_2 q_1) d_x^{(k+1)} + (p_{1x} q_2 + p_1 q_{2x} - p_{2x} q_1 - p_2 q_{1x})\, d^{(k+1)} \\
&\qquad\quad - p_1 c_{1xx}^{(k)} - p_2 c_{2xx}^{(k)} + q_1 b_{1xx}^{(k)} + q_2 b_{2xx}^{(k)}.
\end{aligned} \tag{4.5}
$$

Similarly, we can obtain

$$
\begin{aligned}
&(p_1^2 + p_2^2 + q_1^2 + q_2^2 + 1) d_x^{(k+1)} + (p_1 p_{1x} + p_2 p_{2x} + q_1 q_{1x} + q_2 q_{2x})\, d^{(k+1)} \\
&\qquad = 2(p_1 q_2 - p_2 q_1) a_x^{(k+1)} + (p_{1x} q_2 + p_1 q_{2x} - p_{2x} q_1 - p_2 q_{1x})\, a^{(k+1)} \\
&\qquad\quad - p_1 b_{2xx}^{(k)} + q_2 c_{1xx}^{(k)} - q_1 c_{2xx}^{(k)} + p_2 b_{1xx}^{(k)}.
\end{aligned} \tag{4.6}
$$

Equation (4.5) minus (4.6) yields

$$
\begin{aligned}
&\sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2 + 1} \left[\sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2 + 1} \left(a^{(k+1)} - d^{(k+1)} \right) \right]_x \\
&\qquad = (p_1 + q_2) \left(b_{2xx}^{(k)} - c_{1xx}^{(k)} \right) + (q_1 - p_2) \left(b_{1xx}^{(k)} + c_{2xx}^{(k)} \right),
\end{aligned}
$$

so we have

$$
a^{(k+1)} = d^{(k+1)} + \frac{1}{\sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2 + 1}} \partial^{-1} \left[\frac{(p_1 + q_2) \left(b_{2xx}^{(k)} - c_{1xx}^{(k)} \right) + (q_1 - p_2) \left(b_{1xx}^{(k)} + c_{2xx}^{(k)} \right)}{\sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2 + 1}} \right].
$$

Substituting this equation into (4.6), we arrive at

$$
\begin{aligned}
d^{(k+1)} &= \frac{1}{\sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2 + 1}} \partial^{-1} \left\{ \frac{2\sqrt{p_1 q_2 - p_2 q_1}}{\sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2 + 1}} \right. \\
&\quad \times \partial \left[\frac{\sqrt{p_1 q_2 - p_2 q_1}}{\sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2 + 1}} \partial^{-1} \left(\frac{(p_1 + q_2) \left(b_{2xx}^{(k)} - c_{1xx}^{(k)} \right) + (q_1 - p_2) \left(b_{1xx}^{(k)} + c_{2xx}^{(k)} \right)}{\sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2 + 1}} \right) \right] \\
&\quad \left. + \frac{-p_1 b_{2xx}^{(k)} + q_2 c_{1xx}^{(k)} - q_1 c_{2xx}^{(k)} + p_2 b_{1xx}^{(k)}}{\sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2 + 1}} \right\}.
\end{aligned} \tag{4.7}
$$

Thus, $a^{(k+1)}$ has the following form:

$$
\begin{aligned}
a^{(k+1)} &= \frac{1}{\sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2 + 1}} \partial^{-1} \left\{ \frac{2\sqrt{p_1 q_2 - p_2 q_1}}{\sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2 + 1}} \right. \\
&\quad \times \partial \left[\frac{\sqrt{p_1 q_2 - p_2 q_1}}{\sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2 + 1}} \partial^{-1} \left(\frac{(p_1 + q_2) \left(b_{2xx}^{(k)} - c_{1xx}^{(k)} \right) + (q_1 - p_2) \left(b_{1xx}^{(k)} + c_{2xx}^{(k)} \right)}{\sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2 + 1}} \right) \right] \\
&\quad \left. + \frac{-p_1 c_{1xx}^{(k)} - p_2 c_{2xx}^{(k)} + q_1 b_{1xx}^{(k)} + q_2 b_{2xx}^{(k)}}{\sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2 + 1}} \right\},
\end{aligned} \tag{4.8}
$$

so one can obtain the sequence of $\left\{ a^{(k)}, d^{(k)}, b_i^{(k)}, c_i^{(k)},\ i = 1, 2 \right\}$ by Eqs (4.3), (4.7), (4.8) and a set of given initial values

$$
\begin{aligned}
a^{(0)} &= d^{(0)} = \frac{1}{\sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2 + 1}}, \\
b_1^{(0)} &= -c_2^{(0)} = \frac{p_1 - q_2}{\sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2 + 1}}, \\
b_2^{(0)} &= c_1^{(0)} = \frac{q_1 + p_2}{\sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2 + 1}}.
\end{aligned}
$$

$$
\begin{aligned}
d^{(1)} &= \frac{1}{[(p_1-q_2)^2+(p_2+q_1)^2+1]^{3/2}}\left(p_{1x}p_2-p_{1x}q_1-p_2q_{2x}-q_1q_{2x}-p_{2x}p_1+p_{2x}q_2-q_{1x}p_1+q_{1x}q_2\right),\\
a^{(1)} &= d^{(1)}+\frac{1}{\sqrt{(p_1-q_2)^2+(p_2+q_1)^2+1}},\\
b_1^{(1)} &= \frac{1}{[(p_1-q_2)^2+(p_2+q_1)^2+1]^{3/2}}\left(p_{2x}+q_{1x}+p_1^3-2p_1^2q_2+p_1q_2^2+p_1p_2^2+2p_1p_2q_1+p_1q_1^2+p_1\right),\\
b_2^{(1)} &= \frac{1}{[(p_1-q_2)^2+(p_2+q_1)^2+1]^{3/2}}\left(p_{1x}-q_{2x}-p_1^2p_2+2p_1p_2q_2-p_2q_2^2-p_2^3-2p_2^2q_1-p_2q_1^2-p_2\right),\\
c_1^{(1)} &= \frac{1}{[(p_1-q_2)^2+(p_2+q_1)^2+1]^{3/2}}\left(p_{1x}-q_{2x}-p_1^2q_1+2p_1q_1q_2-q_1q_2^2-p_2^2q_1-2p_2q_1^2-q_1^3-q_1\right),\\
c_2^{(1)} &= \frac{1}{[(p_1-q_2)^2+(p_2+q_1)^2+1]^{3/2}}\left(p_{2x}+q_{1x}-p_1^2q_2+2p_1q_2^2-q_2^3-p_2^2q_2-2p_2q_1q_2-q_1^2q_2-q_2\right).
\end{aligned}
$$

To guarantee the uniqueness of $\left\{a^{(k)},d^{(k)},b_i^{(k)},c_i^{(k)},\ i=1,2\right\}$, we also need to impose the integration conditions

$$
a^{(k)}|_{u=0}=d^{(k)}|_{u=0}=b_i^{(k)}|_{u=0}=c_i^{(k)}|_{u=0}=0,\qquad i=1,2,\quad k\geq 2.
$$

Therefore, we can determine the sequence by means of Maple and prove the localness of the functions $\left\{a^{(k)},d^{(k)},b_i^{(k)},c_i^{(k)},\ i=1,2\right\}$. However, we omit these tedious expressions for the sake of simplification.

Now, taking

$$
\begin{aligned}
V^{[n]} &= \lambda^2(\lambda^n W)_+ + \Delta_n\\
&= \lambda^2(\lambda^n W)_+ + \begin{bmatrix} 0 & \lambda b_{1x}^{(n)} & \lambda b_{2x}^{(n)} & 0\\ -\lambda b_{1x}^{(n)} & 0 & 0 & -\lambda c_{1x}^{(n)}\\ -\lambda b_{2x}^{(n)} & 0 & 0 & -\lambda c_{2x}^{(n)}\\ 0 & \lambda c_{1x}^{(n)} & \lambda c_{2x}^{(n)} & 0 \end{bmatrix},
\end{aligned}
$$

the zero curvature equations $U_{t_n}-V_x^{[n]}+\left[U,V^{[n]}\right]=0$, $n\geq 0$ engender a new soliton hierarchy of WKI type

$$
\begin{bmatrix} p_1\\ p_2\\ q_1\\ q_2 \end{bmatrix}_{t_n} = K_n = \begin{bmatrix} c_{1xx}^{(n)}\\ c_{2xx}^{(n)}\\ -b_{1xx}^{(n)}\\ -b_{2xx}^{(n)} \end{bmatrix},\qquad n\geq 0. \tag{4.9}
$$

5. Conclusions and discussions

There are many soliton hierarchies obtained by means of matrix spectral problems based on the real Lie algebras such as $sl(2,\mathbb{R})$ and $so(3,\mathbb{R})$ [1–19, 23–26, 28, 30, 32–38]. In this paper, we have introduced some new 4×4 matrix spectral problems based on $so(4,\mathbb{R})$ and new soliton hierarchies of KN type AKNS type and WKI type which have been derived. In addition, we use Maple to deal with some complex symbolic computation.

Different from semisimple Lie algebras, there is a growing interest in soliton hierarchies generated from matrix spectral problems associated with non-semisimple Lie algebras. Recently, various bi-integrable couplings and tri-integrable couplings from certain soliton hierarchies bring us ideas to construct and classify integrable systems [20–22, 29, 32]. In a subsequent study, we will consider these issues including the construction of n-coupled integrable coupling systems based on non-semisimple Lie algebras, the classification problem of multi-component soliton hierarchies and the guarantee conditions of variational identity for non-semisimple Lie algebras.

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